1 Local Brouwer degree

Let $D \subset \mathbb{R}^n$ be an open set and $f : S \to \mathbb{R}^n$ be continuous, $D \subset S$ and $c \in \mathbb{R}^n$. Suppose that

$$\text{the set} \quad f^{-1}(c) \cap D \quad \text{is compact.} \tag{1}$$

Then *the local Brouwer degree of f at c in the set D* is defined. We denote it by $\deg(f, D, c)$.

If $\overline{D} \subset \mathrm{dom}(f)$ and $\overline{D}$ is compact, then (1) follows from the condition

$$c \notin f(\partial D) \tag{2}$$

Properties:

Degree is an integer.

$$\deg(f, D, c) \in \mathbb{Z}. \tag{3}$$

Solution property.

$$\text{If} \quad \deg(f, D, c) \neq 0, \quad \text{then there exists } x \in D \text{ with } f(x) = c. \tag{4}$$

Homotopy property. Let $H : [0, 1] \times D \to \mathbb{R}^n$ be continuous. Suppose that

$$\bigcup_{\lambda \in [0,1]} H_\lambda^{-1}(c) \cap D \quad \text{is compact} \tag{5}$$

Then

$$\forall \lambda \in [0, 1] \quad \deg(H_\lambda, D, c) = \deg(H_0, D, c) \tag{6}$$

If $[0, 1] \times \overline{D} \subset \mathrm{dom}(H)$ and $\overline{D}$ is compact, then (5) follows from the following condition

$$c \notin H([0, 1], \partial D) \tag{7}$$

Local degree is a locally constant function. Assume D is bounded and open. If p and q belong to the same component of $\mathbb{R}^n \setminus f(\partial D)$, then

$$\deg(f, D, p) = \deg(f, D, q) \tag{8}$$

Excision property. Suppose that $E \subset D$, E is open and

$$f^{-1}(c) \cap D \subset E \tag{9}$$

Then

$$\deg(f, E, c) = \deg(f, D, c). \tag{10}$$

Local degree for affine maps. Suppose that $f(x) = A(x - x_0) + c$, where A is a linear map and $x_0 \in \mathbb{R}^n$. If the equation $A(x) = 0$ has no nontrivial solutions (i.e. if $Ax = 0$, then $x = 0$) and $x_0 \in D$, then

$$\deg(f, D, c) = \mathrm{sgn}(\det A), \tag{11}$$

Product property Let $U_i \subset \mathbb{R}^{n_i}$, $c_i \in \mathbb{R}^{n_i}$, $f_i : U_i \to \mathbb{R}^{n_i}$, for $i = 1, 2$. The map $(f_1, f_2) : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is given by $(f_1, f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2)$. We have

$$\deg((f_1, f_2), U_1 \times U_2, (c_1, c_2)) = \deg(f_1, U_1, c_1) \cdot \deg(f_2, U_2, c_2), \qquad (12)$$

whenever the right hand side is defined.

Multiplication property Let $D \subset \mathbb{R}^n$ be bounded and open. Let $f : \bar{D} \subset \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ are two continuous mappings and Δ_i the bounded components of $\mathbb{R}^n \setminus f(\partial D)$. Then

$$\deg(g \circ f, D, p) = \sum_{\Delta_i} \deg(g, \Delta_i, p) \deg(f, D, \Delta_i), \qquad (13)$$

where $\deg(f, D, \Delta_i) = \deg(f, D, q_i)$ for some $q_i \in \Delta_i$. From equation (8) it follows that this definition of $\deg(f, D, \Delta_i)$ does not depend on the choice of q_i.

Addition property. If $D = \bigcup_{i \in I} D_i$, where each D_i is open, the family $\{D_i\}$ is disjoint and $\partial D_i \subset \partial D$, then for every $c \notin f(\partial D)$:

$$\deg(f, D, c) = \sum_{i \in I} \deg(f, D_i, c). \qquad (14)$$

∎

From Multiplication property and formula (11) we obtain immediately

Collorary 1 *Let $D \subset \mathbb{R}^n$ be open and bounded. Let $A : D \to \mathbb{R}^n$, be continuous and $0 \notin A(\partial D)$,*

$$\deg(-A, U, 0) = (-1)^n \deg(A, U, 0), \qquad (15)$$

As the consequence of Addition and Excision property we obtain the following

Collorary 2 *Suppose that D is a finite union of open sets $D = \bigcup_{i=1}^n D_i$ such that the sets $f_{|D_i}^{-1}(c)$ are mutually disjoint and $c \notin f(\partial D_i)$. Then*

$$\deg(f, D, c) = \sum_{i=1}^{n} \deg(f_{|D_i}, D_i, c) \qquad (16)$$

Here is another important consequence of above properties

Collorary 3 *Assume $V \subset \mathbb{R}^n$ is bounded and open. Let $f : \overline{V} \to \mathbb{R}^n$ be a C^1-map. Assume that $c \in \mathbb{R}^n \setminus f(\partial V)$ is a regular value for f, i.e. for each $x \in f^{-1}(c)$ the Jacobian matrix of f at x denoted by $Df(x)$ is nonsingular, then*

$$\deg(f, V, c) = \sum_{x \in f^{-1}(c)} \operatorname{sgn}\left(\det Df(x)\right)$$

2 Fixed point index

$U \subset \mathbb{R}^n$ open.

$f : U \to \mathbb{R}^n$ continuous map is *admissible* if

$$\mathrm{Fix}(f) = \{x \in U : \quad f(x) = x\} \tag{17}$$

is compact.

If $\overline{U} \subset \mathrm{dom}(f)$ and $\overline{U}$ is compact, then (17) follows from the condition

$$f(x) \neq x, \quad \text{for } x \in \partial D \tag{18}$$

Assume that $f : U \to X$ is admissible, compact map. Then we define the fixed point index of f in U by

$$I(f, U) = deg(I - f, U, 0) \in \mathbb{Z},$$

Properties:

Excision If $U^{'} \subset U$ and $\mathrm{Fix}(f) \subset U^{'}$, then

$$\mathrm{ind}(f, U) = \mathrm{ind}(f, U^{'}). \tag{19}$$

Fixed points If $\mathrm{ind}(f, U) \neq 0$, then $\mathrm{Fix}(f) \neq \emptyset$.

Homotopy property Continuous homotopy $h_t : U \to X$ is *admissible* when $\bigcup_{t=0}^{1} \mathrm{Fix}(h_t)$ is compact.

If $h_t : U \to X$ is an admissible homotopy, then $\mathrm{ind}(h_0, U) = \mathrm{ind}(h_1, U)$.

Fixed point index for affine maps. Suppose that $f(x) = A(x - x_0) + x_0$, where A is a linear map and $x_0 \in \mathbb{R}^n$. If the equation $(Id - A)(x) = 0$ has no nontrivial solutions (i.e. if $Ax = x$, then $x = 0$) and $x_0 \in D$, then

$$\mathrm{(ind)(f, D) = sgn(det(Id - A))}, \tag{20}$$

Multiplicativity If $f_1 : U_1 \to X_1$ and $f_2 : U_2 \to X_2$ are admissible, then the product $f_1 \times f_2 : U_1 \times U_2 \to X_1 \times X_2$ is admissible and

$$\mathrm{ind}(f_1 \times f_2, U_1 \times U_2) = \mathrm{ind}(f_1, U_1)\,\mathrm{ind}(f_2, U_2). \tag{21}$$

Commutativity Let $U \subset X$ and $U' \subset X'$ be open and assume that $f : U \to X'$, $g : U' \to X$ are continuous. If one of the composites

$$gf : V = f^{-1}(U') \to X,\ fg : V' = g^{-1}(U) \to X',$$

is admissible, then so is the other and, in that case,

$$\mathrm{ind}(gf, V) = \mathrm{ind}(fg, V'). \tag{22}$$

2.1 The degree of maps $S^n \to S^n$

In this section we recall some relevant facts on the degree of maps $S^n \to S^n$ see for example [**?**, Ch. 7.5].

Definition 1 *Let $n \geq 1$. The degree of a map continuous $f : S^n \to S^n$ is a unique integer $d(f)$ such that $f_*(u) = d(f) \cdot u$, for any generator $u \in H_n(S^n)$, where $H_n(S^n)$ is n-th homology group of S^n and $f_* : H_n(S^n) \to H_n(S^n)$ is the induced homomorphism.*

For $n = 0$ we define the degree, $d(f)$, as follows, $S^0 = \{-1, 1\}$. We set

$$d(f) = \begin{cases} 1, & \text{if } f(1) = 1 \text{ and } f(-1) = -1, \\ -1, & \text{if } f(1) = -1 \text{ and } f(-1) = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

Theorem 4 (H. Hopf) *Let $n \geq 1$. Then $f, g : S^n \to S^n$ are homotopic if and only if $d(f) = d(g)$.*

Lemma 5 *Let $u > 0$, Assume that $A : \overline{B_u}(0,1) \to \mathbb{R}^u$ is a continuous map, such that*

$$0 \notin A(\partial B(0,1))$$

Let the map $s_A : S^{u-1} \to S^{u-1}$ be given by

$$s_A(x) = \frac{A(x)}{\|A(x)\|}.$$

Then

$$\deg(A, \overline{B_u}(0,1), 0) = d(s_A) \tag{24}$$

3 Covering relations, the simple case

Notation: For a given norm in $\mathbb{R}^n$, by $B_n(c,r)$ we denote the open ball of radius r centered at $c \in \mathbb{R}^n$. When the dimension n is obvious from the context, we will drop the subscript n. Let $S^n(c,r) = \partial B_{n+1}(c,r)$, by the symbol S^n we will denote $S^n(0,1)$. We set $\mathbb{R}^0 = \{0\}$, $B_0(0,r) = \{0\}$, $\partial B_0(0,r) = \emptyset$.

For a given set Z, by $\operatorname{int} Z$, $\overline{Z}$, ∂Z we denote the interior, the closure and the boundary of Z, respectively. For a map $h : [0,1] \times Z \to \mathbb{R}^n$, we set $h_t = h(t, \cdot)$. By Id we denote the identity map. For a map f, by $\operatorname{dom}(f)$ we will denote the domain of f. If $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map, we say that $X \subset \operatorname{dom}(f^{-1})$ iff the map $f^{-1} : X \to \mathbb{R}^n$ is well defined and continuous.

Definition 2 *An h-set is a quadruple consisting of*

- *a compact subset N of $\mathbb{R}^n$,*
- *a pair of numbers $u(N), s(N) \in \{0, 1, 2, \dots\}$, with $u(N) + s(N) = n$,*

- *a homeomorphism* $c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$*, such that*

$$c_N(N) = \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1).$$

With an abuse of notation, we will denote such a quadruple by N. We denote

$$\begin{aligned} N_c &= \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1), \\ N_c^- &= \partial\overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1), \\ N_c^+ &= \overline{B_{u(N)}}(0,1) \times \partial\overline{B_{s(N)}}(0,1), \\ N^- &= c_N^{-1}(N_c^-), \quad N^+ = c_N^{-1}(N_c^+). \end{aligned}$$

Hence an h-set N is a product of two closed balls with respect to some coordinate system. The numbers $u(N)$ and $s(N)$ stand for the dimensions of nominally unstable and stable directions, respectively. The subscript c refers to the new coordinates given by homeomorphism c_N. Notice that if $u(N) = 0$, then $N^- = \emptyset$ and if $s(N) = 0$, then $N^+ = \emptyset$.

Definition 3 *Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f : N \to \mathbb{R}^n$ be a continuous map. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s$. We say that*

$$N \overset{f}{\Longrightarrow} M$$

(N f-covers M) iff the following conditions are satisfied

1. *There exists a continuous homotopy $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$, such that the following conditions hold true*

$$\begin{aligned} h_0 &= f_c, \\ h([0,1], N_c^-) \cap M_c &= \emptyset, \\ h([0,1], N_c) \cap M_c^+ &= \emptyset. \end{aligned}$$

2. *There exists a linear map $A : \mathbb{R}^u \to \mathbb{R}^u$, such that*

$$h_1(p,q) = (Ap, 0), \quad \text{for } p \in \overline{B_u}(0,1) \text{ and } q \in \overline{B_s}(0,1), \tag{25}$$

$$A(\partial B_u(0,1)) \subset \mathbb{R}^u \setminus \overline{B_u}(0,1). \tag{26}$$

In the context of the above definition we will call the map h_1 *a model map* for the relation $N \overset{f}{\Longrightarrow} M$.

Remark 6 *When $u > 0$, then condition (26) is equivalent to each of the following conditions*

$$\begin{aligned} \overline{B_u}(0,1) &\subset A(B_u(0,1)), \\ \|Ap\| &> 1, \quad \text{for } p \in \partial B_u(0,1), \\ \|Ap\| &> \|p\|, \quad \text{for } p \neq 0. \end{aligned}$$

Remark 7 *When $u = 0$, then $\mathbb{R}^u = \{0\}$ and so $A : \mathbb{R}^u \to \mathbb{R}^u$ is given by $A(0) = 0$. Taking into account that $\partial B_u(0,1) = \emptyset$, we see that the second part of (26) is vacuously satisfied, and so condition (26) is equivalent to $h_1(x) = 0$ for all x. It is easy to see that, in this case, $N \overset{f}{\Longrightarrow} M$ iff $f(N) \subset \operatorname{int} M$.*

Definition 4 *Let N be an h-set. We define the h-set N^T as follows*

- *The compact subset of the quadruple N^T is the compact subset of the quadruple N, also denoted by N,*
- *$u(N^T) = s(N)$, $s(N^T) = u(N)$*
- *The homeomorphism $c_{N^T} : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N^T)} \times \mathbb{R}^{s(N^T)}$ is defined by*

$$c_{N^T}(x) = j(c_N(x)),$$

  where $j : \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)} \to \mathbb{R}^{s(N)} \times \mathbb{R}^{u(N)}$ is given by $j(p,q) = (q,p)$.

Notice that $N^{T,+} = N^-$ and $N^{T,-} = N^+$. This operation is useful in the context of inverse maps, as it was first pointed out in [**?**].

Definition 5 *Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. Assume that $g^{-1} : M \to \mathbb{R}^n$ is well defined and continuous. We say that $N \overset{g}{\Longleftarrow} M$ (N g-backcovers M) iff $M^T \overset{g^{-1}}{\Longrightarrow} N^T$.*

Following [**?**], let us point out that, although covering and backcovering occur often simultaneously, they are not equivalent, for example it can happen that the map f is not defined on N.

Theorem 8 *Let N_i, $i = 0, \ldots, k$ be h-sets and $N_k = N_0$. Assume that for each $i = 1, \ldots, k$ we have either*

$$N_{i-1} \overset{f_i}{\Longrightarrow} N_i, \tag{27}$$

or

$$N_i \subset dom\,(f_i^{-1}) \quad and \quad N_{i-1} \overset{f_i}{\Longleftarrow} N_i. \tag{28}$$

Then there exists a point $x \in \operatorname{int} N_0$, such that

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \in \operatorname{int} N_i, \quad i = 1, \ldots, k, \tag{29}$$

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x. \tag{30}$$

4 Multiple wrapped covering relations

The goal of this section is to generalize the notion of covering relations introduced in Section 3. We will change condition **2** in the definition of covering relations in order to allow for more general maps at the end of homotopy h (this means that we allow for different *model maps*).

Definition 6 *Assume that N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f : N \to \mathbb{R}^n$ be a continuous map. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s$. Let w be a nonzero integer. We say that*

$$N \overset{f,w}{\Longrightarrow} M$$

(N f-covers M with degree w) iff the following conditions are satisfied

1. *there exists a continuous homotopy $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$, such that the following conditions hold true*

$$h_0 = f_c, \tag{31}$$
$$h([0,1], N_c^-) \cap M_c = \emptyset, \tag{32}$$
$$h([0,1], N_c) \cap M_c^+ = \emptyset. \tag{33}$$

2. *There exists a map $A : \mathbb{R}^u \to \mathbb{R}^u$, such that*

$$h_1(p,q) = (A(p), 0), \text{ for } p \in \overline{B_u}(0,1) \text{ and } q \in \overline{B_s}(0,1), \tag{34}$$
$$A(\partial B_u(0,1)) \subset \mathbb{R}^u \setminus \overline{B_u(0,1)}. \tag{35}$$

Moreover, we require that

$$\deg(A, \overline{B_u}(0,1), 0) = w,$$

Note that in the case $u = 0$, an h-set N can cover an h-set M only with degree $w = 1$.

The previous definition of covering relation (Definition 2) is a particular case of the present one, with the degree w equal to $\operatorname{sgn}(\det(A))$ (for $u > 0$). See Figure **??** for an example of a multiple wrapped covering relation. As in Section 3, we will call the map h_1 *a model map* for the relation $N \overset{f,w}{\Longrightarrow} M$.

Remark 9 *In applications, we would like to decide whether two h-sets are correctly aligned based essentially on the information on their boundaries. Condition 1 from the above definition is stated in this spirit. In condition 2, we can express the local Brouwer degree of A as the winding number of $A(\partial B_u(0,1))$ about the origin. More precisely, in the case $u > 0$, since the map $A : \overline{B_n}(0,1) \to \mathbb{R}^n$ satisfies*

$$0 \notin A(\partial B_u(0,1)), \tag{36}$$

we can define a map $s_A : S^{u-1} \to S^{u-1}$ by

$$s_A(x) = \frac{A(x)}{\|A(x)\|}. \tag{37}$$

The degree $d(s_A)$ of a mapping of a sphere is defined in Appendix 2.1. By Lemma 5, we obtain $\deg(A, \overline{B_u}(0,1), 0) = d(s_A)$. *Thus, the degree of a covering $N \overset{f,w}{\Longrightarrow} M$ can be computed as $w = d(s_A)$.*

We define the corresponding notion of backcovering for this new type of covering relation.

Definition 7 *Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. Assume that $g^{-1} : |M| \to \mathbb{R}^n$ is a well defined, continuous map. We say that $N \overset{g,w}{\Longleftarrow} M$ (N g-backcovers M with degree w) iff $M^T \overset{g^{-1},w}{\Longrightarrow} N^T$.*

Theorem 10 *Let N_i, $i = 0, \dots, k$ be h-sets and $N_k = N_0$. Assume that for each $i = 1, \dots, k$ we have either*

$$N_{i-1} \overset{f_i,w_i}{\Longrightarrow} N_i, \tag{38}$$

or

$$N_i \subset dom\,(f_i^{-1}) \quad and \quad N_{i-1} \overset{f_i,w_i}{\Longleftarrow} N_i. \tag{39}$$

Then there exists a point $x \in \operatorname{int} N_0$, such that

$$\begin{aligned} f_i \circ f_{i-1} \circ \cdots \circ f_1(x) &\in \operatorname{int} N_i, \quad i = 1, \dots, k, & (40)\\ f_k \circ f_{k-1} \circ \cdots \circ f_1(x) &= x. & (41) \end{aligned}$$

5 The topological transversality theorem

The goal of this section is to state and prove the main topological transversality theorem for a chain of covering relations. For this end we need first to define the notions of vertical and horizontal disks in an h-set.

Definition 8 *Let N be an h-set. Let $b : \overline{B_{u(N)}}(0,1) \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is* a horizontal disk in N *if there exists a continuous homotopy $h : [0,1] \times \overline{B_{u(N)}}(0,1) \to N_c$, such that*

$$\begin{aligned} h_0 &= b_c & (42)\\ h_1(x) &= (x, 0), \qquad \textit{for all } x \in \overline{B_{u(N)}}(0,1) & (43)\\ h(t,x) &\in N_c^-, \qquad \textit{for all } t \in [0,1] \textit{ and } x \in \partial\overline{B_{u(N)}}(0,1) & (44) \end{aligned}$$

Definition 9 *Let N be an h-set. Let $b : \overline{B_{s(N)}}(0,1) \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is* a vertical disk in N *if there exists a continuous homotopy $h : [0,1] \times \overline{B_{s(N)}}(0,1) \to N_c$, such that*

$$\begin{aligned} h_0 &= b_c\\ h_1(x) &= (0, x), \qquad \textit{for all } x \in \overline{B_{s(N)}}(0,1)\\ h(t,x) &\in N_c^+, \qquad \textit{for all } t \in [0,1] \textit{ and } x \in \partial\overline{B_{s(N)}}(0,1). \end{aligned}$$

It is easy to see that b is a horizontal disk in N iff b is a vertical disk in N^T.

We would like to remark here that a horizontal disk in N can be at the same time also vertical in N. An example of such disk is shown on Fig. 1. In case homotopies used in the definitions of horizontal and vertical disks are different. The existence of such disks, which are both vertical and horizonal will play very important role in our method for detection of an infinite number symmetric periodic orbits for maps with reversal symmetry.

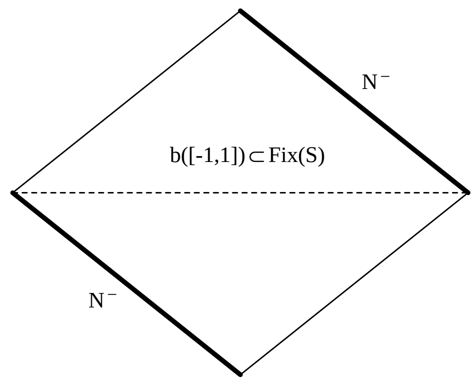

Figure 1: The curve b is both horizontal and vertical disk in N. In this example $u(N) = s(N) = 1$.

Now we are ready to state and prove the main topological transversality theorem. A simplified version of this theorem was given in [?] for the case of one unstable direction and covering relations chain without backcoverings. The argument in [?], which was quite simple and was based on the connectivity only, cannot be carried over to a larger number of unstable directions or to the situation when both covering and backcovering relations are present.

Theorem 11 *Let $k \geq 1$. Assume N_i, $i = 0, \dots, k$, are h-sets and for each $i = 1, \dots, k$ we have either*

$$N_{i-1} \stackrel{f_i, w_i}{\Longrightarrow} N_i \tag{45}$$

or $N_i \subset dom(f_i^{-1})$ and

$$N_{i-1} \stackrel{f_j, w_i}{\Longleftarrow} N_i. \tag{46}$$

Assume that b_0 is a horizontal disk in N_0 and b_e is a vertical disk in N_k.

Then there exists a point $x \in \operatorname{int} N_0$, such that

$$\begin{aligned} x &= b_0(t), \quad \textit{for some } t \in B_{u(N_0)}(0,1) && (47) \\ f_i \circ f_{i-1} \circ \cdots \circ f_1(x) &\in \operatorname{int} N_i, \quad i = 1, \dots, k && (48) \\ f_k \circ f_{k-1} \circ \cdots \circ f_1(x) &= b_e(z), \quad \textit{for some } z \in B_{s(N_k)}(0,1) && (49) \end{aligned}$$

6 How to find a homotopy for the covering relations

The goal of this section is to present sufficient conditions which ensure that $N \overset{f}{\Longrightarrow} M$, solely based on the knowledge of $f(N)$ and M.

Definition 10 *Let N be a h-set. We set*

$$S(N)_c^- = \{(p,q) \in \mathbb{R}^u \times \mathbb{R}^s \mid \|p\| > 1\}. \tag{50}$$

We define $S(N)^- = c_N^{-1}(S(N)_c^-)$.

The above theorem allows to take its assumptions as a definition of covering relation as it was done before by Zgliczyński in [**?**, **?**] for maps with one topologically expanding direction ($u = 1$), and in [**?**] for maps which are close to products of one dimensional maps. Below we will discuss the case of $u = 1$.

6.1 Case of one nominally expanding direction, $u = 1$

In this section we discuss the case of $u = 1$, hence we have only one nominally expanding direction. The basic idea here is that each of the sets N^-, $S(N)^-$ consists of two disjoint components, allowing us to simplify the assumptions of Theorem **??**.

Definition 11 *Let N be an h-set, such that $u(N) = 1$. We set*

$$\begin{aligned} N_c^{le} &= \{-1\} \times \overline{B}_s(0,1), \\ N_c^{re} &= \{1\} \times \overline{B}_s(0,1), \\ S(N)_c^l &= (-\infty, -1) \times \mathbb{R}^s, \\ S(N)_c^r &= (1, \infty) \times \mathbb{R}^s. \end{aligned}$$

We define

$$N^{le} = c_N^{-1}(N_c^{le}), \quad N^{re} = c_N^{-1}(N_c^{re}),$$
$$S(N)^l = c_N^{-1}(S(N)^l), \quad S(N)^r = c_N^{-1}(S(N)^r).$$

We will call N^{le}, N^{re}, $S(N)^l$ and $S(N)^r$ the left edge, the right edge, the left side and right side of N, respectively.

It is easy to see that $N^- = N^{le} \cup N^{re}$ and $S(N)^- = S(N)^l \cup S(N)^r$.

The triple $(N, \overline{S(N)^l}, \overline{S(N)^r})$ represents a t-set, as it was defined in [**?**].

Theorem 12 *Let N, M be two h-sets in $\mathbb{R}^n$, such that $u(N) = u(M) = 1$ and $s(N) = s(M) = s = n - 1$. Let $f : N \to \mathbb{R}^n$ be continuous.*

Assume that there exists $q_0 \in \overline{B}_s(0,1)$, such that the following conditions are satisfied

$$\begin{aligned} f(c_N(\overline{B}_u(0,1) \times \{q_0\})) &\subset \operatorname{int}(S(M)^l \cup M \cup S(M)^r), & (51) \\ f(N) \cap M^+ &= \emptyset, & (52) \end{aligned}$$

and one of the following two conditions holds true

$$f(N^{le}) \subset S(M)^l \quad \textit{and} \quad f(N^{re}) \subset S(M)^r, \tag{53}$$

$$f(N^{le}) \subset S(M)^r \quad \textit{and} \quad f(N^{re}) \subset S(M)^l. \tag{54}$$

Then there exists $w = \pm 1$*, such that*

$$N \overset{f,w}{\Longrightarrow} M.$$