COMPUTER ASSISTED PROOFS IN DYNAMICS

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The outline of the course

- Monday - overview, motivations;

- Rigorous numerics for maps and ODEs (Tue,Wed)
interval arithmetics, Lohner algorithm for ODEs, C^n - Lohner algorithm for ODs

- Topological tools for maps and ODEs (Thu,Fri)
local Brouwer degree, fixed point index, elements of the Conley index theory, topological horseshoes, symbolic dynamics topological entropy

Second Week:

- C^n-methods for maps and ODEs (Mon,Tue)
the interval Newton method, hyperbolicity, shadowing, homo- and heteroclinic solutions, some bifurcation proofs

- Infinite dimensions - dissipative PDEs (Wed-Fri) the Kuramoto-Sivashinski PDE on the line, the algorithm for rigorous integration of orbits, the existence of steady states, the bifurcation of steady states, the existence of periodic orbits

Main points of this course

• the field is wide open: needs both theory (theorems) and algorithms

• inequalities matter

• there is a need for new approaches to 'well understood' phenomena in dynamics, example: horseshoes and symbolic dynamics (will be discussed today)

• for ODEs(of higher dimension) and PDEs there are virtually no rigorous results about the dynamics.

Today

- some existing computer assisted proofs

- what can be proved by computer, interval arithmetics, the role of inequalities

- example PCR3BP, topological horseshoes

- example KS PDEs, topological tools in infinite dimension

Some computer assisted proofs in dynamics

- Langford - 1982, the proof of Feigenbaum universality conjectures

- Eckmann, Koch, Wittwer - 1984, universality for area-preserving maps

- Grebogi, Hammel, Yorke 1987 - rigorous numerical shadowing of trajectories

- Neumaier, Rage, Schlier - 1994, - chaos in the molecular Thiele-Wilson model

- Mischaikow and Mrozek - chaos in Lorenz equations, 1995

- Palmer, Coomes, Kocak, Stoffer, Kichgraber - 1996-2003 - chaos via shadowing for Henon map, PCR3BP

- W. Tucker - 2001 - geometric model for Lorenz attractor

CAPD - Krakow/GT(now Rutgers) group

• Mischaikow (Rutgers), Mrozek, Zgliczynski, Wilczak, Galias, Kapela, Pilarczyk, Arioli (Milan)

•proofs of chaos (semiconjugacy with Bernoulli shift) for Lorenz equations, Rössler equations, Hénon map, Chua circuit, PCR3BP

•homo- and heteroclinic orbits PCR3BP, Hénon map, Kuramoto-Sivashinsky ODE

•Kuramoto-Sivashinsky PDE: existence of multiple steady states and its bifurcations, periodic orbits

•N-body problem: the existence of simple choreographies

General scheme of CAP in dynamics

- a problem - $\mathcal{P}$, for example the question of existence of the horseshoe for Poincaré map for ODE

- abstract theorem, $\mathcal{T}$, implying a solution of problem $\mathcal{P}$, provided we can verify $\mathcal{Z}$ - the assumptions in $\mathcal{T}$

- the reduction $\mathcal{Z}$ to finite computations, $\mathcal{O}$

- finite rigorous computation of $\mathcal{O}$ checking $\mathcal{Z}$

- If $\mathcal{Z}$ is true, then theorem $\mathcal{T}$ gives positive answer to our problem $\mathcal{P}$

Some difficulties:

• computer is finite, the continuum can not be in rigorous way represented in computer (round-off errors)

• not every theorem can be verified in finite computations

• computer can be used to verification of theorems, whose assumptions can be reduced to a finite number of inequalities (strong), which can be verified in finite approximate (but rigorous) computation

Interval arithmetics
a cure for round-off errors

Arithmetics on closed intervals. For example:

- $[1, 3] \langle + \rangle [3, 17] = [4, 20]$

- $[-1, 1] \langle \cdot \rangle [3, 4] = [-4, 4]$

- $1 \langle / \rangle 3 = [0.33333, 0.33334]$

Rigorous interval arithmetics can be realized on the computer i.e. for each arithmetic operator $\diamondsuit \in \{+, -, \cdot, /\}$ the following is true

$$[a_-, a^+] \diamondsuit [b_-, b_+] \subset [a_-, a^+] \langle \diamondsuit \rangle [b_-, b_+]$$

Example: finding zero of an analytic function

$f(x) = \sum_{n=0}^{\infty} a_n x^n$

Problem: Prove that f has a zero in interval $(1,2)$

Numerical simulation: Apparently $f(x)$ is increasing on $[1,2]$ and $f(1) < 0$ i $f(2) > 0$. From the intermediate value thm. it follows that f has a zero in $(1,2)$

Reduction to finite computation:

- $f_M(x) = \sum_{n=0}^{M} a_n x^n$ - a function computable in finite number of steps

- analytical estimate: $|f_M(x) - f(x)| < \epsilon$ dla $x \in [1,2]$, this is done by a mathematican

- rigorous check on the computer that

$f_M(1) < -\epsilon$ i $f_M(2) > \epsilon$

Example: the existence of an attracting periodic orbit

$x' = f(x)$, $x \in R^3$

Two-dimensional Poincaré map, P, on section Θ.

Numerical fact: Apparently, all orbits starting in some open set U converge to periodic orbit γ.

Brouwer Theorem: If D is homeomorphic with the closed ball, $D \subset \Theta$
and $P(D) \subset \text{int}D$ (interior of D), then there exists $x \in D$ such that $P(x) = x$.
In particular, the trajectory of x is periodic.

Reduction to finite computations:

Condition: $P(D) \subset \text{int}D$ - represents a finite number of inequalities, if D - a parallepiped or ball

Phase space discretization: $D \subset \sum_{i=1}^{M} D_i$, D_i small enough, to compute $P(D_i)$ with a reasonable overestimation

$M \approx \frac{L^2 \cdot \text{Area}(D)}{4\epsilon^2}$, where
ϵ - an error margin
L - a Lipschitz constant (rigorous) for P

$$|P(x) - P(y)| < L|x - y|$$

L obtained in interval computations is usually much larger than L seen in nonrigorous simulations (the wrapping effect)

Total computation time: $= M\cdot$ computation time of $P(D_i)$

The sources of errors (overestimations) in rigorous computations of ODEs:

- round-off erros - *interval arithmetics*

- the numerical method error (the time discretization error) - *explict formulas for error terms*

- the space discretizaton error and the propagation error (- *SERIOUS PROBLEM*)

- the errors connected to the intersection with the section in the computation of Poincaré map

Wrapping effect

Harmonic oscillator

$$x' = -y, \qquad y' = x$$

Time shift by h, φ_h , rotation by h - IZOMETRY.

It turns out that when multiple iterations of φ_h realized in the ideal (no round-off error) interval arithmetics yield

$$\lim_{n->\infty} \langle \varphi_{\frac{2\pi}{n}} \rangle ([-\delta, \delta]^2) = e^{2\pi}[-\delta, \delta]^2$$

($e^{2\pi} \approx 536$) will we would expect that

$$\lim_{n->\infty} \langle \varphi_{\frac{2\pi}{n}} \rangle = \varphi_{2\pi} = \mathrm{Id}$$

DISASTER - SERIOUS OVERFLOW SOON

REASON:

- after each step the result is the following form $I_1 \times I_2$, where I_1, I_2 are intervals

These are not the reasons

- round-off errors

- the numerical method error

Hence increasing of the precision of the computations and improvement of numerical method via taking higher order and/or smaller time step does not guarantee any improvement.

Conclusion: Naive application of interval arithmetics to the integration of ODEs if very ineffective.