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A LOHNER-TYPE ALGORITHM FOR CONTROL SYSTEMS AND ORDINARY DIFFERENTIAL INCLUSIONS

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Abstract. We describe a Lohner-type algorithm for the computation of rigorous upper bounds for reachable set for control systems, solutions of ordinary differential inclusions and perturbations of ODEs.

1. **Introduction.** Our goal is to present a Lohner-type algorithm for a rigorous integration of perturbations of ODEs, which can be seen also as an algorithm for an integration of control systems or ordinary differential inclusions. By rigorous integration we mean that we provide verified bounds, taking care of all errors appearing during numerical integration. This paper depends heavily on [25], as the proposed algorithm is a modification running on top of the C^0-Lohner algorithm for ODEs described (after [11, 12]) there.

We study the following nonautonomous ODE

$$x'(t) = f(x(t), y(t)), \quad x(0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is C^1 and $y : \mathbb{R} \supset D \to \mathbb{R}^m$. Assume that we have some knowledge about $y(t)$, for example $|y(t)| < \epsilon$ for $0 \leq t \leq T$. We would like to find a rigorous enclosure for $x(t)$.

The problem of this type arises, for example, in the context of the control theory (see [5, 9, 20]) and in the rigorous integration of dissipative PDEs (see [24, 26, 28] for more details). In this last setting x represents the dominating modes and y is a tail of the Fourier expansion, so that (1) is complemented by the equation for y of the form $y'(t) = g(x(t), y(t))$ for which we are able to produce some a priori bounds. The proposed algorithm works. Using it we were able to prove the existence of multiple periodic orbits for Kuramoto-Sivashinsky PDE [26, 28].

The proposed algorithm can also be used to find rigorous bounds for solutions of differential inclusions

$$x' \in h(x) + [\epsilon(t)], \tag{2}$$

where $h : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-vector field and $[\epsilon(t)] \subset \mathbb{R}^n$. We can cast (2) in the form (1) by setting $f(x, y) = h(x) + y$ and requiring that $y(t) \in [\epsilon(t)]$ for all t. The equation (2) is know as an "inflation" (see [6, 8]).

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Non-autonomous ODEs represent another important class of applications. While one can easily modify the Lohner algorithm to handle a non-autonomous ODE directly, it makes sense to apply the proposed Lohner-type algorithm for perturbed ODEs for (1), because only in this way we can estimate rigorously the Poincaré map on a section $\alpha(x) = 0$ (defined in terms of x only) for any initial conditions (x, t_0). This kind of algorithm shall allow to attack the question of symbolic dynamics for non-autonomous ODEs (see [2]) and ODEs with small delays (see [23]).

Another new element in this paper, besides the proposed algorithm, is a new inequality concerning bounds for perturbations of ODEs. It is contained in Theorem 4.2 and links together the component-wise estimates based on one-sided Lipschitz conditions (see [21]) and the logarithmic norms (see [3, 13]).

The content of the present paper can be described as follows: in Section 2 we define a notion of weak solution of (1) and state some facts from the theory of Lebesgue integration. In Section 3 we recall the notion of the logarithmic norm and state its basic properties. In Sections 4 and 5 we derive basic estimates for comparison of perturbed and unperturbed ODEs. In Section 6 we give a description of one step of the proposed Lohner-type algorithm. In Section 7 we describe how to estimate the trajectory of (1) between time steps which allows to compute the Poincaré map. In the following section we discuss some tests.

The algorithm presented in this paper was implemented as a part of CAPD library (see [1]). This library contains many tools for rigorous computations and computer assisted proofs in the contexts of dynamical systems. All the tests in Section 8 was performed using CAPD library.

1.1. **Basic notation.** We will use the same conventions as in [25]. In the sequel, by arabic letters we denote single valued objects like vectors, real numbers, matrices. Quite often in this paper we will use square brackets, for example $[r]$, to denote sets. Usually this will be some set constructed in the algorithm. Sets will also be denoted by single letters, for example S, when it is clear from the context that it represents a set. In situations when we want to stress (for example in the detailed description of algorithm) that we have a set in a formula involving both single-valued objects and sets we will rather use the square bracket, hence we prefer to write $[S]$ instead of S to represent a set. From this point of view $[S]$ and S are different symbols in the alphabet used to name variables and formally speaking there is no relation between the set represented by $[S]$ and the object represented by S. Quite often in the description of the algorithm we will have a situation that both variables $[S]$ and S are used simultaneously, then usually $S \in [S]$, but this is always stated explicitly.

For a set $[S]$ by $[S]_I$ we denote the interval hull of $[S]$, i.e. the smallest product of intervals containing $[S]$. The symbol $\mathrm{hull}(x_1, \ldots, x_k)$ will denote the interval hull of intervals $x_1, \ldots, x_k$. For any interval set $[S] = [S]_I$ by $\mathrm{m}([S])$ we will denote a center point of $[S]_I$. For any interval $[a, b]$ we define a diameter by $\mathrm{diam}([a, b]) = b - a$. For an interval vector or an interval matrix $[S] = [S]_I$ by $\mathrm{diam}\,([S])$ we will denote the maximum of diameters of its components. For an interval $[x^-, x+]$ we set $right([x^-, x^+]) = x^+$ and $left([x^-, x^+]) = x^-$.

For a function $f(x_1, x_2, \ldots, x_k)$ and sets $X_1, X_2, \ldots, X_k$ we define

$$f(X_1, \ldots, X_k) = \{f(z_1, \ldots, z_k) \mid \text{ where } z_i \in X_i \text{ for } i = 1, \ldots, j\}$$

For a set $X \subset \mathbb{R}^d$ by $\mathrm{int}\, X$ we denote an interior of X. For $\mathbb{R}^n$ we will denote the norm of x by $\|x\|$ and if the formula for the norm is not specified in some context, then it means that it is ok to use any norm there. Let $x_0 \in \mathbb{R}^s$, then $B(x_0, r) = \{z \in \mathbb{R}^s \mid \|x_0 - z\| < r\}$.

For $v, w \in \mathbb{R}^n$ and $A, B \in \mathbb{R}^{n\times n}$ $(n = 1, \dots, \infty)$ we say that

$$\begin{array}{lll} v \leq w & \text{iff} & \forall i \quad v_i \leq w_i, \\ A \leq B & \text{iff} & \forall ij \quad A_{ij} \leq B_{ij}. \end{array}$$

1.2. **Warning.** At the first encounter with the question of an rigorous integration of (1) one may hope that the direct application of any algorithm for rigorous integration of ODEs should be enough for (2). To this end consider a differential inclusion

$$x' \in f(x) + [\epsilon], \qquad [\epsilon] = \Pi_{i=1}^n [-\epsilon_i, \epsilon_i]. \tag{3}$$

and a related ODE

$$x' = f(x) + \epsilon, \qquad \epsilon \in [\epsilon]. \tag{4}$$

One may naively hope that, for example, the Lohner algorithm applied to (4) with $[\epsilon]$ as an interval parameter in the definition of a constant term in $f(x)$ will give an enclosure not only for (4), but also for (3). For this to be true we need the following

Conjecture 1. Assume $x(t)$ satisfies (3) for $t \in [0, T]$.

Then for any $t \in [0, T]$ there exists $\epsilon \in [\epsilon]$ such that $x_\epsilon(t) = x(t)$ and $x_\epsilon(0) = x(0)$, where $x_\epsilon$is a solution of (4).

The above conjecture is false as shown by the following example [18].

Consider a differential inclusion given by

$$\begin{aligned} x' &\in y + [-\epsilon, \epsilon], \\ y' &\in -x + [-\epsilon, \epsilon]. \end{aligned} \tag{5}$$

For fixed $\delta \in [-\epsilon, \epsilon]^2$ we have the following system of ODEs

$$\begin{aligned} x' &= y + \delta_1, \\ y' &= -x + \delta_2, \end{aligned} \tag{6}$$

all solutions with an initial condition in a compact set have a uniform bound independent of δ for $t > 0$, which is given by the energy integral for (6)

$$(x - \delta_2)^2 + (y + \delta_1)^2. \tag{7}$$

This is not the case for the solutions of (5) as it is clearly seen for $\epsilon(t)$ given as a resonant forcing

$$\begin{aligned} x' &= y, \\ y' &= -x + \epsilon \sin t. \end{aligned} \tag{8}$$

The main reason for the difference between equations (3) and (4) lies in the fact that differential inclusion (3) corresponds to the ODE with *time varying perturbation* $\epsilon(t)$ i.e.

$$x' = f(x) + \epsilon(t).$$

2. **Control systems, the notion of the solution.** In this section we define a notion of (weak) solution of (1).

We use some standard notions from the measure theory, see for example [17] for precise definitions. The integral will always mean the Lebesgue integral and the measure of the set is always Lebesgue measure.

2.1. **Some facts from the theory of Lebesgue integral.** We will denote by $m(E)$ the Lebesgue measure of E.

Let D be a measurable subset of $\mathbb{R}^k$. By $L^1(D)$ we will denote a set of measurable functions $f : D \to \mathbb{R}$ such that $\int_D |f| dm < \infty$. If $f : D \to \mathbb{R}^n$ is measurable, then we say that $f \in L^1(D)$ if function $\|f\| \in L^1(D)$.

Definition 2.1. Let $D \subset \mathbb{R}$ be an interval. The function $f : D \to \mathbb{R}^k$ is *absolutely continuous* , if for every $\epsilon > 0$ there exists $\delta > 0$, such that for any family of disjoint intervals $(\alpha_1, \beta_1), \ldots, (\alpha_N, \beta_N)$ with

$$\sum_{i=1}^{N} (\beta_i - \alpha_i) < \delta$$

the following inequality is satisfied

$$\sum_{i=1}^{N} (f(\beta_i) - f(\alpha_i)) < \epsilon$$

The following statement follows directly from results about the differentiability of measures and functions of bounded variation (see [17, Chapter 8]).

Theorem 2.2. *Let $D = [a, b]$, $x : D \to \mathbb{R}^n$.*

There exists a measurable function $g : D \to \mathbb{R}^n$ such that equation

$$x(t) - x(a) = \int_a^t g(s) ds \tag{9}$$

holds for all $t \in [a, b]$ iff x is absolutely continuous. In this situation $x'(t)$ exists almost everywhere in $[a, b]$ and $x'(t) = g(t)$.

From [17, Thm. 8.8] it follows that

Lemma 2.3. Let $f : [a, b] \to \mathbb{R}^k$ be a measurable function. Then for almost all points $x \in [a, b)$ holds

$$\lim_{h \to 0^+} \frac{1}{h} \int_x^{x+h} \|f(s) - f(x)\| ds = 0 \tag{10}$$

2.2. **Weak solutions of ODEs.** Control System is given by equation

$$x'(t) = f(x(t), y(t)) \quad x(t_0) = x_0 \tag{11}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is C^1 and $y : \mathbb{R} \supset D \to \mathbb{R}^m$ is a measurable function from a given class U.

Because the right hand side of (11) can be non-continuous we need to define what we mean by solution of (11).

Definition 2.4. Let $D \subset \mathbb{R}$ be an interval (a connected subset of $\mathbb{R}$) containing t_0.

An absolutely continuous function $x : D \to \mathbb{R}^n$ is a *weak solution* of (11) if for all $t \in D$ holds

$$x(t) = x_0 + \int_{t_0}^t f(x(s), y(s)) ds. \tag{12}$$

We say that a continuous function $x : D \to \mathbb{R}^n$ is a *(classical) solution* of (11) if $x'(t)$ exists for all $t \in \operatorname{int} D$, $x(t_0) = x_0$ and

$$x'(t) = f(x(t), y(t)), \qquad \forall t \in \operatorname{int} D. \tag{13}$$

From Theorem 2.2 it follows that x is a weak solution of (11) iff

$$x'(t) = f(x(t), y(t)), \quad \text{allmost everywhere in } D \tag{14}$$

and the function $t \mapsto f(x(t), y(t))$ is in $L^1(D)$. Hence the weak solution in the sense of Def. 2.4 is a solution of (11) in the sense of Caratheodory [21].

In the remainder of this paper we will always consider the function f on the right hand side of (11) to be of class C^k (for $k \geq 1$) and y to be bounded on compact intervals and measurable. In such situation the integral equation (12) has a unique solution defined for some $h > 0$ on $[t_0, t + h]$. The proof of this fact is a straightforward application of the Banach contraction principle [21].

3. Basic facts on logarithmic norms.

Let $\|\cdot\|$ denote a vector norm on $\mathbb{R}^n$ as well as its subordinate matrix (operator) norm on $\mathbb{R}^{n\times n}$. The classical definition of the *logarithmic norm* of matrix A,

$$\mu(A) = \lim_{h\to 0^+} \frac{\|I + hA\| - 1}{h} \tag{15}$$

was introduced in 1958 independently by Dahlquist [3] and Lozinskii [13].

In this section we will briefly recall some basic facts, with proofs, about the logarithmic norms. A survey regarding the modern developments stemming from this notion the reader is referred to [19] and the literature given there. Our presentation is based on [4, Ch. 1.5], which was based on [3].

Lemma 3.1. For any matrix $A \in \mathbb{R}^{n\times n}$ the limit in (15) exists and

$$\frac{\|I + h_1 A\| - 1}{h_1} \leq \frac{\|I + h_2 A\| - 1}{h_2}, \qquad \text{for } 0 < h_1 < h_2 \tag{16}$$

$$-\|A\| \leq \mu(A) \leq \|A\|. \tag{17}$$

Proof: Let us fix $h > 0$ and let $0 < \theta < 1$, then

$$\|I + \theta hA\| = \|\theta(I + hA) + (1-\theta)I\| \leq \theta\|I + hA\| + (1-\theta)\|I\|.$$

From this immediately obtain

$$\frac{\|I + \theta hA\| - 1}{\theta h} \leq \frac{\|I + hA\| - 1}{h}, \tag{18}$$

which proves (16).

From the triangle inequality one gets

$$-h\|A\| \leq \|I + hA\| - \|I\| \leq h\|A\|, \tag{19}$$

therefore

$$-\|A\| \leq \frac{\|I + hA\| - 1}{h} \leq \|A\|. \tag{20}$$

The monotonicity (16) and the existence of the lower bound imply the existence of $\mu(A)$. □

Theorem 3.2. *The function $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$, which assigns to A its logarithmic norm is continuous and convex. Moreover, functions $\mu(h, A) = \frac{\|I+hA\|-1}{h}$ converge locally uniformly and monotonically to $\mu(A)$ for $h \to 0^+$.*

To be more precise, for any compact set $K \subset \mathbb{R}^{n\times n}$ and any $\epsilon > 0$ there exists $h_0 > 0$, such that for all $0 < h < h_0$ and any $A \in K$ holds

$$\epsilon > \mu(h, A) - \mu(A) \geq 0. \tag{21}$$

Proof: Let $h > 0$. An easy computation show that, for any $0 \leq \lambda \leq 1$ and $A_1, A_2 \in \mathbb{R}^{n\times n}$ holds

$$\mu(h, \lambda A_1 + (1-\lambda)A_2) \leq \lambda\mu(h, A_1) + (1-\lambda)\mu(h, A_2).$$

Therefore, for any $h > 0$ function $\mu(h, \cdot) : \mathbb{R}^{n\times n} \to \mathbb{R}$ is convex.

By taking the limit $h \to 0^+$ from Lemma 3.1 it follows that $\mu(A)$ is a convex function. Observe that on any bounded set $U \subset \mathbb{R}^{n\times n}$ $\mu(A)$ is bounded by $\sup_{A\in U} \|A\| < +\infty$, therefore from the theory of convex functions (see for example [10, Chap. 6]) it follows that μ is continuous. The uniform convergence of $\mu(h, \cdot)$ to μ on compact sets follows from Dini's Theorem on monotone sequences of pointwise converging continuous functions to continuous limit and Lemma 3.1. □

The following lemma follows directly from the convexity of $\mu(A)$

Lemma 3.3. Let $A : [0, 1] \to \mathbb{R}^{n\times n}$ be a bounded measurable function. Then

$$\mu\left(\int_0^1 A(s)ds\right) \leq \int_0^1 \mu(A(s))ds \leq \sup_{s\in[0,1]} \mu(A(s)). \tag{22}$$

4. Bounds for perturbations of ODEs.

In this section we state the basic theorem comparing a solution of an ODE and an approximate solution. Our approach unifies the approach based on logarithmic norms and one-sided Lipschitz condition leading to component-wise bounds from [21, Ch. II.13].

4.1. Estimates for non-autonomous linear equations.

Consider a linear equation

$$x'(t) = A(t) \cdot x(t) + b(t), \tag{23}$$

where $x(t) \in \mathbb{R}^k$, $A(t) \in \mathbb{R}^{k\times k}$, $b(t) \in \mathbb{R}^k$, A and b are bounded and measurable.

We would like to give some bounds on solutions of (23). We consider that a decomposition of the phase space $\mathbb{R}^k$ of the form $\mathbb{R}^k = \oplus_{i=1}^n \mathbb{R}^{k_i}$. Therefore, we have a decomposition of $z \in \mathbb{R}^k$ into $(z_1, \ldots, z_n)$ such that $z_i \in \mathbb{R}^{k_i}$. In this section we will carefully distinguish between the symbol $\|\cdot\|$ and $|\cdot|$. The symbol $\|\cdot\|$ will always denote a norm, but the symbol $|z|$ for $z \in \mathbb{R}^k$ will usually denote a vector of norms of z_i, but this will be always clearly indicated in the text. Observe that, when we have such decomposition, then equation (23) can be written as follows

$$z_i'(t) = \sum_j A_{ij}(t)z_j(t) + b_i(t), \quad i = 1, \ldots, n \tag{24}$$

where $z_i, b_i \in \mathbb{R}^{k_i}$ and $A_{ij}(t) \in L(\mathbb{R}^{k_i}, \mathbb{R}^{k_j})$ is a linear map (a matrix). In this way the matrix A is decomposed into blocks A_{ij}. To each block we will assign a number J_{ij} and collect them in a matrix J. Roughly speaking J_{ij} will estimate the influence of z_j on z_i'.

The fundamental lemma in this section is:

Lemma 4.1. Assume that $z : [0, T] \to \mathbb{R}^k = \oplus_{i=1}^n \mathbb{R}^{k_i}$ is an absolutely continuous map, which is a weak solution of the equation

$$z'(t) = A(t) \cdot z(t) + \delta(t), \tag{25}$$

where $\delta : [0, T] \to \mathbb{R}^k$ and $A : [0, T] \to \mathbb{R}^{k\times k}$ are bounded and measurable.

Assume that a measurable matrix function $J : [0,T] \to \mathbb{R}^{n\times n}$ satisfies the following inequalities for all $t \in [0,T]$

$$J_{ij}(t) \geq \begin{cases} \|A_{ij}(t)\| & \text{for } i \neq j, \\ \mu(A_{ii}(t)) & \text{for } i = j. \end{cases} \tag{26}$$

Let $C_i(t) = \|\delta_i(t)\|$ and $|z|(t) = (\|z_1(t)\|, \|z_2(t)\|, \ldots, \|z_n(t)\|)$.
Then

$$|z|(t) \leq y(t) \tag{27}$$

where $y : [0,T] \to \mathbb{R}^n$ is a weak solution of the problem

$$y'(t) = J(t)y(t) + C(t), \qquad y(0) = |z|(0). \tag{28}$$

Proof: Observe that for all i the function $t \mapsto \|z_i(t)\|$ is absolutely continuous. Therefore from Theorem 2.2 it follows that for almost every $t \in [0,T]$ the derivative of $\|z_i\|$ exists. We will estimate this derivative for such t.

We have

$$\begin{aligned} & z(t+h) \\ =& z(t) + \int_t^{t+h} A(s)z(s)ds + \int_t^{t+h} \delta(s)ds \\ =& z(t) + h\left(A(t)z(t)\right) + h\delta(t)) + \int_t^{t+h} (A(s)z(s) - A(t)z(t)) + (\delta(s) - \delta(t))ds \end{aligned}$$

Let us fix i and $t \in [0,T)$. We consider the projection onto the i-th subspace. We have

$$\begin{aligned} \|z_i(t+h)\| \leq & \|I + hA_{ii}(t)\| \cdot \|z_i\|(t) + h\sum_{j\neq i} \|A_{ij}(t)\| \cdot \|z_j(t)\| + h\|\delta_i(t)\| \\ & + \int_t^{t+h} \|A(s)z(s) - A(t)z(t)\|\, ds + \int_t^{t+h} \|\delta(s) - \delta(t)\|\, ds \end{aligned}$$

and then we obtain for $h > 0$

$$\begin{aligned} & \frac{\|z_i(t+h)\| - \|z_i(t)\|}{h} \\ \leq & \frac{\|I + hA_{ii}(t)\| - 1}{h} \cdot \|z_i\|(t) \\ & + \sum_{j\neq i} \|A_{ij}(t)\| \cdot \|z_j(t)\| + C_i + \frac{1}{h}\int_t^{t+h} \|A(s)z(s) - A(t)z(t)\|\, ds \\ & + \frac{1}{h}\int_t^{t+h} \|\delta(s) - \delta(t)\|\, ds \end{aligned}$$

Observe that from Lemma 2.3 it follows that the last two terms in the above inequality tend to 0 as $h \to 0$ for almost all points in $[0,T)$. From now on we assume that t is such point.

By passing to the limit with $h \to 0^+$ we obtain for almost all points in $t \in [0,T]$

$$\begin{aligned} \frac{d\|z_i\|}{dt}(t) & \leq \mu(A_{ii}(t))\|z_i\|(t) + \sum_{j\neq i} \|A_{ij}(t)\| \cdot \|z_j(t)\| + C_i(t) \\ & \leq \sum_j J_{ij}(t)\|z_j\|(t) + C_i(t) \end{aligned} \tag{29}$$

Let us define

$$x(t) = (x_1(t), x_2(t), \dots, x_n(t)) = (\|z_1(t)\|, \|z_2(t)\|, \dots, \|z_n(t)\|),$$

Inequality (29) can be rewritten in vector form as follows

$$x'(t) \leq J(t) \cdot x(t) + C(t), \quad \text{for almost all } t \in [0,T]. \tag{30}$$

Let $y : [0,T] \to \mathbb{R}^n$ be a weak solution of

$$y'(t) = J(t) \cdot y(t) + C(t), \tag{31}$$

such that $y(0) > |z|(0) = x(0)$.

We want to show that

$$x(t) < y(t), \quad t \in [0,T]. \tag{32}$$

Let us take the diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$, such that $\Lambda_{ii} + J_{ii}(t) \geq 0$ for all $i = 1, \dots, n$ and $t \in [0,T]$. Let us define a matrix-valued function $B : [0,T] \to \mathbb{R}^{n\times n}$ by

$$B(t) = \Lambda + J(t). \tag{33}$$

Obviously $B_{ij}(t) \geq 0$ for all $t \in [0,T]$.

For any $i = 1, \dots, n$ from (30) we obtain for almost all $t \in [0,T]$

$$x_i'(t) + \Lambda_{ii} x_i(t) \leq \sum_j B_{ij}(t) x_j(t) + C_i(t), \tag{34}$$

hence

$$\frac{d}{dt}\left(e^{\Lambda_{ii} t} x_i(t)\right) \leq e^{\Lambda_{ii} t} \left(\sum_j B_{ij} x_j(t) + C_i(t) \right).$$

The last inequality has the following vector form

$$\frac{d}{dt}\left(e^{\Lambda t} x(t)\right) \leq e^{\Lambda t} B(t) x(t) + e^{\Lambda t} C(t). \tag{35}$$

From the above inequality and from Theorem 2.2 it follows that

$$\begin{aligned} e^{\Lambda t} x(t) =& e^{\Lambda \cdot 0} x(0) + \int_0^t \frac{d}{dt}\left(e^{\Lambda t} x(t)\right)(s) ds \leq x(0) \\ &+ \int_0^t e^{\Lambda s} B(s) x(s) + e^{\Lambda s} C(s) ds. \end{aligned}$$

Hence we obtain

$$x(t) \leq e^{-\Lambda t} x(0) + \int_0^t e^{-\Lambda(t-s)} \left(B(s)x(s) + C(s)\right)\, ds \qquad \text{for } t \in [0,T] \tag{36}$$

An analogous computation applied to (31) shows that y satisfies the following integral equation

$$y(t) = e^{-\Lambda t} y(0) + \int_0^t e^{-\Lambda(t-s)} \left(B(s)y(s) + C(s)\right)\, ds. \tag{37}$$

Now we are ready to prove (32). Let

$$t_0 = \sup\{t \in [0,T] \mid y(s) > x(s), \quad s \in [0,t)\}. \tag{38}$$

Obviously from the continuity of $y(t) - x(t)$ it follows that $t_0 > 0$. From (37) and (36) we obtain

$$y(t_0) - x(t_0) \geq e^{-\Lambda t_0}(y(0) - x(0)) + \int_0^{t_0} e^{-\Lambda(t_0 - s)} B(s)(y(s) - x(s))\, ds > 0.$$

By the continuity inequality $y(t) > x(t)$ will hold for $t \in [t_0, t_0 + \epsilon)$ for some $\epsilon > 0$. Therefore $t_0 = T$.

Hence condition (32) holds. By passing to the limit $y(0) \to x(0)$ we obtain our assertion. □

Theorem 4.2. *Let $h > 0$. Assume that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be C^1 and $y : [t_0, t_0 + h] \to \mathbb{R}^m$ is bounded and measurable.*

Let $[W_y] \subset \mathbb{R}^m$ be convex and such that, $y([t_0, t_0 + h]) \subset [W_y]$.

Let $y_c \in [W_y]$. Assume that $x_1, x_2 : [t_0, t_0 + h] \to \mathbb{R}^n$, both absolutely continuous, are weak solutions of the following problems, respectively

$$x_1' = f(x_1, y_c), \quad x_1(t_0) = x_0, \tag{39}$$

$$x_2' = f(x_2, y(t)), \quad x_2(t_0) = \bar{x}_0. \tag{40}$$

Let $[W_1] \subset [W_2] \subset \mathbb{R}^n$ be convex and compact and such that

$$x_1(t) \in [W_1], \qquad x_2(t) \in [W_2], \qquad \text{for } t \in [t_0, t_0 + h].$$

Then the following inequality holds for $t \in [t_0, t_0 + h]$ and $i = 1, \dots, n$

$$|x_{1,i}(t) - x_{2,i}(t)| \leq \left(e^{J(t-t_0)} \cdot (x_0 - \bar{x}_0)\right)_i + \left(\int_{t_0}^t e^{J(t-s)} C \, ds\right)_i, \tag{41}$$

where

$$\begin{aligned} [\delta] &= \{f(x, y_c) - f(x, y) \mid x \in [W_1], y \in [W_y]\}, \\ C_i &\geq \sup |[\delta_i]|, \quad i = 1, \dots, n \\ J_{ij} &\geq \begin{cases} \sup \mu(\frac{\partial f_i}{\partial x_j}([W_2], [W_y])) & \text{if } i = j, \\ \sup \left\| \frac{\partial f_i}{\partial x_j}([W_2], [W_y]) \right\| & \text{if } i \neq j. \end{cases} \end{aligned}$$

Proof: Let $z(t) = x_1(t) - x_2(t)$. We have for $t \in [t_0, t_0 + h]$

$$\begin{aligned} z(t) &= \left(x_1(t_0) + \int_{t_0}^t f(x_1(s), y_c) ds\right) - \left(x_2(t_0) + \int_{t_0}^t f(x_2(s), y(s)) ds\right) \\ &= z(t_0) + \int_{t_0}^t (f(x_1(s), y_c) - f(x_2(s), y(s))) \, ds. \end{aligned}$$

Now observe that

$$\begin{aligned} &f(x_1(t), y_c) - f(x_2(t), y(t)) = f(x_1(t), y_c) - f(x_1(t), y(t)) + \\ &f(x_1(t), y(t)) - f(x_2(t), y(t)) = \delta(t) + A(t) \cdot (x_1(t) - x_2(t)), \end{aligned}$$

where $\delta(t) \in [\delta]$ is bounded and measurable and

$$A_{ij}(t) = \int_0^1 \frac{\partial f_i}{\partial x_j} (x_2(t) + s(x_1(t) - x_2(t)), y(t)) \, ds$$

is bounded and measurable matrix.

We obtain

$$z(t) = z(t_0) + \int_{t_0}^t (A(s) z(s) + \delta(s)) \, ds \tag{42}$$

To apply Lemma 4.1 to the function $z = x_1 - x_2$ to obtain (49) we need to show that

$$J_{ij} \geq \begin{cases} \sup_{t \in [t_0, t_0 + h]} \|A_{ij}(t)\| & \text{for } i \neq j, \\ \sup_{t \in [t_0, t_0 + h]} \mu(A_{ii}(t)) & \text{for } i = j. \end{cases} \tag{43}$$

For the off-diagonal terms we have

$$\begin{aligned}\|A_{ij}(t)\| &\leq \int_0^1 \left\| \frac{\partial f_i}{\partial x_j}(x_2(t)+s(x_1(t)-x_2(t)),y(t)) \right\| ds \\ &\leq \sup_{x\in[W_2],y\in[W_y]} \left\| \frac{\partial f_i}{\partial x_j}(x,y) \right\| \leq J_{ij}.\end{aligned}$$

For the diagonal case we use Lemma 3.3.

The result now follows from Lemma 4.1. □

It is possible to organize the error estimates slightly differently, namely estimate $[\delta]$ on $[W_2]\times[W_y]$ instead of on $[W_1]\times[W_y]$, which will produce larger $[\delta]$, but in the same time estimate J on $[W_2]\times\{y_c\}$ instead of $[W_2]\times[W_y]$, which should result in better J, to obtain the following variant of the above theorem.

Theorem 4.3. *The same assumptions and notations as in Theorem 4.2.*

Then the following inequality holds for $t\in[t_0,t_0+h]$ and $i=1,\dots,n$

$$|x_{1,i}(t)-x_{2,i}(t)| \leq \left(e^{J(t-t_0)}\cdot(x_0-\bar{x}_0)\right)_i + \left(\int_{t_0}^t e^{J(t-s)}C\,ds\right)_i, \tag{44}$$

where

$$\begin{aligned}[\delta] &= \{f(x,y_c)-f(x,y)\mid x\in[W_2], y\in[W_y]\},\\ C_i &\geq \sup|[\delta_i]|, \quad i=1,\dots,n\\ J_{ij} &\geq \begin{cases}\sup\mu(\frac{\partial f_i}{\partial x_j}([W_2],y_c)) & \text{if } i=j,\\ \sup\left\|\frac{\partial f_i}{\partial x_j}([W_2],y_c)\right\| & \text{if } i\neq j.\end{cases}\end{aligned}$$

Proof: We proceed as in the proof of Theorem 4.2. But the difference between $f(x_1(t),y_c)$ and $f(x_2(t),y(t))$ is computed differently. Namely,

$$\begin{aligned}&f(x_1(t),y_c)-f(x_2(t),y(t)) = f(x_1(t),y_c)-f(x_2(t),y_c)+\\ &f(x_2(t),y_c)-f(x_2(t),y(t)) = A(t)\cdot(x_1(t)-x_2(t))+\delta(t),\end{aligned}$$

where $\delta(t)\in[\delta]$ and

$$A_{ij}(t) = \int_0^1 \frac{\partial f_i}{\partial x_j}(x_2(t)+s(x_1(t)-x_2(t)),y_c)\,ds.$$

We continue as in the proof of Theorem 4.2. □

5. **Formulas for various cases.** In this section we rewrite Theorems 4.2 and 4.3 in the form, which will be later used in our algorithm for the integration of differential inclusions.

5.1. **The estimation of perturbations of ODEs based on logarithmic norms.** From Theorem 4.3 using the trivial decomposition consisting of the whole space we obtain the following lemma.

Lemma 5.1. Let $h>0$. Assume that $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ be C^1 and $y:[t_0,t_0+h]\to\mathbb{R}^m$ be bounded and measurable.

Let $[W_y]\subset\mathbb{R}^m$ be convex and such that, $y([t_0,t_0+h])\subset[W_y]$.

Let $y_c \in [W_y]$. Assume that $x_1, x_2 : [t_0, t_0+h] \to \mathbb{R}^n$ both absolutely continuous, are weak solutions of the following problems, respectively

$$x_1' = f(x_1, y_c), \quad x_1(t_0) = x_0, \tag{45}$$
$$x_2' = f(x_2, y(t)), \quad x_2(t_0) = \bar{x}_0. \tag{46}$$

Let $[W_1] \subset [W_2] \subset \mathbb{R}^n$ be convex and compact and such that

$$x_1(t) \in [W_1], \quad x_2(t) \in [W_2], \qquad \text{for } s \in [t_0, t_0+h].$$

Then for any $t \in [0, h]$ holds

$$\begin{aligned} &\|x_2(t_0+t) - x_1(t_0+t)\| \\ &\le \exp(lt)\|x_1(t_0) - x_2(t_0)\| + \exp(lt)\int_{t_0}^{t_0+t} \exp(-ls)\|[\delta]\| ds \\ &= \exp(lt)\|x_1(t_0) - x_2(t_0)\| + \frac{\|[\delta]\|}{l}(\exp(lt) - 1) \end{aligned}$$

where $l = \sup\left(\mu(\frac{\partial f}{\partial x}([W_2], y_c))\right)$, and μ is the logarithmic norm of the matrix (see [7] for the definition) and

$$[\delta] = \{f(x, y_c) - f(x, y) \mid x \in [W_2], y \in [W_y]\}.$$

5.2. **A component-wise estimate.** From Theorem 4.2 using the trivial decomposition $\mathbb{R}^m = \bigoplus_{i=1}^m \mathbb{R}$ we obtain the following lemma.

Lemma 5.2. Let $h > 0$. Assume that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be C^1 and $y : [t_0, t_0+h] \to \mathbb{R}^m$ is bounded and measurable.

Let $[W_y] \subset \mathbb{R}^m$ be convex and such that, $y([t_0, t_0+h]) \subset [W_y]$.

Let $y_c \in [W_y]$. Assume that $x_1, x_2 : [t_0, t_0+h] \to \mathbb{R}^n$, both absolutely continuous, are weak solutions of the following problems, respectively

$$x_1' = f(x_1, y_c), \quad x_1(t_0) = x_0, \tag{47}$$
$$x_2' = f(x_2, y(t)), \quad x_2(t_0) = \bar{x}_0. \tag{48}$$

Let $[W_1] \subset [W_2] \subset \mathbb{R}^n$ be convex and compact and such that

$$x_1(t) \in [W_1], \quad x_2(t) \in [W_2], \qquad \text{for } s \in [t_0, t_0+h].$$

Then the following inequality holds for $t \in [t_0, t_0+h]$ and $i = 1, \ldots, n$

$$|x_{1,i}(t) - x_{2,i}(t)| \le \left(e^{Jt} \cdot (x_0 - \bar{x}_0)\right)_i + \left(\int_{t_0}^{t} e^{J(t-s)} C\, ds\right)_i, \tag{49}$$

where

$$\begin{aligned} [\delta] &= \{f(x, y_c) - f(x, y) \mid x \in [W_1], y \in [W_y]\}, \\ C_i &\ge \sup|[\delta_i]|, \quad i = 1, \ldots, n \\ J_{ij} &\ge \begin{cases} \sup \frac{\partial f_i}{\partial x_j}([W_2], [W_y]) & \text{if } i = j, \\ \sup\left|\frac{\partial f_i}{\partial x_j}([W_2], [W_y])\right| & \text{if } i \ne j. \end{cases} \end{aligned}$$

6. **The Lohner-type algorithm for perturbations of ODEs.** For a given measurable function $y : [0,\infty) \to \mathbb{R}^m$ which is bounded on compact intervals let $\varphi(t, x_0, y)$ denotes a weak solution of equation (1) with initial condition $x(0) = x_0$. For a given $y_0 \in \mathbb{R}^m$ let $\overline{\varphi}(t, x_0, y_0)$ be a solution of the following Cauchy problem

$$x' = f(x, y_0), \quad x(0) = x_0 \tag{50}$$

with the same initial condition $x(0) = x_0$. Observe that system (50) is a particular case of (1) with $y(t) = y_0$.

Let U be a some family of functions $y : [0,\infty) \to \mathbb{R}^m$ which are measurable and are uniformly bounded on any compact interval, i.e. for any $T > 0$ there exists $M(T)$, such that for every $y \in U$ and every $t \in [0, T]$ holds $\|y(t)\| \leq M(T)$.

We are interested in finding rigorous bounds for $\phi(t, [x_0], [y_0])$, where $[x_0] \subset \mathbb{R}^n$ and $[y_0] \subset U$. The set $[y_0]$ might be defined as some dynamical process, in this case we may need to compute something for each time step, or it can be just given by the specifying the bounds, for example $y \in [y_0]$ iff $y(t) \in [-\epsilon, \epsilon]^m$ and y is measurable.

Below we propose a modification of the original Lohner algorithm [11, 12] to treat problem (1). Our presentation follows the description of the C^0-Lohner algorithm presented in [25].

6.1. **One step of the algorithm.** In the description below the objects with an index k refer to the current values and those with an index $k+1$ are the values after the next time step.

We define

$$[y_k] = \{y \in U \mid y(t) = z(t_k + t) \quad \text{for some } z \in [y_0]\}.$$

For given $[y] \subset U$ we will also use the following notation

$$[y]([t_1, t_2]) = \{z(t) \mid z \in [y], t \in [t_1, t_2]\}.$$

One step of the Lohner algorithm is a shift along the trajectory of system (1) with following input and output data:

Input data:

- t_k is a current time
- h_k is a time step
- $[x_k] \subset \mathbb{R}^n$, such that $\varphi(t_k, [x_0], [y_0]) \subset [x_k]$
- possibly some bounds for $[y_k]$

Output data:

- $t_{k+1} = t_k + h_k$ is a new current time
- $[x_{k+1}] \subset \mathbb{R}^n$, such that $\varphi(t_{k+1}, [x_0], [y_0]) \subset [x_{k+1}]$
- possibly some bounds for $[y_0][0, t_{k+1})$.

We do not specify here a form (a representation) of sets $[x_k]$. They can be interval sets, balls, doubletons etc. (see [15, 25]). This issue is very important in handling of the wrapping effect and is discussed in detail in [11, 12] (see also Section 3 in [25]).

One step of the algorithm consists from the following parts:

1.: Generation of a priori bounds for φ **and** $[y_0]([t_k, t_{k+1}])$.

We find a convex and compact set $[W_2] \subset \mathbb{R}^n$ and a convex set $[W_y] \subset \mathbb{R}^m$, such that

$$\varphi([0, h_k], [x_k], [y_k]) \subset [W_2] \tag{51}$$

$$[y_k]([0, h_k]) \subset [W_y] \tag{52}$$

2.: We fix $y_c \in [W_y]$.

3.: Computation of an unperturbed x-projection. We apply one step of the C^0-Lohner algorithm to (50) with a time step h_k and an initial condition given by $[x_k]$ and $y_0 = y_c$. As a result we obtain $[\overline{x}_{k+1}] \subset \mathbb{R}^n$ and a convex and compact set $[W_1] \subset \mathbb{R}^n$, such that

$$\begin{aligned} \overline{\varphi}(h_k, [x_k], y_c) &\subset [\overline{x}_{k+1}] \\ \overline{\varphi}([0, h_k], [x_k], y_c) &\subset [W_1] \end{aligned}$$

4.: Computation of the influence of the perturbation. Using formulas from Lemmas 5.2 or 5.1 we find a set $[\Delta] \subset \mathbb{R}^n$, such that

$$\varphi(t_{k+1}, [x_0], [y_0]) \subset \overline{\varphi}(h_k, [x_k], y_c) + [\Delta]. \tag{53}$$

Hence

$$\varphi(t_{k+1}, [x_0], [y_0]) \subset [x_{k+1}] = [\overline{x}_{k+1}] + [\Delta] \tag{54}$$

5.: If needed we do some computation to obtain $[y_{k+1}]$

6.2. **Part 1 - comments.** In the context of an nonautonomous ODE with small and uniformly bounded $[\delta]$ we can set $[W_y] = \mathbb{R}$. To obtain $[W_2]$ any rough enclosure procedure devised for ODEs should work. In the context of a dissipative PDE the whole story is more complicated and we refer the interested reader to [26].

6.3. **Part 4 - details.** In Lemmas 5.1 and 5.2 we have presented two ways to compute $[\Delta] = [\Delta](h)$ for $0 \leq h \leq h_k$.

An approach based on component-wise estimates

1. We set

$$\begin{aligned} [\delta] &= [\{f(x, y_c) - f(x, y) \mid x \in [W_1], y \in [W_y]\}]_I \\ C_i &= \text{right}(|[\delta_i]|), \quad i = 1, \ldots, n \\ J_{ij} &= \begin{cases} \text{right}\left(\frac{\partial f_i}{\partial x_i}([W_2], [W_y])\right) & \text{if } i = j, \\ \text{right}\left(\left|\frac{\partial f_i}{\partial x_j}([W_2], [W_y])\right|\right). & \text{if } i \neq j. \end{cases} \end{aligned}$$

2. $D = \int_0^h e^{J(h-s)} C \, ds$
3. $[\Delta_i] = [-D_i, D_i]$, for $i = 1, \ldots, n$

It remains to explain how we compute $\int_0^t e^{A(t-s)} C \, ds$. First observe that

$$\int_0^t e^{A(t-s)} C \, ds = t \left(\sum_{n=0}^{\infty} \frac{(At)^n}{(n+1)!} \right) \cdot C. \tag{55}$$

We fix any norm $\|\cdot\|$, such that for any matrix $A = (a_{ij})$ we have $|a_{ij}| \leq \|A\|$. It is not true for a general norm, for example if we take vector norm on $\mathbb{R}^2$ defined by $\|(x_1, x_2)\| = max\{\frac{1}{100}x_1, x_2\}$ then the associated matrix norm of a matrix $\begin{pmatrix} 0 & 100 \\ 0 & 0 \end{pmatrix}$ is equal to 1. We may take for example L^∞-norm, i.e. $\|x\|_\infty = \max_i |x_i|$ (*although we should chose a norm for which $\|At\|$ becomes as small as possible*). Let us set

$$\tilde{A} = At, \qquad A_m = \frac{\tilde{A}^m}{(m+1)!}.$$

In this notation

$$\sum_{m=0}^{\infty} \frac{(At)^m}{(m+1)!} = \sum_{m=0}^{\infty} A_m$$

$$A_0 = \mathrm{Id}, \qquad A_{m+1} = A_m \cdot \frac{\tilde{A}}{m+2}$$

For the remainder term we will use the following estimate

$$\|A_{N+k}\| \leq \|A_N\| \cdot \left\| \frac{\tilde{A}}{N+2} \right\|^k$$

Hence if $\left\| \frac{\tilde{A}}{N+2} \right\| < 1$, then

$$\begin{aligned} \left\| \sum_{m>N} A_m \right\| &\leq \|A_N\| \cdot \left\| \frac{\tilde{A}}{N+2} \right\| \cdot \left(1 - \left\| \frac{\tilde{A}}{N+2} \right\| \right)^{-1} \\ &= \|A_N\| \cdot \frac{\|\tilde{A}\|}{N+2-\|\tilde{A}\|} = r \end{aligned}$$

And finally,

$$\sum_{m=0}^{\infty} A_m = \sum_{m=0}^{N} A_m + [-r, r]^n \tag{56}$$

An approach based on logarithmic norms:(compare Lemma 5.1) We fix any norm $\|\cdot\|$, for example the L^∞-norm: $\|x\|_\infty = \max_i |x_i|$ (one should chose the norm which gives the smallest l in 3., below)

1. $[\delta] = [\{f(x, y_c) - f(x, y) \mid x \in [W_1], y \in [W_y]\}]_I$.
2. $C = \|[\delta]\|$
3. $l = \text{right}\left(\mu(\frac{\partial f}{\partial x}([W_2], y_c))\right)$
4. If $l \neq 0$, then $D = \frac{C(e^{lh}-1)}{l}$.
   If $l = 0$, then $D = Ch$
5. $[\Delta] = [-D, D]^n$

Remark. In both cases we compute

$$[\delta] = [\{f(x, y_c) - f(x, y) \mid x \in [W_1], y \in [W_y]\}]_I. \tag{57}$$

One need to be very careful in the computation of $[\delta]$ using (57), because direct interval evaluation of $[\{f(x, y_c) - f(x, y) \mid x \in [W_1], y \in [W_y]\}]_I$ yields big overestimation. Namely, when there is no perturbations at all, i.e. $[W_y] = \{y_c\}$, then $[\delta] = 0$. On the other hand if $f([W_1]) = [\{f(x, y_c) \mid x \in [W_1]\}]_I = [a^-, a^+]$ then the naive interval computation give $[\delta] = [a^- - a^+, a^+ - a^-]$, so $\operatorname{diam}[\delta] = 2 \operatorname{diam} f([W_1])$ and this can be big because $[W_1]$ is an enclosure of a solution during the whole time step.

6.4. **Rearrangement.** It is well known that a direct interval evaluation of (54) leads to huge overestimates, which are mainly due to the wrapping effect [14, 11], hence an essential part of the Lohner algorithm is designed to reduce it. Sets $[x_k]$ are not stored as interval sets (i.e. products of intervals), but using various representations. For instance we can represent set $[X] \subset \mathbb{R}^n$ as

$$[X] = x + [B][R] \text{ where } x \in \mathbb{R}^n, [B] \subset \mathbb{R}^{n \times n}, [R] \subset \mathbb{R}^n.$$

Each representation treats (54) differently. Below we include all necessary formulas for representations from [11] (see [25] for more details and the motivation).
Evaluations 2 and 3. In this representation

$$[x_k] = x_k + [B_k][\tilde{r}_k]. \tag{58}$$

In the context of our algorithm in part 3 we obtain

$$[\overline{x}_{k+1}] = \overline{x}_{k+1} + [B_{k+1}][\overline{r}_{k+1}]. \tag{59}$$

Now we have to take into account equation (54). We set

$$x_{k+1} = \mathrm{m}(\overline{x}_{k+1} + [\Delta]) \tag{60}$$

$$[\tilde{r}_{k+1}] = [\overline{r}_{k+1}] + [B_{k+1}^{-1}]\left(\overline{x}_{k+1} + [\Delta] - x_{k+1}\right). \tag{61}$$

Evaluation 4. In this representation

$$[x_k] = x_k + C_k[r_0] + [B_k][\tilde{r}_k]. \tag{62}$$

In the context of our algorithm in part 3 we obtain

$$[\overline{x}_{k+1}] = \overline{x}_{k+1} + C_{k+1}[r_0] + [B_{k+1}][\overline{r}_{k+1}]. \tag{63}$$

Equation (54) is taken into account exactly in the same way as in previous evaluations, i.e., we use (60) and (61).

7. **Rigorous estimates between time steps.** Let $S \subset \mathbb{R}^n$ be a manifold (we call it a section). Let the differential inclusion be defined by (2) with $[\epsilon(t)] \subset \mathbb{R}^n$ for $t \in D$. We define $[y] = \{y : D \to \mathbb{R}^n | y(t) \in \epsilon(t)\}$. For a fixed $\epsilon(t) = y \in [y]$ differential inclusion (2) becomes an ODE. For this ODE we define $P_y : S \supset S_0 \to S$ to be a Poincaré map (fist return map to section S). For each point $x_0 \in S$ such that $P_y(x_0)$ exists for all $y \in [y]$ the Poincaré map of differential inclusion (2) is defined by

$$P(x_0) = \{x \in S | x = P_y(x_0) \text{ for some } y \in [y]\} \tag{64}$$

In order to compute the Poincaré map for differential inclusion it not enough to know x_k before section and x_{k+1} after section but we also need estimates for $x(t)$ during whole time step (i.e for all times $t \in [t_k, t_k + h_k]$) to be able to estimate the intersection of the trajectory with the section (see [25] for more details). Below we present an algorithm which deals with this task.

Input parameters:

- h_k is a time step
- $[x_k] \subset \mathbb{R}^n$, such that $\varphi(t_k, [x_0], [y_0]) \subset [x_k]$
- $[x_{k+1}] \subset \mathbb{R}^n$, such that $\varphi(t_k + h_k, [x_0], [y_0]) \subset [x_{k+1}]$
- convex and compact set $[W_2] \subset \mathbb{R}^n$ and convex set $[W_y] \subset \mathbb{R}^m$, such that

$$\varphi([t_k, t_k + h_k], [x_0], [y_0]) \subset [W_2] \tag{65}$$

$$[y_0]([t_k, t_{k+1}]) \subset [W_y]. \tag{66}$$

- $y_c \in [W_y]$
- $[\overline{x}_{k+1}] \subset \mathbb{R}^n$, such that $\overline{\varphi}(h_k, [x_k], y_c) \subset [\overline{x}_{k+1}]$
- $[W_1] \subset \mathbb{R}^n$ compact and convex, such that $\overline{\varphi}([0, h_k], [x_k], y_c) \subset [W_1]$

Output:
We compute $[E_k] \subset \mathbb{R}^n$ such that

$$\varphi(t_k + [0, h_k], [x_0], [y_0]) \subset [E_k],$$

Algorithm:

- We compute $[\overline{E}_k] \subset \mathbb{R}^n$, such that
$$\overline{\varphi}([0, h_k], [x_k], y_c) \subset [\overline{E}_k] \tag{67}$$
using a procedure for an ODE described in [25]. This procedure requires as input data: h_k, $[x_k]$, $[\overline{x}_{k+1}]$ and $[W_1]$.
- we compute a set $[\Delta] \subset \mathbb{R}^n$, such that
$$\varphi(t_k + h, [x_0], [y_0]) \subset \overline{\varphi}(h, [x_k], y_c) + [\Delta], \qquad \text{for } 0 \leq h \leq h_k. \tag{68}$$
Observe that this requires y_c, $[W_1]$, $[W_2]$ and $[W_y]$.
- finally we obtain
$$\varphi(t_k + [0, h_k], [x_0], [y_0])_i \subset [E_k]_i = [\overline{E}_k]_i + [\Delta]_i. \tag{69}$$

Slightly better algorithm:

- if $0 \notin f_i([W_2], [W_y])_i$, then the i-th coordinate is strictly monotone on $[W_2] \times [W_y]$, hence we set
$$[E_k]_i = \text{hull}([x_k]_i, [x_{k+1}]_i)$$
- if $0 \in f_i([W_2], [W_y])$, then we compute $[\overline{E}_k] \subset \mathbb{R}^n$, such that
$$\overline{\varphi}([0, h_k], [x_k], y_c) \subset [\overline{E}_k] \tag{70}$$
using a procedure for an ODE described in [25]. This procedure requires as input data: h_k, $[x_k]$, $[\overline{x}_{k+1}]$and $[W_1]$.
We have
$$\varphi(t_k + [0, h_k], [x_0], [y_0])_i \subset [E_k]_i = [\overline{E}_k]_i + [\Delta]_i. \tag{71}$$

A drawback of this approach:
if we have to perform several time steps during which the computed enclosure for the trajectory has a nonempty intersection with the section, then Δ is added twice.

7.1. **Computation of the Poincaré map.** If as in [25] we assume that the section is given by $\alpha(x) = 0$ then an algorithm discussed in Section 5 in [25] also applies in the present context.

8. **Some tests, discussion.** In Section 6 we presented two variants of an algorithm for rigorous integration of differential inclusions: first based on the logarithmic norms and the second one that uses component-wise estimates. To shorten the notation in this section we call them the *LN method* and the *CW method* correspondingly. Observe that Theorem 4.2 offers also the possibility of mixed approach, treating some groups of coordinates componentwise and other groups by using logarithmic norms. We expect that for generic systems the CW method will be by far better that then the LN method or the mixture of two. Tests performed for Rössler equation or Kuramoto-Sivashinsky PDE confirm this. On the other hand for systems having some special properties one can expect that with a suitably chosen norm the LN method might be better than the CW method. We tested this on the harmonic oscillator example, which preserves the distance between points. It turns

initial set $[X]$ perturbations $[\epsilon]$	$(0.0, -10.3, 0.03) + \{0\} \times [-10^{-4}, 10^{-4}]^2$ $[-10^{-4}, 10^{-4}]^3$
CW method	
$P([X])$	$\begin{pmatrix} [-0.2115046, 0.2088673] \\ [-3.6978105, -3.4735188] \\ [0.0311734, 0.0332674] \end{pmatrix}^T$
diam $P([X])$	(0.4203719,0.2242916,0.0020940)
LN method	
$P([X])$	$\begin{pmatrix} [-0.3889325, 0.3773784] \\ [-3.7822709, -3.3744466] \\ [0.0303242, 0.0341994] \end{pmatrix}^T$
diam $P([X])$	(0.7663109 ,0.4078243 ,0.0038752)

TABLE 1. Perturbed Rössler equation: Value of a Poincaré map on section $\Theta = \{x = 0, x' > 0\}$

out that even in this case the CW method works better in most of tests, loosing only marginally in some cases.

8.1. **Rössler equations.** Rössler equations [16] are given by

$$\begin{aligned} x' &= -(y+z) \\ y' &= x + 0.2y \\ z' &= 0.2 + z(x-a), \end{aligned} \tag{72}$$

where a is a real parameter. In our tests we set $a = 5.7$ - the 'classical' parameter value for which numerical simulation display a strange attractor [16].

In our test we focus on computation of a Poincaré map, P, on section $\Theta = \{x = 0, x' > 0\}$ around a point $x_0 = (0.0, -10.3, 0.03)$. This is a point from the attractor (or close to the attractor, which we have found numerically difficult in [27]).

In Table 1 we list the results of a computation of Poincaré map on section Θ for a differential inclusion $x' \in f(x) + [\epsilon]$, where $f(x)$ is the vector field in Rössler equations (72) and $[\epsilon] = [-10^{-4}, 10^{-4}]^3$. The initial condition was $x_0 + \{0\} \times [-10^{-4}, 10^{-4}]^2$. In computations the method based on the component-wise estimates and the Lohner algorithm - 4th evaluation was used.

We see that our algorithm can provide good estimates even for perturbed system and for set of initial data containing numerically difficult points from attractor.

8.2. **Kuramoto-Sivashinsky PDE.** Assuming odd and periodic boundary conditions the Kuramoto-Sivashinsky equation can be reduced [24] to the following infinite system of ordinary differential equations

$$\dot{a}_k = k^2(1 - \nu k^2)a_k - k\sum_{n=1}^{k-1} a_n a_{k-n} + 2k\sum_{n=1}^{\infty} a_n a_{n+k} \quad k = 1, 2, 3, \ldots \tag{73}$$

where $\nu > 0$. In [26, 28] using the algorithm based on component-wise estimates described in this paper to handle the dominant modes and the method of self-consistent bounds developed in [24] to deal with the tail (the remaining modes) the existence of multiple periodic orbits has been proved for a range for $\nu \in [0.032, 0.127]$. Some of these orbits were attracting, while others were unstable with one unstable direction.

For example, for $\nu = 0.1212$ the computation of the Poincaré map using the LN method based on the maximum norm yielded results with the largest diameter around 50 times as large as the one based on the CW method.

8.3. **Perturbed harmonic oscillator.** We use the harmonic oscillator to compare the LN method and the CW method. Although in general we expect CW method to perform better, in the case of harmonic oscillator (where appropriate logarithmic norm is equal to 0) we expect LN method to give better estimates.

Both methods first find the solution of the unperturbed system and then they add the influence of perturbation denoted (following Section 6) by Δ. In LN method Δ is "measured" by one number, i.e $\Delta = [-D, D]^2$. It is not the case for CW method, here estimates can differ for coordinates, so we have $\Delta = [-D_1, D_1] \times [-D_2, D_2]$. Therefore in comparison of this two methods the result can be ambiguous, it can occur that estimates on some coordinates are better in LN method but on the other are worse.

The equations of the perturbed harmonic oscillator are given by

$$\begin{aligned} x' &= y + \epsilon_1 \\ y' &= -x + \epsilon_2 \end{aligned} \tag{74}$$

and we will always use the initial condition $(1, 0) + [-\delta, \delta]^2$.

For this simple system we are able to compute Δ for both methods by hand. Let h denote time step used.
For LN method we used the euclidean logarithmic norm μ_e because it is optimal for this case. Namely, we have

$$l = \mu_e\left(\frac{\partial f}{\partial x}([W_2], y_c\right) = \mu_e\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = 0. \tag{75}$$

Therefore, we obtain $\Delta = [-D, D]^2$ where

$$D = h\sqrt{\epsilon_1^2 + \epsilon_2^2}.$$

For CW method we obtain $\Delta = ([-D_1, D_1], [-D_2, D_2])$, where

$$\begin{aligned} D_1 &= \epsilon_1 \sinh h + \epsilon_2(\cosh h - 1), \\ D_2 &= \epsilon_1(\cosh h - 1) + \epsilon_2 \sinh h. \end{aligned}$$

Suppose that $\epsilon_1 = \epsilon_2 := \epsilon$, then the LN method is better than the CW method if

$$\sqrt{2}h\epsilon < \epsilon(\sinh h + \cosh h - 1) = \epsilon(\exp(h) - 1) \tag{76}$$

Inequality (76) holds for $h > 0.657275$. As it can be seen in Table 2 results of computations agree with this theoretical estimate and the LN method is better for $h > 0.657275$. We were not able to use time steps $h > 0.8$ because for such a big time steps our rough enclosure procedure (the first part of the algorithm) fails.

The situation is quite different, when we perturb on one coordinate, only. Suppose that $\epsilon_1 = 0$ and $\epsilon_2 = \epsilon$. Now, for LN method we have

$$D = h\epsilon,$$

and for CW method

$$\begin{aligned} D_1 &= \epsilon(\cosh h - 1) = \epsilon(\frac{h^2}{2!} + \frac{h^4}{4!} + \dots), \\ D_2 &= \epsilon \sinh h = \epsilon(h + \frac{h^3}{3!} + \frac{h^5}{5!} + \dots). \end{aligned}$$

time step h	LN method D	CW method D_1, D_2
0.799	0.112996	0.122332
0.7	0.0989949	0.101375
0.66	0.0933381	0.0934792
0.658	0.0930553	0.0930927
0.657	0.0929138	0.0928997
0.65	0.0919239	0.0915541
0.5	0.0707107	0.0648721
0.25	0.0353553	0.0284025
0.1	0.0141421	0.0105171
0.01	0.00141421	0.00100502
0.001	0.000141421	0.00010005

TABLE 2. Perturbed harmonic oscillator $\epsilon_1 = \epsilon_2 = 0.1$: Estimates of perturbations for various time steps - comparison between LN and CW method

time step	LN method	CW method	
h	D	D_1	D_2
0.8	0.08	0.0337435	0.0888106
0.5	0.05	0.0127626	0.0521095
0.25	0.025	0.0031413	0.0252612
0.1	0.01	0.0005004	0.0100167
0.01	0.001	5.0e-06	0.0010001
0.001	0.0001	5.002e-08	0.0001

TABLE 3. Perturbed harmonic oscillator $\epsilon_1 = 0, \epsilon_2 = 0.1$: Estimates of perturbations for various time steps - comparison between LN and CW method

From the above formulas it follows that for time steps up to 1.616137 value of D_1 is smaller than D, but D_2 is always bigger than D. In Table 3 we list values of perturbations for LN an CW method for various time steps. Again for time steps bigger than 0.8 our implementation could not find a rough enclosure. For small time steps the ratio $\frac{D}{D_1}$ is quite big, while the ratio $\frac{D}{D_2}$ is slightly less than one. So overall it is better to use the CW method.

In Table 4 we compare diameters of computed rigorous estimates of solutions of (74) after time $T = 2\pi$ for these two methods using various values of h, ϵ and δ. Again we perturb only second coordinate i.e. $\epsilon_1 = 0, \epsilon_2 = \epsilon$. As expected, we see that decreasing time steps results in the increase of the accuracy of the estimates, but at the price of an increased computational cost. In the second part of the table we were changing set sizes and in the third one we were changing the size of the perturbation. It can be seen that our algorithm is capable to provide estimates even for perturbations much bigger than values of the vector field. Observe that with the time steps used in these experiments the CW method is better than the LN method. The biggest time step h used was approximately equal to 0.785.

ϵ	δ	number of steps	size of the set after time $T = 2\pi$ LN method	CW method
0.1	0.01	9	1.615936	1.178825
0.1	0.01	100	1.619474	0.8453958
0.1	0.01	1000	1.619995	0.8225159
0.1	0.01	10000	1.62	0.8202514
0.1	0.01	100000	1.62	0.8200251
0.1	0	100	1.599474	0.8253958
0.1	0.01	100	1.619474	0.8453958
0.1	0.1	100	1.799474	1.025396
0.01	0.01	100	0.1799474	0.1025396
0.1	0.01	100	1.619474	0.8453958
1	0.01	100	16.01474	8.273958
10	0.01	100	159.9674	82.55958

TABLE 4. Perturbed harmonic oscillator $\epsilon_1 = 0, \epsilon_2 = \epsilon$: Estimates of perturbations for various values of the parameters - comparison between LN and CW method

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