Cone conditions and covering relations

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Abstract

We show how to effectively link covering relations with cone conditions. We give a new, 'dynamical', proof of the stable manifold theorem.

1 Introduction

2 Notation

By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ we denote the set of natural, integer, rational, real and complex numbers, respectively. $\mathbb{Z}_-$ and $\mathbb{Z}_+$ are negative and positive integers, respectively. By S^1 we will denote a unit circle on the complex plane.

For $\mathbb{R}^n$ we will denote the norm of x by $\|x\|$ and if the formula for the norm is not specified in some context, then it means that it is ok to use any norm there. Let $x_0 \in \mathbb{R}^s$, then $B_s(x_0, r) = \{z \in \mathbb{R}^2 \mid \|x_0 - z\| < r\}$ and $B_s = B(0, 1)$.

For $z \in \mathbb{R}^u \times \mathbb{R}^s$ we will call usually the first coordinate, x, and the second one y. Hence $z = (x, y)$, where $x \in \mathbb{R}^u$ and $y \in \mathbb{R}^s$. We will use the projection maps $\pi_1(z) = \pi_x(z) = x(z) = x$ and $\pi_2(z) = \pi_y(z) = y(z) = y$.

Let $z \in \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ be a compact set and $f : U \to \mathbb{R}^n$ be continuous map, such that $z \notin f(\partial U)$. Then the local Brouwer degree [S] of f on U at z is defined and will be denoted by $\deg(f, U, z)$.

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. By $\mathrm{Sp}(A)$ we denote the spectrum of A, which is the set of $\lambda \in \mathbb{C}$, such that there exists $x \neq 0$, such that $Ax = \lambda x$.

3 Covering relations, horizontal and vertical disks

Definition 1 *[GiZ, Definition 1] An h-set, N, is a quadruple $(|N|, u(N), s(N), c_N)$ such that*

- *$|N|$ is a compact subset of $\mathbb{R}^n$*
- *$u(N), s(N) \in \{0, 1, 2, \dots\}$ are such that $u(N) + s(N) = n$*
- *$c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$ is a homeomorphism such that*

$$c_N(|N|) = \overline{B_{u(N)}} \times \overline{B_{s(N)}}.$$

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We set

$$\begin{aligned}
\dim(N) &:= n,\\
N_c &:= \overline{B_{u(N)}} \times \overline{B_{s(N)}},\\
N_c^- &:= \partial B_{u(N)} \times \overline{B_{s(N)}},\\
N_c^+ &:= \overline{B_{u(N)}} \times \partial B_{s(N)},\\
N^- &:= c_N^{-1}(N_c^-), \quad N^+ = c_N^{-1}(N_c^+).
\end{aligned}$$

Hence an h-set, N, is a product of two closed balls in some coordinate system. The numbers $u(N)$ and $s(N)$ are called the nominally unstable and nominally stable dimensions, respectively. The subscript c refers to the new coordinates given by homeomorphism c_N. Observe that if $u(N) = 0$, then $N^- = \emptyset$ and if $s(N) = 0$, then $N^+ = \emptyset$. In the sequel to make notation less cumbersome we will often drop the bars in the symbol $|N|$ and we will use N to denote both the h-sets and its support.

Definition 2 *[GiZ, Definition 3] Let N be a h-set. We define a h-set N^T as follows*

- $|N^T| = |N|$
- $u(N^T) = s(N)$, $s(N^T) = u(N)$
- *We define a homeomorphism $c_{N^T} : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N^T)} \times \mathbb{R}^{s(N^T)}$, by*

  $$c_{N^T}(x) = j(c_N(x)),$$

  where $j : \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)} \to \mathbb{R}^{s(N)} \times \mathbb{R}^{u(N)}$ is given by $j(p, q) = (q, p)$.

∎

Observe that $N^{T,+} = N^-$ and $N^{T,-} = N^+$. This operation is useful in the context of inverse maps.

Definition 3 *[W2, Definition 2.2] Assume that N, M are h-sets, such that $u(N) = u(M) = u$ and let $f : N \to \mathbb{R}^{\dim(M)}$ be continuous. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^{s(M)}$.*

Let w be a nonzero integer. We say that

$$N \overset{f,w}{\Longrightarrow} M$$

(N f-covers M with degree w) iff the following conditions are satisfied

1. *there exists a continuous homotopy $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$, such that the following conditions hold true*

$$h_0 = f_c, \tag{1}$$
$$h([0,1], N_c^-) \cap M_c = \emptyset, \tag{2}$$
$$h([0,1], N_c) \cap M_c^+ = \emptyset. \tag{3}$$

2. *If $u > 0$, then there exists a map $A : \mathbb{R}^u \to \mathbb{R}^u$, such that*

$$\begin{aligned} h_1(p,q) &= (A(p),0), \text{ for } p \in \overline{B_u}(0,1) \text{ and } q \in \overline{B_s}(0,1), && (4)\\ A(\partial B_u(0,1)) &\subset \mathbb{R}^u \setminus \overline{B_u}(0,1). && (5)\end{aligned}$$

Moreover, we require that

$$\deg(A, \overline{B_u}(0,1), 0) = w, \tag{6}$$

Observe that in the above definition $s(N)$ and $s(M)$ can be different, this is the only difference compared to [GiZ, Definition 6].

Remark 1 *Observe, that since for any norm in $\mathbb{R}^n$ the closed unit ball is homeomorphic to $[-1,1]^n$, therefore for h-sets and covering relations we will use different norms in different contexts.*

Remark 2 *If the map A in condition 2 of Def. 3 is a linear map, then condition (5) implies, that*

$$\deg(A, \overline{B_u}(0,1), 0) = \pm 1.$$

Hence condition (6) is in this situation automatically fulfilled with $w = \pm 1$.

In fact, this is the most common situation in the applications of covering relations.

Most of the time we will not interested in the value of w in the symbol $N \overset{f,w}{\Longrightarrow} M$ and we will often drop it and write $N \overset{f}{\Longrightarrow} M$, instead. Sometimes we may even drop the symbol f and write $N \Longrightarrow M$.

Definition 4 *[GiZ, Definition 7] Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $g : \mathbb{R}^n \supset \Omega \to \mathbb{R}^n$. Assume that $g^{-1} : |M| \to \mathbb{R}^n$ is well defined and continuous. We say that $N \overset{g}{\Longleftarrow} M$ (N g-backcovers M) iff $M^T \overset{g^{-1}}{\Longrightarrow} N^T$.*

Definition 5 *[WZ, Definition 10] Let N be an h-set. Let $b : \overline{B_{u(N)}} \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is* a horizontal disk in N *if there exists a homotopy $h : [0,1] \times \overline{B_{u(N)}} \to N_c$, such that*

$$\begin{aligned} h_0 &= b_c && (7)\\ h_1(x) &= (x,0), \qquad \text{for all } x \in \overline{B_{u(N)}} && (8)\\ h(t,x) &\in N_c^-, \qquad \text{for all } t \in [0,1] \text{ and } x \in \partial B_{u(N)} && (9)\end{aligned}$$

Definition 6 *[WZ, Definition 11] Let N be an h-set. Let $b : \overline{B_{s(N)}} \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is* a vertical disk in N *if there exists a homotopy $h : [0,1] \times \overline{B_{s(N)}} \to N_c$, such that*

$$\begin{aligned} h_0 &= b_c \\ h_1(x) &= (0,x), \qquad \text{for all } x \in \overline{B_{s(N)}} \\ h(t,x) &\in N_c^+, \qquad \text{for all } t \in [0,1] \text{ and } x \in \partial B_{s(N)}. && (10)\end{aligned}$$

Definition 7 *Let N be an h-set in $\mathbb{R}^n$ and b be a horizontal (vertical) disk in N.*
We will say that $x \in \mathbb{R}^n$ belongs to b, *when $b(z) = x$ for some $z \in \mathrm{dom}(b)$.*

By $|b|$ we will denote the image of b. Hence $z \in |b|$ iff z belongs to b.

The theorem below contains a slight generalization of Theorem 9 in [GiZ]

Theorem 3 *Assume N_i, $i = 0, \dots, k$, $N_k = N_0$ are h-sets and for each $i = 1, \dots, k$ we have*

$$N_{i-1} \overset{f_i, w_i}{\Longrightarrow} N_i. \tag{11}$$

Then there exists a point $x \in \mathrm{int} N_0$, such that

$$f_i \circ f_{i-1} \circ \dots \circ f_1(x) \in \mathrm{int} N_i, \quad i = 1, \dots, k \tag{12}$$

$$f_k \circ f_{k-1} \circ \dots \circ f_1(x) = x \tag{13}$$

Proof: Under additional assumption that $s(N_i) = s$ for $i = 1, \dots, k$ this theorem was proved in [GiZ].

The situation of different $s(N_i)$ can be reduced to the previous one as follows. Let $s = \max_{i=1,\dots,k-1} s_i$.

Let us fix the norm $\|x\| = \max_i |x_i|$.

We define new h-sets $\tilde{N}_i$ and maps $\tilde{f}_i$ as follows

$$|\tilde{N}_i| = |N_i| \times [-1, 1]^{s - s_i}, \quad u(\tilde{N}_i) = u(N_i), \quad s(\tilde{N}_i) = s \tag{14}$$

$$c_{\tilde{N}_i}(x, y, \tilde{y}) = (c_{N_i}(x, y), \tilde{y}), \qquad \text{where } (x, y) \in \mathbb{R}^{\dim(N_i)},\ \tilde{y} \in \mathbb{R}^{s - s_i} \tag{15}$$

Let h_i be the homotopy from the covering relation $N_{i-1} \overset{f_i}{\Longrightarrow} N_i$. We define a new homotopy $\tilde{h}_i$ and $\tilde{f}_i$ by

$$\tilde{h}_i(t, (x, y, \tilde{y}_{i-1})) = h_i(t, (x, y)) \times \{0\}^{s - s(N_i)}$$

$$\tilde{f}_i(x, y, \tilde{y}_{i-1}) = f_i(x, y) \times \{0\}^{s - s(N_i)}$$

Observe that for $i = 1, \dots, k$ we have

$$\tilde{N}_{i-1} \overset{\tilde{f}_i, w_i}{\Longrightarrow} \tilde{N}_i \tag{16}$$

The assertion now follows from Theorem 9 in [GiZ]. ∎

Theorem 4 *Let $k \geq 1$. Assume N_i, $i = 0, \dots, k$, are h-sets and for each $i = 1, \dots, k$ we have*

$$N_{i-1} \overset{f_i, w_i}{\Longrightarrow} N_i \tag{17}$$

Assume that b_0 is a horizontal disk in N_0 and b_e is a vertical disk in N_k.

Then there exists a point $x \in \mathrm{int} N_0$, such that

$$x = b_0(t), \quad \text{for some } t \in B_{u(N_0)}(0, 1) \tag{18}$$

$$f_i \circ f_{i-1} \circ \dots \circ f_1(x) \in \mathrm{int} N_i, \quad i = 1, \dots, k \tag{19}$$

$$f_k \circ f_{k-1} \circ \dots \circ f_1(x) = b_e(z), \quad \text{for some } z \in B_{s(N_k)}(0, 1) \tag{20}$$

Proof: Just as in the case of Theorem 3, the assertion was proved in [WZ, Thm. 4] under the assumption that $s(N_i) = s$ is independent of i.

We can reduce the current case exactly in the same way as in the proof of Theorem 3. We define $\tilde{f}_i$ and $\tilde{N}_i$ as it was done there. For disks let h_0 and h_e be the homotopies from definitions of b_0 and b_e, respectively. We define the horizontal disk $\tilde{b_0}$ and the vertical disk $\tilde{b_e}$ and their homotopies $\tilde{h}_0$ and $\tilde{h}_e$ as follows

$$\begin{aligned}
\mathrm{dom}(\tilde{b}_0) &= \mathrm{dom}(b_0), \quad \tilde{b}_0(x) = b_0(x) \times \{0\}^{s-s(N_0)} \\
\tilde{h}_0(t,x) &= h(t,x) \times \{0\}^{s-s(N_0)} \\
\mathrm{dom}(\tilde{b}_e) &= \mathrm{dom}(b_e) \times [-1,1]^{s-s(N_k)}, \qquad \tilde{b}_e(y,\tilde{y}) = (b_e(y),\tilde{y}), \\
\tilde{h}_e(t,y,\tilde{y}) &= (h_e(y),\tilde{y}).
\end{aligned}$$

Now we apply Theorem 4 from [WZ]. ∎

4 Cone conditions

The goal of this section is to introduce a method, which will allow to handle relatively easily the hyperbolic structure on h-sets.

Definition 8 *Let $N \subset \mathbb{R}^n$ be an h-set and $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form*

$$Q((x,y)) = \alpha(x) - \beta(y), \qquad (x,y) \in \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}, \tag{21}$$

where $\alpha : \mathbb{R}^{u(N)} \to \mathbb{R}$, and $\beta : \mathbb{R}^{s(N)} \to \mathbb{R}$ are positive definite quadratic forms.

The pair (N,Q) we be called an h-set with cones.

Quite often we will drop Q in the symbol (N,Q) and we will say that N is an h-set with cones.

4.1 Cone conditions for horizontal and vertical disks

Definition 9 *Let (N,Q) be a h-set with cones.*

Let $b : \overline{B_u} \to |N|$ be a horizontal disk.

We will say that b satisfies the cone condition (with respect to Q) *iff for any $x_1, x_2 \in \overline{B_u}$, $x_1 \neq x_2$ holds*

$$Q(b_c(x_1) - b_c(x_2)) > 0. \tag{22}$$

Definition 10 *Let (N,Q) be a h-set with cones.*

Let $b : \overline{B_s} \to |N|$ be a vertical disk.

We will say that b satisfies the cone condition (with respect to Q) *iff for any $y_1, y_2 \in \overline{B_s}$, $y_1 \neq y_2$ holds*

$$Q(b_c(y_1) - b_c(y_2)) < 0. \tag{23}$$

Lemma 5 *Let (N,Q) be a h-set with cones and let $b : \overline{B_u} \to |N|$ be a horizontal disk satisfying the cone condition.*

Then there exists a Lipschitz function $y : \overline{B_u} \to \overline{B_s}$ such that

$$b_c(x) = (x, y(x)). \tag{24}$$

Analogously, if $b : \overline{B_s} \to |N|$ is a vertical disk satisfying the cone condition, then there exists a Lipschitz function $x : \overline{B_s} \to \overline{B_u}$

$$b_c(y) = (x(y), y)). \tag{25}$$

Proof: We will prove only the first assertion, the proof of the other one is analogous.

In the first part of this proof we will show that for any $x \in \mathrm{int} B_{u(N)}$ there exists $z \in \mathrm{int} B_{u(N)}$ and $y_x \in \overline{B}_{s(N)}$, such that

$$b_c(z) = (x, y_x). \tag{26}$$

For this we will use the local Brouwer degree.

In the second part using the cone condition we will show that y_x is uniquely defined and its dependence on x is Lipschitz. Then we extend the definition of $y(x)$ to $x \in \partial B_u$.

Let h be the homotopy from the definition of the horizontal disk b.

To prove (26) consider the homotopy $\pi_1 \circ h : [0,1] \times \overline{B}_{u(N)} \to \overline{B}_{u(N)}$, where $\pi_1 : \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)} \to \mathbb{R}^{u(N)}$ is a projection on the first component. Let us fix $x \in \mathrm{int} B_{u(N)}$. It is easy to see that, since $x \notin \pi_1 \circ h(t, \partial B_{u(N)}$ the local Brower degrees in the formula below are defined and the stated equalities are satisfied by the homotopy property

$$\deg(\pi_1 \circ b_c, \overline{B}_{u(N)}, x) = \deg(\pi_1 \circ h_1, \overline{B}_{u(N)}, x) = \deg(\mathrm{Id}, \overline{B}_{u(N)}, x) = 1. \tag{27}$$

This proves (26).

To prove the uniqueness of y_x, assume that there exists $z_1, z_2 \in \mathrm{int} B_{u(N)}$ and $y_1, y_2 \in \overline{B}_{s(N)}$, $y_1 \neq y_2$ such that

$$b_c(z_1) = (x, y_1), \qquad b_c(z_2) = (x, y_2). \tag{28}$$

From the cone condition for b it follows that

$$0 < Q(b_c(z_1) - b_c(z_2)) = \alpha(0) - \beta(y_1 - y_2) < 0 \tag{29}$$

which is a contradiction. Hence we have a well defined function

$$y(x) = y_x, \qquad \text{for } x \in \mathrm{int} B_{u(N)}. \tag{30}$$

Observe that from the cone condition it follows that for any $x_1, x_2 \in \mathrm{int} B_{u(N)}$, $x_1 \neq x_2$ holds

$$A\|x_1 - x_2\|^2 \geq \alpha(x_1 - x_2) > \beta(y(x_1) - y(x_2)) \geq B\|y(x_1) - y(x_2)\|^2, \tag{31}$$

where A, B are some positive constants related to quadratic forms α and β, respectively.

This proves the Lipschitz condition.

It is easy to see that the function $y(x)$ can be extended also to the boundary of $B_{u(N)}$. Observe that from the closeness of $|b|$ it follows that $(x, y(x)) \in |b|$ for $x \in \partial B_{u(N)}$. ∎

4.2 Cone conditions for maps

Definition 11 *Assume that $(N, Q_N), (M, Q_M)$ are h-sets with cones, such that $u(N) = u(M) = u$ and let $f : N \to \mathbb{R}^{\dim(M)}$ be continuous. Assume that $N \overset{f}{\Longrightarrow} M$. We say that f satisfies the cone condition (with respect to the pair (N, M)) iff for any $x_1, x_2 \in N_c$, $x_1 \neq x_2$ holds*

$$Q_M(f_c(x_1) - f_c(x_2)) > Q_N(x_1 - x_2). \tag{32}$$

The basic theorem relating covering relations and cone conditions is

Theorem 6 *Assume that*

$$N_0 \overset{f_0}{\Longrightarrow} N_1 \overset{f_1}{\Longrightarrow} N_2 \overset{f_2}{\Longrightarrow} \cdots \overset{f_{k-1}}{\Longrightarrow} N_k, \tag{33}$$

where all h-sets are h-sets with cones and f_i for $i = 0, \dots, k-1$ satisfies the cone condition.

Assume that $b : \overline{B_{s(N_k)}} \to N_k$ is a vertical disk in N_k satisfying the cone condition.

Then the set of points $z \in N_0$ satisfying the following two conditions

$$f_{i-1} \circ f_{i-2} \circ \cdots \circ f_0(z) \in N_i, \qquad \text{for } i = 1, \dots, k \tag{34}$$

$$f_{k-1} \circ \cdots \circ f_0(z) \in |b| \tag{35}$$

is a vertical disk satisfying the cone condition.

Proof: For the proof it is enough to consider the case of $k = 1$, only. For $k > 1$ the result follows by induction.

Without any loss of the generality we can assume that $N_0 = N_{0,c} = \overline{B}_{u(N_0)} \times \overline{B}_{s(N_0)}$, $N_1 = N_{1,c} = \overline{B}_{u(N_1)} \times \overline{B}_{s(N_1)}$, $f_0 = f_{0,c}$. Consider a family of horizontal disks in N_0 $d_y : \overline{B}_{u(N_0)} \to N_0$ for $y \in \overline{B}_{s(N_0)}$

$$d_y(x) = (x, y). \tag{36}$$

From Theorem 4, applied to chain $N_0 \overset{f_0}{\Longrightarrow} N_1$ and disks d_y in N_0 and b in N_1 it follows that each $y \in \overline{B_{s(N_0)}}$ there exists $x \in \overline{B_{u(N_0)}}$, such that

$$f_0(x, y) \in |b| \tag{37}$$

Let us fix $y \in \overline{B_{s(N_0)}}$. We will show that there exists only one x satisfying (37). For the proof assume the contrary, hence we have $x_1 \neq x_2$ and x_1, x_2 both satisfy (37).

Observe that $Q_{N_0}((x_1,y)-(x_2,y))>0$, hence from the fact that f_0 satisfies the cone condition it follows that

$$Q_{N_1}(f_0(x_1,y)-f_0(x_2,y)))>0. \tag{38}$$

But the above inequality is in a contradiction with the cone condition for b. Hence (37) defines a function $x(y)$ in a unique way.

It is easy to see that function $x(y)$ is continuous. Namely, from the compactness argument it follows that it is enough to prove that if we have a sequence of pairs (x_n,y_n), where $y_n \in \overline{B_s}$, $y_n \to \bar{y}$ for $n \to \infty$ and $x_n = x(y_n)$, $x_n \to \bar{x}$, then $f_0(\bar{x},\bar{y}) \in |b|$, but this is an obvious consequence of the continuity of f_0 and the compactness of $|b|$.

Obviously, $b_0 : \overline{B_s} \to \overline{B_u} \times \overline{B_s}$ defined by $b_0(y)=(x(y),y)$ is a vertical disk in N_0. It remains to show that it satisfies the cone condition.

We will prove this by a contradiction. Assume that we have y_1 and y_2 such that

$$Q_{N_0}((x(y_1),y_1)-(x(y_2),y_2)) \geq 0, \tag{39}$$

then

$$Q_{N_1}(f_0(x(y_1),y_1)-f_0(x(y_2),y_2))>0, \tag{40}$$

hence the points $f_0(x(y_1),y_1)$ and $f_0(x(y_2),y_2)$ cannot both belong to b, because the cone condition is violated. ∎

4.3 Rigorous numerical verification of cone conditions

Assume that (N,Q_N) and (M,Q_M) are h-sets with cones and a map $f : N \to \mathbb{R}^{\dim(M)}$ is C^1.

Observe that for $x_2 \to x_1$

$$Q_M(f(x_2)-f(x_1)) - Q_N(x_2-x_1) \to 0. \tag{41}$$

Hence there is no chance that the cone condition can be verified rigorously on computer [N, KWZ], by direct evaluation in interval arithmetics of $Q_M(f(x_2)-f(x_1)) - Q_N(x_2-x_1)$.

Our intention is to give a condition, which will imply the cone condition and will be verifiable on computer.

Definition 12 *Let $U \subset \mathbb{R}^n$ and let $g : U \to \mathbb{R}^n$ be a C^1 map. The we define* the interval enclosure of $Dg(U)$ *by*

$$[Dg(U)] = \left\{ A \in \mathbb{R}^{n\times n} \mid \forall_{ij} A_{ij} \in \left[\inf_{x\in U} \frac{\partial g_i}{\partial x_j}(x), \sup_{x\in U} \frac{\partial g_i}{\partial x_j}(x) \right] \right\} \tag{42}$$

Let $[df_c(N_c)]$ be the interval enclosure of df_c on N_c. Observe that when $\dim(M) \neq \dim(N)$ this is not a square matrix.

Lemma 7 *Assume that for any $B \in [df_c(N_c)]$, the quadratic form*

$$V(x) = Q_M(Bx) - Q_N(x) \tag{43}$$

is positive definite, then for any $x_1, x_2 \in N_c$ such that $x_1 \neq x_2$ holds

$$Q_M(f_c(x_1) - f_c(x_2)) > Q_N(x_1 - x_2). \tag{44}$$

Proof: Let us fix x_1, x_2 in N_c. We have

$$f_c(x_2) - f_c(x_1) = \int_0^1 df_c(x_1 + t(x_2 - x_1))dt \cdot (x_2 - x_1). \tag{45}$$

Let $B = \int_0^1 df_c(x_1 + t(x_2 - x_1))dt$. Obviously $B \in [df_c]$. Hence

$$f_c(x_2) - f_c(x_1) = B(x_2 - x_1). \tag{46}$$

We have

$$Q_M(f_c(x_2) - f_c(x_1)) - Q_N(x_2 - x_1) =$$
$$Q_M(B(x_2 - x_1)) - Q_N(x_2 - x_1) = V(x_2 - x_1) > 0.$$

∎

In the light of the above lemma the verification of cone conditions can be reduced to checking that the interval matrix corresponding to the quadratic form V for various choices of $B \in [df_c(N_c)]$ given by

$$V = [df_c(N_c)]^T Q_M [df_c(N_c)] - Q_N \tag{47}$$

is positive definite.

5 Stable and unstable manifolds for hyperbolic fixed point of a map

Definition 13 *Consider the map $f : X \to X$.*

Let $x \in X$. Any sequence $\{x_k\}_{k \in I}$, where $I \subset \mathbb{Z}$ is a set containing 0 and for any $l_1 < l_2 < l_3$ in $\mathbb{Z}$ if $l_1, l_3 \in I$, then $l_2 \in I$, such that

$$x_0 = x, \qquad f(x_i) = x_{i+1}, \qquad \text{for } i, i+1 \in I \tag{48}$$

will be called an orbit through x. If $I = \mathbb{Z}_-$, then we will say that $\{x_k\}_{k \in I}$ is a full backward orbit through x.

Definition 14 *Let X be a topological space and let the map $f : X \to X$ be continuous.*

Let $Z \subset \mathbb{R}^n$, $x_0 \in Z$, $Z \subset \mathrm{dom}(f)$. We define

$$\begin{aligned}
W^s_Z(z_0,f) &= \{z \mid \forall_{n\geq 0} f^n(z) \in Z, \quad \lim_{n\to\infty} f^n(z) = z_0\} \\
W^u_Z(z_0,f) &= \{z \mid \exists\ \{x_n\} \subset Z \textit{ a full backward orbit through } z\textit{, such that} \\
&\qquad \lim_{n\to-\infty} x_n = z_0\} \\
W^s(z_0,f) &= \{z \mid \lim_{n\to\infty} f^n(z) = z_0\} \\
W^u(z_0,f) &= \{z \mid \exists\ \{x_n\} \textit{ a full backward orbit through } z\textit{, such that} \\
&\qquad \lim_{n\to-\infty} x_n = z_0\} \\
Inv^+(Z,f) &= \{z \mid \forall_{n\geq 0} f^n(z) \in Z\} \\
Inv^-(Z,f) &= \{z \mid \exists\ \{x_n\} \subset Z \textit{ a full backward orbit through } z\ \}
\end{aligned}$$

If f is known from the context, then we will usually drop it and use $W^s(z_0), W^s_Z(z_0)$ etc instead.

The following fact is well know in the Conley index theory (see [M])

Lemma 8 *Let X be a topological space and let the map $f : X \to X$ be continuous. If Z is compact, then the sets $Inv^{\pm}(Z,f)$ are compact.*

Lemma 9 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map. Assume that z_0 is a fixed point of f.*

Assume that there exists an h-set N with cones, such that $z_0 \in N$,

$$N \overset{f}{\Longrightarrow} N, \tag{49}$$

and f satisfies the cone condition with respect to the pair (N,N).

Then

$$Inv^+ N = W^s_N(z_0), \tag{50}$$
$$Inv^- N = W^u_N(z_0). \tag{51}$$

Proof: To prove (50) it is enough to show that, if $f^n(z) \in N$ for all $n \geq \mathbb{N}$, then $\lim_{n\to\infty} f^n(z) = z_0$.

Observe that the function $V(z) = Q(z - z_0)$ is a Lapunov function on N, i.e. is increasing on orbits in N, therefore there is a unique fixed point of f in N. Hence $f^n(z)$for $n \to \infty$ must converge toward z_0. It is easy to see, by the Lapunov function argument that there is only one fixed point in N. This finishes the proof of (50).

To prove (51) it is enough to show, that any backward orbit in N, $\{x_k\}_{k\in\mathbb{Z}_-}$ converges to z_0. But this is true by the same Lapunov function argument as in the previous paragraph. ∎

Theorem 10 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map. Assume that z_0 is a fixed point of f.*

Assume that there exists an h-set N with cones, such that $z_0 \in N$

$$N \overset{f}{\Longrightarrow} N, \tag{52}$$

and f satisfies the cone condition with respect to the pair (N, N).

Then $W^s_N(z_0)$ is a vertical disk in N satisfying the cone condition.

Therefore, $W^s_N(z_0)$ can be represented as a graph of a Lipschitz function over the nominally stable space in N.

Proof: First we show that for all $y \in \overline{B_s}$ there exists $x \in \overline{B_u}$, such that

$$z = c_N^{-1}(x, y) \in W^s_N(z_0). \tag{53}$$

By Lemma 9 it is equivalent to showing that

$$f^n(z) \in N, \quad \text{for } n \in \mathbb{N}. \tag{54}$$

Consider a family of horizontal disks in N $d_y : \overline{B_{u(N)}} \to N$ for $y \in \overline{B_{s(N)}}$

$$d_y(x) = (x, y). \tag{55}$$

Consider an infinite chain of covering relations

$$N \overset{f}{\Longrightarrow} N \overset{f}{\Longrightarrow} N \overset{f}{\Longrightarrow} \cdots N \overset{f}{\Longrightarrow} \cdots \tag{56}$$

From Theorem 4 applied to d_y, b_v, an arbitrary vertical disk in N and finite chains $N \overset{f}{\Longrightarrow} N \overset{f}{\Longrightarrow} N \overset{f}{\Longrightarrow} \cdots N$ of increasing length using the compactness argument one can show (see [W2, Col. 3.10]) that for every $y \in \overline{B_s}$ there exists $x \in \overline{B_u}$, such that (54) holds for $z = c_N^{-1}(x, y)$.

The next step is to prove that such x is unique. Let us assume the contrary, then there exists $y \in \overline{B_s}$ and $x_1, x_2 \in \overline{B_u}$, $x_1 \neq x_2$, such that $z_i = c_N^{-1}(x_i, y)$ for $i = 1, 2$ satisfies condition (54). Observe that

$$Q(z_1 - z_2) = \alpha(x_1 - x_2) > 0, \tag{57}$$

hence from the cone condition and (54) it follows that

$$Q(f^n(z_1) - f^n(z_2) > \alpha(x_1 - x_2), \qquad \text{for } n \in \mathbb{N}. \tag{58}$$

Passing to the limit $n \to \infty$ we obtain

$$0 = Q(z_0 - z_0) = \lim_{n \to \infty} Q(z_1(t) - z_2(t)) > \alpha(x_1 - x_2) > 0. \tag{59}$$

This is a contradiction. Hence we have a well defined function $x(y)$ on $\overline{B_s}$.

From the uniqueness of $x(y)$ the continuity of $x(y)$ follows easily. Namely, from the compactness argument it follows that it is enough to prove that if we have a sequence of pairs $(x_n = x(y_n), y_n)$, where $y_n \in \overline{B_s}$, $y_n \to \bar{y}$ for $n \to \infty$ and $x_n \to \bar{x}$, then $c_N^{-1}(\bar{x}, \bar{y}) \in Inv^+ N$, but this is an obvious consequence of the closeness of $Inv^+ N$.

Hence, $W_N^s(x_0) = |b|$, where b is a vertical disk in N, given by $b(x,y) = c_N^{-1}(x(y),y)$. Now we prove the cone condition for this disk.

We have to check whether

$$Q_N((x(y_1),y_1) - (x(y_2),y_2)) < 0, \qquad \text{for all } y_1, y_2 \in \overline{B_s},\ y_1 \neq y_2 \tag{60}$$

Assume that it does not hold. Then for some $z_1, z_2 \in W_N^s(x_0)$, $z_1 \neq z_2$ we have

$$Q(z_1 - z_2) \geq 0. \tag{61}$$

From the cone condition it follows that

$$Q(f^n(z_1) - f^n(z_2) > Q(f(z_1) - f(z_2)) > 0, \qquad \text{for } n > 1. \tag{62}$$

Passing to the limit $n \to \infty$ we obtain

$$0 = Q(z_0 - z_0) = \lim_{n\to\infty} Q(f^n(z_1) - f^n(z_2)) > Q(f(z_1) - f(z_2)) > 0. \tag{63}$$

Which is a contradiction. This proves (60). ∎

Theorem 11 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map. Assume that z_0 is a fixed point of f.*

Assume that there exists an h-set N with cones, such that $z_0 \in N$,

$$N \overset{f}{\Longrightarrow} N, \tag{64}$$

and f satisfies the cone condition with respect to the pair (N,N).

Then $W_N^u(z_0)$ is a horizontal disk in N satisfying the cone condition.

Therefore, $W_N^u(z_0)$ can be represented as a graph of a Lipschitz function over the nominally unstable space in $|N|$.

Proof: Without any loss of the generality we can assume that $N = \overline{B}_u \times \overline{B}_s$ and $c_N = id$.

We will prove that for any $x \in B_u$ there exists $y \in B_s$, such that $(x,y) \in W_N^u(x_0)$. For any $x \in \overline{B}_u$ let v_x be a vertical disk given by

$$v_x(y) = (x,y).$$

Let $h : \overline{B}_u \to \overline{B}_u \times \overline{B}_s$ be a horizontal disk given by $h(x) = (x,0)$.

Consider a chain of covering relations consisting of k replicas of $N \overset{f}{\Longrightarrow} N$. It follows from Theorem 4 it follows that there exists a finite orbit $\{w_{-k}^k, w_{-k+1}^k, \dots, w_{-1}^k, w_0^k\}$, such that

$$\begin{aligned} w_{-k}^k, w_{-k+1}^k, \dots, w_{-1}^k, w_0^k &\in N \\ f(w_l^k) = w_{l+1}^k, &\qquad l = -k, \dots, -1 \\ w_{-k}^k \in |h|, &\qquad w_0^k \in |v_x|. \end{aligned}$$

By applying a diagonal argument we can find an infinite backward orbit $\{w_l\}_{l\in\mathbb{Z}_-\cup\{0\}}$, such that

$$\begin{array}{rcll} w_l & \in & N, & l=0,-1,-2,\dots \\ f(w_l) & = & w_{l-1}, & l<0 \\ w_0 & \in & |v_x|. & \end{array}$$

Since $V(z)=Q_N(z-z_0)$ is increasing on orbits for $z\neq z_0$, therefore

$$\lim_{l\to-\infty} w_l = z_0. \tag{65}$$

We have proved that

$$w_0 \in W^u_N(z_0)\cap|v_x|. \tag{66}$$

We will prove that w_0 in (66) is uniquely defined. Let p_0 also satisfies the above condition, hence there exists a backward orbit in N through p_0 $\{p_l\}_{l\in\mathbb{Z}_-\cup\{0\}}$. We have

$$Q_N(p_0-w_0) = -\beta(y(p_0)-y(w_0))<0. \tag{67}$$

From the cone condition for map f it follows that the function $Q_N(p_l-w_l)$ is increasing for $l<0$, hence

$$0>Q_N(p_0-w_0)>Q_N(p_l-w_l)>\lim_{l\to-\infty}Q(p_l-w_l)=Q_N(z_0-z_0)=0. \tag{68}$$

Which is a contradiction, therefore w_0 in (66) is uniquely defined.

We define a horizontal disk $d:\overline{B}_u\to\overline{B}_u\times\overline{B}_s$, by $d(x)=(x,w_0)$. From the above considerations it follows that

$$W^u_N(z_0)=|d|. \tag{69}$$

We will show that d is satisfy the cone condition

$$Q_N(w-p)>0, \qquad \text{for all } w,p\in|d|,\ w\neq p. \tag{70}$$

Assume that (70) does not hold. Then there exists two full backward orbits $\{w_l\},\{p_l\}$ in N through w and p and

$$Q_N(w-p)\leq 0. \tag{71}$$

We have for any $l\in\mathbb{Z}_-$

$$0\geq Q_N(w_0-p_0)>Q_N(w_l-p_l)>\lim_{l\to-\infty}Q_N(w_l-p_l)=Q_N(z_0-z_0)=0.$$

But this is a contradiction, hence (70) is satisfied. ∎

Below we present a theorem about the existence of the unstable and unstable manifold for hyperbolic fixed points. In our opinion the most interesting feature, in fact probably the only one, is that the proof uses the arguments from the dynamics, only. The result, concerning the smoothness, is rather weak, when compared to classical results in the literature [HPS], as we have only the Lipschitz condition and a suitable tangency at the fixed point.

Theorem 12 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 local diffeomorphism. Assume that z_0 is a hyperbolic fixed point of f ($Sp(Df(z_0)) \cap S^1 = \emptyset$).*

Let $Z \subset \mathbb{R}^n$ be an open set, such that $z_0 \in Z$.

Then there exists an h-set N with cones, such that $z_0 \in \mathrm{int} N$, $N \subset Z$ and

- $N \overset{f}{\Longrightarrow} N$, $N^T \overset{f^{-1}}{\Longrightarrow} N^T$
- *$W^u_N(z_0)$ is a horizontal disk in N satisfying the cone condition*
- *$W^s_N(z_0)$ is a vertical disk in N satisfying the cone condition.*

Moreover, $W^u_N(z_0)$ can be represented as a graph of a Lipschitz function over the unstable space for the linearization of f at z_0 and tangent to it at z_0. Analogous statement is also valid for $W^s_N(z_0)$.

Proof: Let L ba a linearization of f at z_0, hence $L(z) = z_0 + df(z_0)(z - z_0)$. Let u be the dimension of the unstable manifold and s of the stable manifold of L at z_0.

Then there exists a coordinate system on $\mathbb{R}^n$ and a scalar product $(\cdot, \cdot)$ such that following holds

$$df(z_0) = \left[\begin{array}{cc} A & 0 \\ 0 & U \end{array} \right], \tag{72}$$

where $A : \mathbb{R}^u \to \mathbb{R}^u$ and $U : \mathbb{R}^s \to \mathbb{R}^s$ are linear isomorphisms, such that

$$W^u(z_0, L) = \{z_0\} + \mathbb{R}^u \times \{0\}^s, \quad W^s(z_0, L) = \{z_0\} + \{0\}^u \times \mathbb{R}^s \tag{73}$$

$$\|Ax\| > \|x\|, \quad \text{for } x \in \mathbb{R}^u \setminus \{0\} \tag{74}$$

$$\|Uy\| < \|y\|, \quad \text{for } y \in \mathbb{R}^s \setminus \{0\}, \tag{75}$$

where the norms are $\|x\| = \sqrt{x^2}$ and $\|y\| = \sqrt{y^2}$. We will use these coordinates in our proof.

Observe that (74) and (75) imply that matrices $A^T A - Id$ and $Id - U^T U$ are positive definite.

For any $r > 0$ we define

$$N(r) = \{z_0\} + \overline{B}_u(0, r) \times \overline{B}_s(0, r). \tag{76}$$

We define the homotopy

$$f_\lambda(z) = (1-\lambda) f(z) + \lambda(df(z_0)(z - z_0) + z_0), \quad \text{where } \lambda \in [0, 1] \text{ and } z \in \mathbb{R}^n. \tag{77}$$

It is easy to see that $f_0 = f$ and $f_1(x, y) = df(z_0)(z - z_0) + z_0$.

Let $Q((x, y)) = \alpha x^2 - \beta y^2$, where $x \in \mathbb{R}^u$ and $y \in \mathbb{R}^s$ and $\alpha > 0$, $\beta > 0$ are arbitrary positive reals.

We will need the following lemma, which will be proved after we complete the current proof.

Lemma 13 *There exists $r_0 > 0$, such that for any $0 < r \leq r_0$ for all $z_1, z_2 \in N(r_0)$, $z_1 \neq z_2$ holds*

$$Q(f_\lambda(z_1) - f_\lambda(z_2)) > Q(z_1 - z_2). \tag{78}$$

Moreover, for any $z \in N(r)$ holds

$$(\pi_x f_\lambda(z) - \pi_x z_0)^2 > r, \qquad \text{if } \|\pi_x(z - z_0)\| = r \tag{79}$$

$$(\pi_y f_\lambda(z) - \pi_y z_0)^2 < r, \qquad \text{if } \|\pi_y(z - z_0)\| = r \tag{80}$$

Continuation of the proof of Theorem 12: Let us fix any $r \leq r_0$, where r_0 is as in Lemma 13.

We define an h-set N with cones as follows: we set $|N| = N(r)$, $c_N(z) = \frac{1}{r}(z - z_0)$, $u(N) = u$, $s(N) = s$ and $Q_N(z') = Q(c_N^{-1}(z'))$ for $z' \in N_c$.

From Lemma 13 it follows that the following conditions are satisfied for any $\lambda \in [0, 1]$

$$Q_N(f_{\lambda,c}(z_1) - f_{\lambda,c}(z_2)) > Q_N(z_1 - z_2), \qquad z_1, z_2 \in N_c, z_1 \neq z_2 \tag{81}$$

$$\pi_x f_\lambda(N) \subset \mathbb{R}^n \setminus \pi_x N = \mathbb{R}^n \setminus \overline{B}_u(\pi_x z_0, r) \tag{82}$$

$$\pi_y f_\lambda(N) \subset B_s(\pi_y z_0, r) \tag{83}$$

We will prove that

$$N \overset{f}{\Longrightarrow} N. \tag{84}$$

For this we need a suitable homotopy. We define $H : [0, 1] \times N \to \mathbb{R}^{u+s}$ as follows

$$H(\lambda, z) = \begin{cases} f_{2\lambda}(z) & \text{for } \lambda \in \left[0, \frac{1}{2}\right], \\ (A(\pi_x(z - z_0), (-2\lambda + 2)U\pi_y(z - z_0)) + z_0 & \text{for } \lambda \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Observe that

$$H_0 = f, \qquad H_1(z) = (A(\pi_x(z - z_0)), 0) + z_0 \tag{85}$$

$$\pi_x H_\lambda(N) \subset \mathbb{R}^n \setminus \pi_x N = \mathbb{R}^n \setminus \overline{B}_u(\pi_x z_0, r) \tag{86}$$

$$\pi_y H_\lambda(N) \subset B_s(\pi_y z_0, r). \tag{87}$$

It is immediate to check that the homotopy $h(\lambda, z) = c_N(H(\lambda, c_N^{-1})(z))$ satisfies all conditions for the covering relation $N \overset{f,w}{\Longrightarrow} N$, where $w = \pm 1$ due to linearity of h_1.

Analogous reasoning leads to $N^T \overset{f^{-1}}{\Longrightarrow} N^T$ (we may need to decrease further r in the construction.)

The remaining assertions, with the exception of the one the concerning the tangency to $W^{u,s}(z_0, L)$ at z_0, follow directly from Theorems 10 and 11.

To prove the tangency of $W^u(z_0, f)$ to $z_0 + \mathbb{R}^u \times \{0\}^s$ at $z_0 = (x_0, y_0)$ it is enough to prove that for any $\epsilon > 0$, there exists $r > 0$, such that for any $z = (x, y(x)) \in W^u_{N(r)}(z_0, f)$ holds

$$\|y(x) - y_0\| \leq \epsilon \|x - x_0\|. \tag{88}$$

For given α, β the set $W^u_{N(r)}(z_0, f)$ for r sufficiently small is a horizontal disk satisfying the condition with respect to the quadratic form $Q(x, y) = \alpha x^2 - \beta y^2$. Therefore we have

$$
\begin{aligned}
Q((x, y(x)) - (x_0, y_0)) &> 0 \\
\beta \|y(x) - y_0\|^2 &< \alpha \, \|x - x_0\|^2 \\
\|y(x) - y_0\| &< \sqrt{\alpha/\beta} \, \|x - x_0\|,
\end{aligned}
$$

which proves (88).

The proof of the tangency for $W^s(z_0, f)$ to $z_0 + \{0\}^u \times \mathbb{R}^s$ at z_0 is analogous.

∎

Proof of Lemma 13: To see that (78) is indeed satisfied for z_i close to z_0, we derive some other condition, which forces it (compare Lemma 7). For this end let Q be a symmetric matrix corresponding the quadratic form Q. Then

$$
\begin{aligned}
& Q(f_\lambda(z_1) - f_\lambda(z_2)) - Q(z_1 - z_2) = \\
& (f_\lambda(z_1) - f_\lambda(z_2))^T Q (f_\lambda(z_1) - f_\lambda(z_2)) - (z_1 - z_2)^T Q (z_1 - z_2) = \\
& (z_1 - z_2)^T C^T Q C (z_1 - z_2) - (z_1 - z_2)^T Q (z_1 - z_2) = \\
& (z_1 - z_2)^T (C^T Q C - Q)(z_1 - z_2),
\end{aligned}
$$

where

$$
\begin{aligned}
C = C(\lambda, z_1, z_2) = \int_0^1 df_\lambda(z_1 + t(z_2 - z_1)) dt = \\
(1 - \lambda) \int_0^1 df_\lambda(z_1 + t(z_2 - z_1)) dt + \lambda df(z_0)
\end{aligned}
$$

Observe that for $z_1, z_2 \to z_0$ the matrix $C(\lambda, z_1, z_2)$ converges to $df(z_0)$ uniformly with respect to $\lambda \in [0, 1]$. Therefore it is enough to show that the symmetric matrix $V = df(x_0)^T Q df(x_0) - Q$ is positive definite.

We have

$$
V = \begin{bmatrix} \alpha(A^T A - Id), & 0 \\ 0, & \beta(Id - U^T U) \end{bmatrix}
$$

Since $\alpha > 0$, $\beta > 0$ and $A^T A - Id$ and $Id - U^T U$ are positive definite, hence V is positive definite. From this is follows that there is r_0, such that (78) holds for $z_1, z_2 \in N(r_0)$, $z_1 \neq z_2$.

Now we prove condition (79). We have

$$
\begin{aligned}
(\pi_x f_\lambda(z) - \pi z_0)^2 = (\pi_x f_\lambda(z_1) - \pi_x f_\lambda(z_0))^2 = \\
(C_{11}(\pi_x z - \pi_x z_0) + C_{12}(\pi_y z - \pi_y z_0))^2,
\end{aligned}
\tag{89}
$$

where

$$C_{11} = C_{11}(\lambda, z_1, z_0) = \int_0^1 \frac{\partial \pi_x f_\lambda}{\partial x}(z_0 + t(z - z_0))dt = \\ \frac{\partial \pi_x f_\lambda}{\partial x}(z_0) + O(\|z - z_0\|) = A + O(\|z - z_0\|),$$

$$C_{12} = C_{12}(\lambda, z_1, z_0) = \int_0^1 \frac{\partial \pi_x f_\lambda}{\partial y}(z_0 + t(z - z_0))dt = \\ \frac{\partial \pi_x f_\lambda}{\partial y}(z_0) + O(\|z - z_0\|) = O(\|z - z_0\|).$$

Let us fix $0 < r \leq r_0$ and $\lambda \in [0, 1]$. Let $z = (x, y) \in N(r)$, $z_0 = (x_0, y_0)$ and $\|x - x_0\| = r$. We have

$$(\pi_x f_\lambda(z) - x_0)^2 = (C_{11}(x - x_0))^2 + (C_{12}(y - y_0))^2 + \\ 2(x - x_0)^T C_{11}^T C_{12}(y - y_0) \geq (1 + a - O(r))r^2 - \\ O(r)^2 r^2 - 2(\|A\| + O(r))O(r)r^2 = (1 + a - O(r))r^2$$

where $a > 0$ is such that $x^T A^T A x \geq (1 + a)x^2$. Hence (79) holds provided r_0 is small enough.

The justification of (80) is analogous. ∎

5.1 Continuous dependence of invariant manifolds of a hyperbolic fixed point on parameters

Theorem 14 *Let $\Lambda \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^n$ be open sets. Assume that $f : \Lambda \times V \to \mathbb{R}^n$, where $\Lambda \subset \mathbb{R}^k$ be such that*

- *$\forall \lambda \in \Lambda$ f_λ is C^1-diffeomorphism*
- *f and $\frac{\partial f}{\partial z}$ are continuous on $\Lambda \times \mathbb{R}^n$*
- *z_0 is a hyperbolic fixed point of f_{λ_0}.*

Then there exists sets $C \subset \Lambda$ and $U \subset V$, such $(\lambda_0, z_0) \in \mathrm{int}(C \times U)$ and a continuous function $p : C \to U$, such that $p(\lambda)$ is a hyperbolic fixed point for f_λ, $p(\lambda_0) = z_0$ and $W_U^{u,s}(p(\lambda), f_\lambda)$ depend continuously on λ, for $\lambda \in C$.

The continuity of sets $W^{u,s}(p(\lambda), f_\lambda)$ with respect to $\lambda \in C$ means that they are given as graphs of some functions depending continuously on λ.

Proof: The existence of $p(\lambda)$ follows immediately from the implicit function theorem.

By proceeding as in the proof of Theorem 12, namely by using the diagonalizing coordinates for $\frac{\partial f_{\lambda_0}}{\partial z}(z_0)$ we can construct arbitrarily small h-set with cones (N, Q), $N = N(r)$, such that

$$N \overset{f_{\lambda_0}}{\Longrightarrow} N \tag{90}$$

and the interval quadratic form given by

$$V = [df_{\lambda_0,c}]^T Q [df_{\lambda_0,c}] - Q \tag{91}$$

is positive definite.

Observe that conditions (90,91) are both stable with respect to small change of map f_{λ_0}, therefore there exists a set $C \subset \Lambda$, such that $\lambda_0 \in \mathrm{int} C$ and

$$N \overset{f_\lambda}{\Longrightarrow} N \tag{92}$$

and the interval quadratic form given by

$$V = [df_{\lambda,c}]^T Q [df_{\lambda,c}] - Q \tag{93}$$

is positive definite.

Theorems 10 and 11 imply that $W^{u,s}(p(\lambda), f_\lambda)$ are horizontal or vertical disks in N, respectively.

It remains to prove the continuity of $W^{u,s}(p(\lambda), f_\lambda)$. Let us focus on the unstable manifold. Observe first that $f_\lambda^{-1}(z)$ is continuous on $C \times N$.

From the previous reasoning it follows that there exists a function $y : C \times \overline{B_u}(0,r) \to \overline{B_s}(0,r)$, such that

$$z \in W^u_N(p(\lambda), f_\lambda)) \quad \text{iff} \quad z = z_0 + (x, y(\lambda, x)), \quad \text{for some } x \in \overline{B_u}(0,r). \tag{94}$$

It is enough to prove that the function $y(\lambda, x)$ is continuous with respect to both arguments. Let $(\lambda_k, x_k) \in C \times \overline{B_u}(0,r)$ for $k \in \mathbb{N}$ be a sequence converging to $(\bar{\lambda}, \bar{x}) \in C \times \overline{B_u}(0,r)$. Let us define $\bar{y} = y(\bar{\lambda}, \bar{x})$, $z_k = z_0 + (x_k, y(\lambda_k, x_k))$.

Obviously we have

$$f_{\lambda_k}^{-i}(z_k) \in N, \qquad \text{for } i \in \mathbb{N} \tag{95}$$

Consider the sequence $y_k = y(\lambda_k, x_k)$ we need to show that $\lim_{k\to\infty} y_k = \bar{y}$. Observe that $y_k \in \overline{B_s}(0,r)$, hence we can pick up convergent subsequences. The proof will be completed, when we show that any convergent subsequence of $\{y_k\}$ converges to $\bar{y}$.

Let $\{y_{k_n}\}$ be a subsequence of $\{y\}$ convergent to u_0. We will show that $\bar{z} = z_0 + (\bar{x}, u_0)$ belongs to $W^u(p(\bar{\lambda}), f_{\bar{\lambda}})$.

Since $z_{k_n} \to \bar{z}$ for $n \to \infty$ then from the continuity argument applied to (95) it follows that for any $i \in \mathbb{N}$

$$f_{\bar{\lambda}}^{-i}(\bar{z}) \in N. \tag{96}$$

Therefore $\bar{z} \in \mathrm{Inv}^-(N, f_{\bar{\lambda}})$. From Lemma 9 it follows that $\bar{z} \in W^u_N(p(\bar{\lambda}), f_{\bar{\lambda}})$. Now from (94) it follows that $u_0 = \bar{y}$.

∎

6 The stable and unstable manifolds of hyperbolic fixed points for ODEs.

Consider an ordinary differential equation

$$z' = f(z), \qquad z \in \mathbb{R}^n, \quad f \in C^2(\mathbb{R}^n, \mathbb{R}^n). \tag{97}$$

Let us denote by $\varphi(t,p)$ the solution of (97) with the initial condition $z(0) = p$. For any $t \in \mathbb{R}$ by we define a map $\varphi(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ by $\varphi(t,\cdot)(x) = \varphi(t,x)$. We ignore here the question whether $\varphi(t,x)$ is defined for every (t,x), but this can be achieved by modification of f outside a large ball containing the phenomena under consideration.

Let $Z \subset \mathbb{R}^n$, $z_0 \in Z$. We define

$$W^s_Z(z_0,\varphi) = \{z \mid \forall_{t\geq 0}\varphi(t,z) \in Z, \quad \lim_{t\to\infty} \varphi(t,z) = z_0\} \tag{98}$$
$$W^u_Z(z_0,\varphi) = \{z \mid \forall_{t\leq 0}\varphi(t,z) \in Z, \quad \lim_{t\to-\infty} \varphi(t,z) = z_0\} \tag{99}$$
$$W^s(z_0,\varphi) = \{z \mid \lim_{t\to\infty} \varphi(t,z) = z_0\} \tag{100}$$
$$W^u(z_0,\varphi) = \{z \mid \lim_{t\to-\infty} \varphi(t,x) = z_0\} \tag{101}$$
$$Inv^+(Z,\varphi) = \{z \mid \forall_{t\geq 0}\varphi(t,z) \in Z\} \tag{102}$$
$$Inv^-(Z,\varphi) = \{z \mid \forall_{t\leq 0}\varphi(t,z) \in Z\} \tag{103}$$

Sometimes, when φ is known from the context it will be dropped and we will write $W^s_Z(z_0), Inv^\pm(Z)$ etc.

The goal of this section is to prove the following theorem.

Theorem 15 *Assume that $z_0 = (x_0, y_0)$ is an hyperbolic fixed point for (97), i.e. $Re\lambda \neq 0$ for all $\lambda \in Sp(df(z_0))$.*

Let $Z \subset \mathbb{R}^n$ be an open set, such that $z_0 \in Z$.

Then there exists an h-set N with cones, such that $z_0 \in N$, $N \subset Z$, $W^u_N(z_0)$ is a horizontal disk in N satisfying the cone condition and $W^s_N(z_0)$ is a vertical disk in N satisfying the cone condition.

Moreover, $W^u_N(z_0)$ can be represented as a graph of a Lipschitz function over the unstable space for the linearization of f at z_0 and tangent to it at z_0. Analogous statement is also valid for $W^s_N(z_0)$.

Proof: Consider a flow obtained from (97) by linearization

$$x' = df(z_0)(x - z_0). \tag{104}$$

Let φ_L denotes the flow induced by (104) and let u and s be the dimension of the unstable and stable manifolds for (104) at z_0. It well known that there exists a coordinate system and the scalar product $(\cdot,\cdot)$ such that following holds

$$df(z_0) = \begin{bmatrix} A & 0 \\ 0 & U \end{bmatrix} \tag{105}$$

where $A \in \mathbb{R}^{u\times u}$, $U \in \mathbb{R}^{s\times s}$, such that $A + A^T$ is positive definite and $U + U^T$ is negative definite. In this coordinate system $W^u(z_0, \varphi_L) = \{z_0\} + \mathbb{R}^u \times \{0\}^s$ and $W^s(z_0, \varphi_L) = \{z_0\} + \{0\}^u \times \mathbb{R}^s$. We will use these coordinates in our proof.

Let us fix $\alpha, \beta \in \mathbb{R}_+$. Let us define a quadratic form $Q((x, y)) = \alpha x^2 - \beta y^2$, where $x \in \mathbb{R}^u$ and $y \in \mathbb{R}^s$.

For any $\lambda \in [0, 1]$ let φ_λ be the flow induced by

$$z' = f_\lambda(z) := (1-\lambda)f(z) + \lambda(df(z_0)(z - z_0)) \tag{106}$$

For any $r > 0$ we define $N(r)$ by

$$N(r) = \{z_0\} + \overline{B_u(0, r)} \times \overline{B_s(0, r)}. \tag{107}$$

To proceed further we need the following Lemma, which will be proved after we complete the current proof.

Lemma 16 *There exists $r_0 > 0$, such that for $\lambda \in [0, 1]$ and for any $0 < r \leq r_0$ the following conditions are satisfied.*

$$\frac{d}{dt}Q(\varphi_\lambda(t, z_1) - \varphi_\lambda(t, z_2))_{|t=0} > 0, \qquad \text{for all } z_1, z_2 \in N(r),\ z_1 \neq z_2 \tag{108}$$

$$\frac{d(\pi_x(\varphi_\lambda(t, z)) - x_0)^2}{dt}(z) > 0, \qquad z \in N(r) \text{ and } \|\pi_x(z - z_0)\| \geq \tfrac{r}{2} \tag{109}$$

$$\frac{d(\pi_y(\varphi_\lambda(t, z)) - y_0)^2}{dt}(z) < 0, \qquad z \in N(r) \text{ and } \|\pi_y(z - z_0)\| \geq \tfrac{r}{2} \tag{110}$$

Continuation of the proof of Theorem 15. Let us fix $r = r_0/2$, where r_0 is as in Lemma 16. We define the h-set N with cones as follows: we set $|N| = N(r)$, $c_N(z) = \frac{1}{r}(z - z_0)$, $u(N) = u$, $s(N) = s$ and $Q_N(z') = Q(c_N^{-1}(z'))$ for $z' \in N_c$.

Observe that from Lemma 16 it follows immediately, that in the sense of the Conley index theory [S] the pair (N, N^-) is an isolating block.

From Lemma 16 if follows that for $h > 0$ small enough the following conditions are satisfied for every $\lambda \in [0, 1]$

$$\text{if } z \in N, \text{ then } \varphi_\lambda([-h, h], z) \in N(r_0) \tag{111}$$

$$\text{if } z \in N^-, \text{ then } \varphi_\lambda((0, h], z) \notin N, \tag{112}$$

$$\text{if } z \in N^+, \text{ then } \varphi_\lambda([-h, 0), z) \notin N, \tag{113}$$

$$\text{if } z, \varphi_\lambda(h, z) \in N, \text{ then } \varphi_\lambda([0, h], z) \in N \tag{114}$$

$$\text{if } z, \varphi_\lambda(-h, z) \in N, \text{ then } \varphi_\lambda([-h, 0], z) \in N. \tag{115}$$

From Lemma 16 and condition (111) it follows that

$$Q(\varphi(h, z_1) - \varphi(h, z_2)) > Q(z_1 - z_2), \quad \text{for } z_1, z_2 \in N, z_1 \neq z_2 \tag{116}$$

We will prove that

$$N \overset{\varphi(h,\cdot)}{\Longrightarrow} N. \tag{117}$$

For the proof of (117) we need a suitable homotopy. First consider $H(\lambda, h) = \varphi_\lambda(h, \cdot)$. Obviously, $H_0 = \varphi(h, \cdot)$ and $H_1 = \varphi_L(h, \cdot)$.

From Lemma 16 if follows that

$$\pi_x(H([0,1], N^-)) \subset \mathbb{R}^u \setminus \overline{B}_u(x_0, r), \tag{118}$$

$$\pi_y(H([0,1], N)) \subset B_s(y_0, r). \tag{119}$$

Observe that the above conditions imply that

$$H([0,1], N^-) \cap N = \emptyset, \tag{120}$$

$$H([0,1], N) \cap N^+ = \emptyset. \tag{121}$$

We have $H_1(x, y) = (\exp(Ah)(x - x_0), \exp(Uh)(y - y_0)) + z_0$. Let us define the homotopy $G : [0,1] \times \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}^u \times \mathbb{R}^s$ by

$$G(\lambda, x, y) = (\exp(Ah)(x - x_0), (1 - \lambda) \exp(Uh)(y - y_0)) + z_0. \tag{122}$$

Let F be the homotopy obtained by concatenation of H and G, this means that

$$F(\lambda, z) = \begin{cases} H(2\lambda, z) & \text{for } 0 \leq \lambda \leq 1/2, \\ G(2(\lambda - 1/2), z) & \text{otherwise.} \end{cases} \tag{123}$$

It is easy to see the homotopy $F_c(\lambda, z) = c_N(F(\lambda, c_N^{-1}(z)))$ for $z \in N_c$ satisfies all conditions for the covering relation $N \overset{\varphi(h,\cdot),w}{\Longrightarrow} N$, where $w = \pm 1$ (this follows from the linearity of F_1.)

Now we apply Theorems 11 and 10 to (N, Q) and $\varphi(h, \cdot)$ to infer that $W^u_N(z_0, \varphi(h, \cdot))$ and $W^s_N(z_0, \varphi(h, \cdot))$ are horizontal and vertical disks, respectively.

To finish the proof we need to show that

$$W^u_N(z_0, \varphi(h, \cdot)) = W^u_N(z_0, \varphi) \tag{124}$$

$$W^s_N(z_0, \varphi(h, \cdot)) = W^s_N(z_0, \varphi) \tag{125}$$

Let us prove (124), the proof of (125) is analogous.

Observe first, that the inclusion $W^u_N(z_0, \varphi(h, \cdot)) \supset W^u_N(z_0, \varphi)$ is obvious. For the opposite direction let us take $z \in W^u_N(z_0, \varphi(h, \cdot))$, then from condition (115) it follows that $\varphi((-\infty, 0], z) \subset N$. From Lemma 16 if follows that $V(z) = Q(z - z_0)$ is decreasing (in strong sense) as long as the orbit stays in N. Hence $\lim_{t \to -\infty} \varphi(t, z) = z_0$.

The tangency of the stable (unstable) manifolds of φ and φ_L at z_0 is obtained as in the map case - see the conclusion of the proof of Theorem 12 for more details. ∎

Proof of Lemma 16

Let us fix $\lambda \in [0,1]$. For $z_i \in \mathbb{R}^n$ and $t \in \mathbb{R}$ let $z_i(t) = \varphi_\lambda(t, z_i)$.

Let Q be a symmetric matrix corresponding the quadratic form Q. Then

$$\frac{d}{dt}Q(z_1(t)-z_2(t))_{|t=0} =$$
$$(f_\lambda(z_1)-f_\lambda(z_2))^T Q(z_1-z_2) + (z_1-z_2)^T Q(f_\lambda(z_1)-f_\lambda(z_2)) =$$
$$(z_1-z_2)^T C^T Q(z_1-z_2) + (z_1-z_2)^T QC(z_1-z_2) =$$
$$(z_1-z_2)^T (C^T Q + QC)(z_1-z_2),$$

where

$$C = C(\lambda, z_1, z_2) = \int_0^1 df_\lambda(z_1 + t(z_2-z_1))dt =$$
$$(1-\lambda)\int_0^1 df(z_1+t(z_2-z_1))dt + \lambda df(z_0).$$

Observe that for $z_1, z_2 \to z_0$ the matrix $C(\lambda, z_1, z_2)$ converges to $df(z_0)$ uniformly with respect to $\lambda \in [0,1]$, hence it is enough to show that the symmetric matrix $df(z_0)^T Q + Qdf(z_0)$ is positive definite.

We have

$$df(z_0)^T Q + Qdf(x_0) = \begin{bmatrix} A^T & 0 \\ 0 & U^T \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & -\beta \end{bmatrix} +$$
$$\begin{bmatrix} \alpha & 0 \\ 0 & -\beta \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} \alpha(A+A^T) & 0 \\ 0 & -\beta(U+U^T) \end{bmatrix}$$

Since matrices $\alpha(A+A^T)$ and $-\beta(U+U^T)$ are positive definite, then the same is true about $df(z_0)^T Q + Qdf(x_0)$.

Consider condition (109). Let $z = (x,y)$. We have for $t = 0$

$$\frac{d(\pi_x(\varphi_\lambda(t,z)) - x_0)^2}{dt} = 2(x-x_0)^T \pi_x f_\lambda(z) =$$
$$2(x-x_0)^T C_{11}(x-x_0) + 2(x-x_0)^T C_{12}(y-y_0),$$

where

$$C_{11} = C_{11}(\lambda, z, z_0) = \int_0^1 \frac{\partial \pi_x f_\lambda}{\partial x}(z_0 + t(z-z_0))dt =$$
$$\frac{\partial \pi_x f}{\partial x}(z_0) + O(\|z-z_0\|) = A + O(\|z-z_0\|),$$
$$C_{12} = C_{12}(\lambda, z, z_0) = \int_0^1 \frac{\partial \pi_x f_\lambda}{\partial y}(z_0 + t(z-z_0))dt =$$
$$\frac{\partial \pi_x f}{\partial y}(z_0) + O(\|z-z_0\|) = O(\|z-z_0\|).$$

Now let $z = (x,y) \in N(r)$ and $\|x - x_0\| \geq \frac{r}{2}$. We have for $t = 0$

$$\frac{d(\pi_x(\varphi_\lambda(t,z)) - x_0)^2}{dt} =$$
$$(x-x_0)^T(A+A^T)(x-x_0) + 2(x-x_0)^T O(r)(x-x_0) +$$
$$2(x-x_0)^T O(r)(y-y_0) \geq a(r/2)^2 - O(r)r^2 = (a/4 - O(r))r^2,$$

where $a > 0$ is such that $x^T(A + A^T)x \geq ax^2$. Hence (109) holds provided r_0 is small enough.

The justification of (110) is analogous.

∎

7 Non-hyperbolic example

Consider the following map $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x, y) = (x + x^3, y - y^3) + P(x, y), \tag{126}$$

where $P(x, y)$ is a polynomial, such that the degree of all nonzero terms in P is at least 4.

Observe that $z_0 = (0, 0)$ is a non-hyperbolic fixed point, but a look at the dominant terms $(x + x^3, y - y^3)$, suggests that nevertheless z_0 will have a one dimensional stable and unstable manifolds tangent at z_0 to the coordinate axes.

We will prove the following theorem

Theorem 17 *Consider the map f given by (126).*

There exists an h-set N with cones, such that $z_0 \in \mathrm{int} N$, $N \subset Z$ and

- $N \overset{f}{\Longrightarrow} N$,
- *$W^u_N(z_0)$ is a horizontal disk in N satisfying the cone condition*
- *$W^s_N(z_0)$ is a vertical disk in N satisfying the cone condition.*

Moreover, $W^u_N(z_0)$ is at z_0 tangent to the line $y = 0$ and $W^s_N(z_0)$ is at z_0 tangent to the line $x = 0$.

Let us fix $\alpha > 0$, $\beta > 0$ and consider a quadratic form $Q_{\alpha,\beta} : \mathbb{R}^2 \to \mathbb{R}$

$$Q_{\alpha,\beta}(x, y) = \alpha x^2 - \beta y^2. \tag{127}$$

The first step in the proof of Theorem 17 is the following lemma showing the cone condition for small z_1, z_2.

Lemma 18 *There exists $\delta > 0$, such that if $|x_i| \leq \delta$ and $|y_i| \leq \delta$ for $i = 1, 2$, then*

$$Q_{\alpha,\beta}(f(z_1) - f(z_2)) > Q_{\alpha,\beta}(z_1 - z_2), \tag{128}$$

where $z_i = (x_i, y_i)$ for $i = 1, 2$.

Proof: Let us denote $f(z) = (f_1(z), f_2(z))$ and let us set

$$N(a, b) = a^2 + ab + b^2. \tag{129}$$

Obviously we have

$$\frac{a^2 + b^2}{2} \leq N(a, b) \leq \frac{3(a^2 + b^2)}{2} \tag{130}$$

Observe that

$$\begin{aligned} f_1(z_1) - f_1(z_2) = \\ x_1 - x_2 + (x_1^3 - x_2^3) + C_{1,1}(z_1, z_2)(x_1 - x_2) + C_{1,2}(z_1, z_2)(y_1 - y_2) = \\ (x_1 - x_2)\left(1 + N(x_1, x_2) + C_{1,1}(z_1, z_2)\right) + C_{1,2}(z_1, z_2)(y_1 - y_2), \end{aligned}$$

and

$$\begin{aligned} f_2(z_1) - f_2(z_2) = \\ y_1 - y_2 - (y_1^3 - y_2^3) + C_{2,1}(z_1, z_2)(x_1 - x_2) + C_{2,2}(y_1 - y_2) = \\ (y_1 - y_2)\left(1 - N(y_1, y_2) + C_{2,2}(z_1, z_2))\right) + C_{2,1}(z_1, z_2)(x_1 - x_2) \end{aligned}$$

where

$$\begin{aligned} C_{j,1}(z_1, z_2) &= \int_0^1 \frac{\partial P_j}{\partial x}(z_2 + t(z_1 - z_2))dt \\ C_{j,2}(z_1, z_2) &= \int_0^1 \frac{\partial P_j}{\partial y}(z_2 + t(z_1 - z_2))dt. \end{aligned}$$

It is easy to see that

$$C_{j,i}(z_1, z_2) = O(r^3), \tag{131}$$

where $r = \max_{i=1,2} |x_i|, |y_i|$.

Hence there exists constants $D_k > 0$, for $k = 1, 2, \ldots$, such that for $\|z_i\|_\infty \leq r$ holds

$$\begin{aligned} (f_1(z_1) - f_2(z_2))^2 - (x_1 - x_2)^2 \geq \\ (x_1 - x_2)^2 \left(\left(1 + \frac{r^2}{2} - D_1 r^3\right)^2 - 1 \right) - D_2 r^3 |x_1 - x_2| \cdot |y_1 - y_2| \geq \\ (x_1 - x_2)^2 D_3 r^2 - D_4 r^3 \left((x_1 - x_2)^2 + (y_1 - y_2)^2\right) \geq \\ (x_1 - x_2)^2 D_3 r^2 - D_5 r^5 \end{aligned}$$

Observe that $D_3 \approx 1/2$.

Analogously for the second coordinate of f we obtain, for some positive constants H_i and r sufficiently small

$$\begin{aligned} (y_1 - y_2)^2 - (f_2(z_1) - f_2(z_2))^2 \geq \\ (y_1 - y_2)^2 \left(1 - (1 - \frac{r^2}{2} + H_1 r^3)^2\right) - H_2 r^3 |x_1 - x_2| \cdot |y_1 - y_2| \geq \\ (y_1 - y_2)^2 \left(1 - (1 - H_3 r^2)^2\right) - H_2 r^3 |x_1 - x_2| \cdot |y_1 - y_2| \geq \\ (y_1 - y_2)^2 H_4 r^2 - H_2 r^3 |x_1 - x_2| \cdot |y_1 - y_2| \geq \\ (y_1 - y_2)^2 H_4 r^2 - H_5 r^5 \end{aligned}$$

Observe that $H_4 \approx 1/2$.

Now we are ready to verify the cone condition

$$\begin{aligned} Q_{\alpha,\beta}(f(z_1) - f(z_2)) - Q(z_1 - z_2) = \\ \alpha\left((f_1(z_1) - f_1(z_2))^2 - (x_1 - x_2)^2\right) + \beta\left((y_1 - y_2)^2 - (f_2(z_1) - f_2(z_2))^2\right) \geq \\ \alpha D_3 r^2 (x_1 - x_2)^2 - \alpha D_5 r^5 + \beta H_4 r^2 (y_1 - y_2)^2 - \beta H_5 r^5 \geq \\ \min(\alpha D_3, \beta H_4) r^4 - (\alpha D_5 + \beta H_5) r^5 > 0 \end{aligned}$$

for $r > 0$ sufficiently small. ∎

For $r > 0$ we define an h-set $N(r) \subset \mathbb{R}^2$ as follows: $u = s = 1$, $|N(r)| = [-r, r]^2$, $c_N(z) = \frac{z}{r}$.

Lemma 19 *For r sufficiently small $N(r) \overset{f}{\Longrightarrow} N(r)$.*

Proof: Since we have only one unstable direction, then from [GiZ, ktore tw] it follows that it is enough to prove that

$$f_1(r, y) > r, \qquad f_1(-r, y) < -r, \qquad \text{for } |y| \leq r \tag{132}$$

$$|f_2(x, y)| < r, \qquad \text{for } (x, y) \in N(r) \tag{133}$$

Let r be such that, the following inequalities hold for any $(x, y) \in N(r)$

$$|P_i(x, y)| < r^3, \quad i = 1, 2 \tag{134}$$

$$1 - 3y^2 + \frac{\partial P_2}{\partial y}(x, y) > 0. \tag{135}$$

It is easy to see that (134) implies (132).

To prove (133) observe that from (135) it follows that $|f_2(x, y)|$ achieves its maximum value on $N(r)$ at $(x_0, \pm r)$. Condition (133) now follows immediately from (134). ∎

Proof of Theorem 17 Let us choose $\alpha = \beta = 1$. From the above lemmas it follows that we can take $N = N(r)$ for r sufficiently small. The statements about the existence and conditions on $W^{u,s}(0, f)$ follow directly from Theorems 10 and 11.

The tangency of $W^{u,s}(0, f)$ to coordinate axes is obtained as in the proof of Theorem 12, because we have a freedom to choose any α and β (we may need to decrease further an r). ∎

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