N-BODY CHOREOGRAPHIES WITH A REFLECTIONAL SYMMETRY - COMPUTER ASSISTED EXISTENCE PROOFS

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We proof the existence of symmetric choreographies - solutions of the planar N-body problem on which all bodies travel on the same curve. We describe general method of computer assisted proofs using reflectional symmetry of the orbit to isolate a solution. Using this method we proved, as an example, the existence of many choreographies of 4 and 5 bodies.

1. Introduction

The N-body problem with N equal unit masses is given by a differential equation

$$\ddot{q}_i = \sum_{j \neq i} \frac{q_j - q_i}{r_{ij}^3} \tag{1.1}$$

where $q_i \in \mathbb{R}^n$, $i = 1, ..., N$, $r_{ij} = ||q_i - q_j||$.

We consider planar case (n=2) only, we set $q_i = (x_i, y_i)$, $\dot{q}_i = (\dot{x}_i, \dot{y}_i)$.

By a *simple choreography* [S] we mean a collision-free solution of the N-body problem in which all masses move on the same curve with a constant phase shift. This means that there exists $q : \mathbb{R} \longrightarrow \mathbb{R}^2$ a T-periodic function of time, such that the position of i-th body ($i = 0, \ldots, N-1$) is given by $q_i(t) = q(t + i\frac{T}{N})$ and $(q_0, q_1, \ldots, q_{N-1})$ is solution of the N-body problem.

In Section 2 we present general method of computer assisted proofs of the existence of choreographies. We use the Krawczyk method which needs a solution to be isolated. In Section 3.1 we show how to use a reflectional symmetry of the orbit to isolate a solution. As an example of using this

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method we proved the existence of all choreographies shown on Figures 1 and 2. The described method can be used with choreographies of any number of bodies. But the main difficulty is to compute good bounds for Krawczyk operator (2.2) what is sometimes hard for larger number of bodies and for more complicated orbits.

2. Outline of the method

In this paper restrict our attention to choreographies having a reflectional symmetry (for proofs of the existence of choreographies with more symmetries see [KZ]). Using this we can reduce phase space what helps remove first integrals what is needed to isolate the solution and for the Krawczyk method (see [K] or end of this section) succeed.

Let d denote dimension of the reduced space (see Section 3.1), $4N$ is dimension of the full phase space. We define a map $E : \mathbb{R}^d \longrightarrow \mathbb{R}^{4N}$ which expands, using symmetries of the orbit, the initial condition from the reduced to the full space. Then we define Poincaré map P which to each expanded initial condition x_0 following solution of (1.1) associate, if it exists, a condition $x_1 \in \mathbb{R}^{4N}$ in which for the first time $S(x_1) = 0$. Then we define a map $R : \mathbb{R}^{4N} \longrightarrow \mathbb{R}^d$ which reduces the final configuration to the reduced space. Function S and map R are chosen such that if $S(x_1) = 0$ and $R(x_1) = 0$ then configuration x_1 satisfies some conditions which guarantee that from trajectory between x_0 and x_1, using symmetries and the phase shift of the particles, we can construct the whole periodic orbit (these conditions say that pieces of the orbit fit together smoothly).

We define map $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ by

$$\Phi = R \circ P \circ E. \tag{2.1}$$

Our goal is now to find zeros of map Φ. For this we use the Krawczyk method ([A, K, N]). For a set of initial conditions $[X] = \prod_{i=1}^{d}[a_i, b_i] \subset \mathbb{R}^d$ and $\bar{x} \in [X]$, a good approximation of zero of Φ, we compute rigorous bounds for the Krawczyk operator

$$K(\bar{x}, [X], \Phi) := \bar{x} - C\Phi(\bar{x}) + (Id - C[D\Phi([X])])([X] - \bar{x}), \tag{2.2}$$

where $C \in \mathbb{R}^{n\times n}$ is well chosen linear isomorphism and $[D\Phi([X])]$ denote an interval set that contains the Jacobian of Φ computed on the set $[X]$.

Then we conclude the proof of the existence using

Theorem 2.1. *1. If $x^* \in [X]$ and $\Phi(x^*) = 0$, then $x^* \in K(\bar{x}, [X], F)$.*
2. If $K(\bar{x}, [X], \Phi) \subset int[X]$, then there exists in [X] exactly one solution of equation $\Phi(x) = 0$.

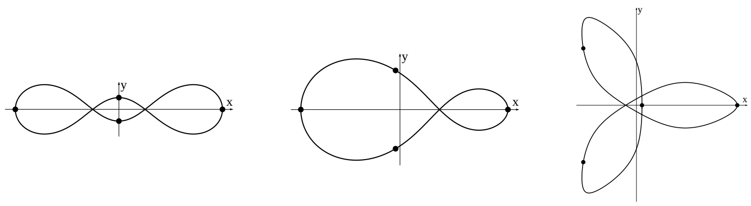

Figure 1. Choreographies of 4 bodies with reflectional symmetry proved to exist

3. Symmetries

Consider a finite group G, a 2-dimensional representations $\tau, \rho : G \longrightarrow O(2)$, constant $T > 0$. We define the action of the group G both on the time circle $\mathbb{T} = \mathbb{R}/T$ and on the space $\mathbb{R}^2$ by

$$\forall g \in G : gt = \tau(g)(t), \quad gx = \rho(g)(x), \tag{3.1}$$

where $t \in \mathbb{T}$ and $x \in \mathbb{R}^2$.

Let $q(t) : \mathbb{R}/T\mathbb{Z} \longrightarrow \mathbb{R}^2$. We say that $q(t)$ is invariant (equivariant) with respect to the action of G if $\forall g \in G : gq(t) = q(gt)$ for all t.

3.1. *Reflectional symmetry*

Let group $\rho(G)$ consists of two elements: the identity and the reflectional symmetry. We change coordinate frame so that the symmetry axis is the X axis. We denote by S_x (S_y) a reflection with respect to the X (resp. Y) axis. For simplicity of a notation by S_x we denote also generator of G.

Let the the action on the time circle be defined by $\tau(S_x)(t) = \varphi - t$, where $\varphi \in [0, 2\pi)$. We set $\varphi = 0$ (it corresponds to the reparametrization of curves). From this definition of the symmetry we have

$$q_i(0) = S_x q_{N-i}(0), \qquad \dot{q}_i(0) = S_y \dot{q}_{N-i}(0), \tag{3.2}$$

$$q_i(\frac{T}{2N}) = S_x q_{N-1-i}(\frac{T}{2N}), \qquad \dot{q}_i(\frac{T}{2N}) = S_y \dot{q}_{N-1-i}(\frac{T}{2N}), \tag{3.3}$$

where $i = 0, 1, \ldots, N-1$ and $q_N = q_0$.

Because we fix the center of the mass at the origin then to reconstruct the configuration of the bodies we need to know a value of $4N-4$ variables. From equation (3.2), (3.3) it follows that in time $t = 0$ and $t = \frac{T}{2N}$ we need to know only half of them. To isolate the solution instead of period we fix the size of the orbit by fixing one variable (in the below formulas it is denoted by a). When we prove the existence of one orbit, then we can get an orbit with any period just by rescaling. Hence the dimension of the reduced space is $2N-3$ ($d = 2N-3$).

For the definition of map R we use

Remark 3.1. If $q_i(t) = S_x q_j(t)$ and $\dot{q}_i(t) = S_y \dot{q}_j(t)$ for some time t then and only then

(i) $(x_i(t) - x_j(t), y_i(t) + y_j(t), \dot{x}_i(t) + \dot{x}_j(t), \dot{y}_i(t) - \dot{y}_j(t)) = (0,0,0,0)$ for $i \neq j$,

(ii) $y_i = \dot{x}_i = 0$ for $i = j$.

For even number of bodies ($N = 2k$) we set

$$E\left(\dot{y}_0 \times \prod_{i=1}^{\frac{N}{2}-1} (x_i, y_i, \dot{x}_i, \dot{y}_i)\right) = (a, 0, 0, \dot{y}_0) \times \prod_{i=1}^{\frac{N}{2}-1} (x_i, y_i, \dot{x}_i, \dot{y}_i)$$

$$\times (x_{\frac{N}{2}}, 0, 0, \dot{y}_{\frac{N}{2}}) \times \prod_{i=\frac{N}{2}+1}^{N-1} (x_{N-i}, -y_{N-i}, -\dot{x}_{N-i}, \dot{y}_{N-i}),$$

where $x_{\frac{N}{2}} = -a - 2\sum_{i=1}^{\frac{N}{2}-1} x_i$ and $\dot{y}_{\frac{N}{2}} = -\dot{y}_0 - 2\sum_{i=1}^{\frac{N}{2}-1} \dot{y}_i$, a is a constant,

$$S\left(\prod_{i=0}^{N-1} (x_i, y_i, \dot{x}_i, \dot{y}_i)\right) = x_0 - x_{N-1},$$

$$R\left(\prod_{i=0}^{N-1} (x_i, y_i, \dot{x}_i, \dot{y}_i)\right) = (\dot{y}_0 - \dot{y}_{N-1}) \times \prod_{i=1}^{\frac{N}{2}-1} (x_i - x_{N-i-1}, y_i + y_{N-i-1}, \dot{x}_i + \dot{x}_{N-i-1}, \dot{y}_i - \dot{y}_{N-i-1}).$$

For odd number of bodies ($N = 2k+1$) we set

$$E\left(\dot{y}_0 \times \prod_{i=1}^{\frac{N-3}{2}} (x_i, y_i, \dot{x}_i, \dot{y}_i) \times (y_{\frac{N-1}{2}}, \dot{x}_{\frac{N-1}{2}})\right) = (a, 0, 0, \dot{y}_0) \times \prod_{i=1}^{\frac{N-1}{2}} (x_i, y_i, \dot{x}_i, \dot{y}_i)$$

$$\times \prod_{i=\frac{N+1}{2}}^{N-1} (x_{N-i}, -y_{N-i}, -\dot{x}_{N-i}, \dot{y}_{N-i}),$$

where $x_{\frac{N-1}{2}} = -\frac{a}{2} - \sum_{i=1}^{\frac{N-3}{2}} x_i$ and $\dot{y}_{\frac{N-1}{2}} = -\frac{\dot{y}_0}{2} - \sum_{i=1}^{\frac{N-3}{2}} \dot{y}_i$, a is a constant,

$$S\left(\prod_{i=0}^{N-1}(x_i, y_i, \dot{x}_i, \dot{y}_i)\right) = y_{\frac{N-1}{2}},$$

$$R\left(\prod_{i=0}^{N-1}(x_i, y_i, \dot{x}_i, \dot{y}_i)\right) = (\dot{x}_{\frac{N-1}{2}}, x_0 - x_{N-1}, \dot{y}_0 - \dot{y}_{N-1})$$
$$\times \prod_{i=1}^{\frac{N-3}{2}}(x_i - x_{N-i-1}, y_i + y_{N-i-1}, \dot{x}_i + \dot{x}_{N-i-1}, \dot{y}_i - \dot{y}_{N-i-1}).$$

Theorem 3.2. *If for some $x \in \mathbb{R}^{2N-3}$ $\Phi(x) = R \circ P \circ E(x) = 0$ then there exists a choreography of N bodies which is invariant with respect to the action of G.*

Now we use a computer to prove the existence of zero of the function Φ. First of all we need to have x_0 a good approximation of the initial condition for the orbit in the reduced space. We want to thank Carles Simó for numerical data we used in proofs. Then we define set $[X] = x_0 + [-\delta, \delta]^d$ and compute rigorous bounds for the Krawczyk operator. Finally we check assumption of Theorem 2.1.

4. Choreographies with 5 bodies

Using approach described in Section 3.1 we proved the existence of several choreographies with 4 and 5 bodies. Some of them are shown on Figures 1 and 2. In case of 5 bodies we have

$E(\dot{y}_0, x_1, y_1, \dot{x}1, \dot{y}_1, y_2, \dot{x}_2) = (a, 0, 0, \dot{y}_0, x_1, y_1, \dot{x}_1, \dot{y}_1, -a/2 - x_1, y_2, \dot{x}_2, -\dot{y}_0/2 - u1, -a/2 - x_1, -y_2, -\dot{x}_2, -\dot{y}_0/2 - \dot{y}_1, x_1, -y_1, -\dot{x}_1, \dot{y}_1)$

$S\left(\prod_{i=0}^{4}(x_i, y_i, \dot{x}_i, \dot{y}_i)\right) = y_2$

$R\left(\prod_{i=0}^{4}(x_i, y_i, \dot{x}_i, \dot{y}_i)\right) = (\dot{x}_2, x_0 - x_4, \dot{x}_0 + \dot{x}_4, x_1 - x_3, y_1 + y_3, \dot{x}_1 + \dot{x}_3, \dot{y}_1 - \dot{y}_3)$

To compute the Poincaré map and its derivative we use a C^1-Lohner algorithm, implemented in the CAPD package [Capd]. For example in computation which proved the existence of the choreography shown in Figure 2 in the bottom right corner we use a time step $h = 0.01$ and the order of the Taylor method $r = 13$ for a computation in the point x_0 and respectively $t = 0.001$ and $r = 5$ for the set $[X]$.Time of computation on PC with AMD Duron 700 MHz procesor in the point x_0 was about 3 minutes and for the set $[X]$ less than 1 minute.

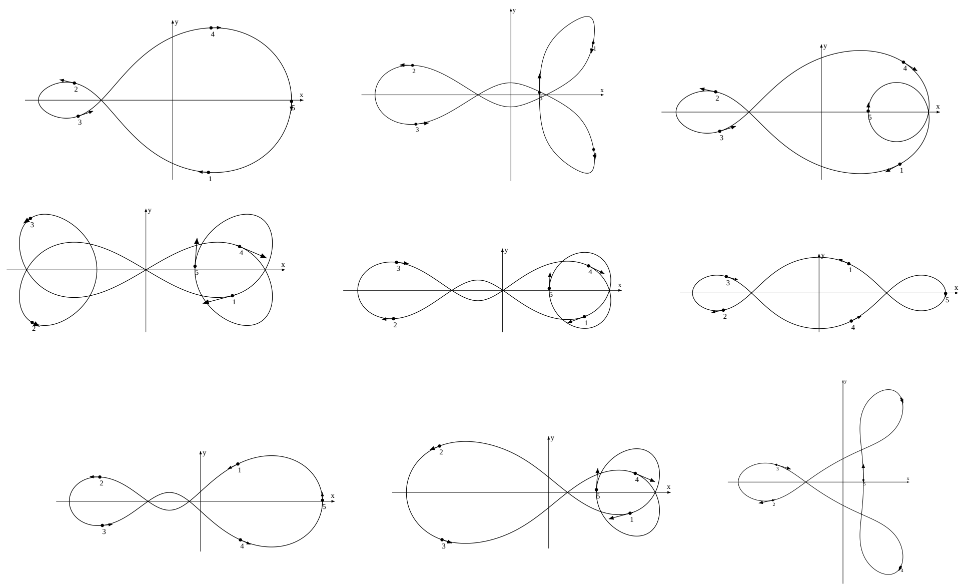

Figure 2. Choreographies of 5 bodies with reflectional symmetry

More details on results of the computation and the source code performing the computation can be found on the web page [Ka].

References

A. G. Alefeld, *Inclusion methods for systems of nonlinear equations - the interval Newton method and modifications.* in *Topics in Validated Computations* J. Herzberger (Editor), 1994 Elsevier Science B.V.

Capd. CAPD - Computer assisted proofs in dynamics, a package for rigorous numerics, *http://limba.ii.uj.edu.pl/˜capd*

KZ. T. Kapela, P. Zgliczyński, *The existence of simple choreographies for N-body problem - a computer assisted proof*, Nonlinearity, vol. 16(2003), 1899-1918.

Ka. Tomasz Kapela webpage, *http://www.ap.krakow.pl/˜tkapela*

K. R. Krawczyk, *Newton-Algorithmen zur Bestimmung von Nullstellen mit Fehlerschanken*, Computing 4, 187–201 (1969)

Mo. R.E. Moore, *Interval Analysis,* Prentice Hall, Englewood Cliffs, N.J., 1966

N. A. Neumeier, *Interval methods for systems of equations*, Cambrigde University Press, 1990.

S. C. Simó, *New families of solutions in N-body problems.* European Congress of Mathematics, Vol. I (Barcelona, 2000), 101–115, Progr. Math., 201, Birkhäuser, Basel, 2001.

ZLo. P. Zgliczyński, C^1*-Lohner algorithm*, Foundations of Computational Mathematics, (2002) 2:429–465