Hyperbolicity and Averaging for the Srzednicki-Wójcik equation

Piotr Zgliczyński[1]

Institute of Computer Science, Faculty of Mathematics and Computer Science,
Jagiellonian University
ul. S. Łojasiewicza 6, 30-348 Kraków, Poland

umzglicz@cyf-kr.edu.pl, piotr.zgliczynski@ii.uj.edu.pl

July 9, 2016

Abstract

For the Srzednicki-Wójcik equation, the planar nonautonomous ODE parameterized by $\kappa \in \mathbb{R}$,

$$z' = \overline{z}(1 + |z|^2 \exp(i\kappa t)), \qquad z(t) \in \mathbb{C}$$

using averaging we show how the region of hyperbolicty grows with $|\kappa|$. Based on this we give bounds on the sizes of bounded orbits.

Keywords: rapid oscillations; averaging; cone conditions

AMS classification: 34C29, 37D05

1 Introduction

In this paper we investigate some aspect of dynamics of Srzednicki-Wøjcik equation [SW]

$$z' = \overline{z}(1 + |z|^2 \exp(i\kappa t)), \qquad z(t) \in \mathbb{C}. \tag{1}$$

This equation has been studied by topological methods [SW, WZ, OW, MS], with the goal of establishing the existence of symbolic dynamics. Main tool was the method of isolating segments introduced in [S1, SW] with papers [WZ, OW] improving estimates so finally the existence of symbolic dynamics was established for $0 < \kappa < 0.5044$. In [MS] using related ideas a computer assisted proof was obtained for $0 < \kappa \leq 1$.

For equation (1) we establish the following results

1. all solutions of (1) with sufficiently large $\|z(0)\|(\geq O(|\kappa|^{1/2}))$ go to infinity in finite time either forward or backward in time. This is the content of Theorem 3.4 in Section 3.

[1]Research has been supported by Polish National Science Centre grant 2015/19/B/ST1/01454

2. in the neighborhood of 0 there is a hyperbolic region for the dynamics induced by (1), which contains a ball of size $O(|\kappa|^{1/4})$ i.e. it grows with $|\kappa|$. This is a consequence of Theorem 3.14 in Section 3.

3. we give a lower bound $O(|\kappa|^{1/4})$ and an upper bound $O(|\kappa|^{1/2})$ of the norm for any bounded nonzero orbits. This is discussed in Section 3.6.

Items 2 and 3 depend on developing explicit estimates for the difference between solutions of

$$z' = \overline{z} \tag{2}$$

and (1), and also for the solutions of corresponding variational equations on large domains.

We treat (1) as the perturbation of problem (2). Since (1) is not a small perturbation of (2) we cannot apply to the vector fields (1) or (2) topological tools such as the isolating blocks or isolating segments originating from the Conley index theory [MM, SW], or the continuation based on the implicit function theorem. Instead we use the averaging method to obtain bounds on the difference of solutions of these systems.

Our approach to averaging differs from the dominant one [AKN97, Nei84, GH, Hale, SVM, JS, L], which following the pioneering work of ([BM, BZ]) used suitable coordinates changes to control the influence of the rapidly oscillating perturbation.

In this work we directly estimate the influence of the rapidly oscillating perturbation on the solution, as it was proposed in the PDE context by Henry [He] (who attribute this approach in the ODE context to Gikhman). We exploit the phenomenon known in the numerical analysis community [I], that fast oscillations are your friend, if one integrates them first. In this work following this advice we consider shifts along the trajectory by a small time step. It turns out that the time shift map for (1) is a small perturbation of the time shift of (2) if $|\kappa| \to \infty$. In literature on averaging for ODEs somewhat analogous methods can be found in [BL, Wa]. In [Wa] an idea is developed that in a suitable weak topology the averaged equation is the limit of the problem with fast oscillation with the oscillation frequency going to infinity. In a sense this observation underlines our approach.

In contrast to other works on averaging we insist on explicit estimates. In the context of (1) this requires obtaining a priori bounds on solutions, to obtain the sets on which we can compute difference between solutions of (1) and (2) based on the averaging.

The approach has been originally developed with the aim to be applicable to dissipative PDEs. We applied it to viscous Burgers equation and Navier-Stokes equations with periodic boundary conditions in paper [CyZ] and in [CMTZ] to Navier-Stokes equations and the damped-Euler equation.

1.1 Notation

Consider nonautonomous ODE

$$x'(t) = f(t, x(t)), \tag{3}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is regular enough to guarantee the uniqueness of the initial value problem $x(t_0) = x_0$. We set $\varphi(t_0, t, x_0) = x(t_0 + t)$, where $x(t)$ is a solution of (3) with initial condition $x(t_0) = x_0$. Obviously in each context it will be clearly stated what is the ordinary differential equation generating φ. We will sometimes refer to φ as to the local process generated by (3).

For matrix U by U^t we will denote its transpose. For a square matrix U we will denote its spectrum by $\mathrm{Sp}(U) = \{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of } U\}$. If $A \in \mathbb{R}^{n\times n}$ by $\mu(A)$ we will denote its logarithmic norm, which is defined by

$$\mu(A) = lim_{h\to 0^+} \frac{\|I + hA\| - 1}{h}. \tag{4}$$

For the properties of logarithmic norm and its relation with the Lipschitz constant for the flow induced by ODEs see [HNW, KZ] and literature cited there. The logarithmic norm depends on the norm used. We will always assume that we are using the euclidean norm.

For a function of several variables we will often use $D_t f$ and $D_z f$ to denote the partial derivatives. For example, $D_z f(t, z) = \frac{\partial f}{\partial z}(t, z)$.

2 Averaging - basic estimates

The results from Sections 2.1 and 2.2 are also contained in [CyZ]. We include them here for the sake of making the paper reasonably self-contained.

2.1 Linear nonautonomous equations

Assume that $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is continuous and for $k = 1, \ldots, m$, $v_k : \mathbb{R} \to \mathbb{R}$ are C^1, $g_k : \mathbb{R} \to \mathbb{R}$ are continuous.

Let us consider the following non-autonomous non-homogenous linear ODE

$$x'(t) = A(t)x(t) + \sum_{k\in J_V} g_k(\omega_k t)v_k(t), \quad x \in \mathbb{R}^n. \tag{5}$$

The set J_V in the sum in (5) might be finite or infinite, or the sum might be an integral over some measure on J_V.

For each $k \in J_V$ let $G_k(t)$ be a primitive of g_k, so

$$G'_k(t) = g_k(t). \tag{6}$$

We will assume later that $|G_k(t)|$'s are bounded. This is the reflection of the oscillating nature of g_k.

Let $M(t,t_0)$ be a fundamental matrix of solutions of the homogenous version of (5)

$$x'(t) = A(t)x(t). \tag{7}$$

This means that for any $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ the function $x(t) = M(t,t_0)x_0$ solves (7) with the initial condition $x(t_0) = x_0$.

It is well known that M has the following properties

$$M(t_0,t_0) = I, \tag{8}$$
$$M(t,t_0)^{-1} = M(t_0,t), \tag{9}$$
$$\frac{\partial}{\partial t}M(t,t_0) = A(t)M(t,t_0), \tag{10}$$
$$\frac{\partial}{\partial t_0}M(t,t_0) = -M(t,t_0)A(t_0). \tag{11}$$

The general solution of (5) is given by

$$\begin{aligned}\varphi(t_0,t,x_0) &= M(t_0+t,t_0)x_0 + \\ &\int_0^t M(t_0+t,t_0+s)\sum_{k\in J_V} g_k(\omega_k(t_0+s))v_k(t_0+s)ds.\end{aligned} \tag{12}$$

We compute the integral in the above formula as follows. Using the integration by parts and (11) we obtain for $k \in J_V$

$$\begin{aligned}I_k(t+t_0) &:= \int_0^t g_k(\omega_k(t_0+s))M(t_0+t,t_0+s)v_k(t_0+s)ds = \\ &\frac{G(\omega_k(t_0+s))}{\omega_k}M(t_0+t,t_0+s)v_k(t_0+s)\Big|_{s=0}^{s=t} + \\ &-\frac{1}{\omega_k}\int_0^t G_k(\omega_k(t_0+s))\frac{\partial}{\partial s}\left(M(t_0+t,t_0+s)v_k(t_0+s)\right)ds = \\ &\frac{1}{\omega_k}\left(G_k(\omega_k(t_0+t))v_k(t_0+t) - G_k(\omega_k t_0)M(t_0+t,t_0)v_k(t_0)\right) + \\ &\frac{1}{\omega_k}\int_0^t G_k(\omega_k(t_0+s))M(t_0+t,t_0+s)A(t_0+s)v_k(t_0+s)ds + \\ &-\frac{1}{\omega_k}\int_0^t G_k(\omega_k(t_0+s))M(t_0+t,t_0+s)v_k'(t_0+s)ds\end{aligned} \tag{13}$$

For Galerkin projections of dissipative PDEs, while $\|A\|$ will not have any uniform bound independent of the projection dimension, we expect $\|Av_k\|$ to be uniformly bounded.

Therefore, we have proved that for the process generated by (5) it holds that

$$\varphi(t_0,t,x_0) = M(t_0+t,t_0)x_0 + \sum_{k\in J_V} I_k(t+t_0). \tag{14}$$

2.2 Estimates for nonlinear problem

Assume that $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1 function and for $k = 1, \dots, m$, $v_k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are C^1, $g_k : \mathbb{R} \to \mathbb{R}$ are continuous.

Consider problem

$$x' = \tilde{F}(t, x) := F(t, x) + \sum_{k \in J_V} g_k(\omega_k t) v_k(t, x) \tag{15}$$

and its oscillation-free version

$$y' = F(t, y). \tag{16}$$

Lemma 2.1. *Let $x : [t_0, t_0 + h] \to \mathbb{R}^n$ and $y : [t_0, t_0 + h] \to \mathbb{R}^n$ be solutions to (15) and (16), respectively, such that $x(t_0) = y(t_0)$.*

Let W be a compact set, such that for any $t \in [0, h]$, the segment joining $x(t_0 + t)$ and $y(t_0 + t)$ is contained in W.

Assume that for $k \in J_V$ $G_k'(t) = g_k(t)$.

Assume that there exist constants l, $C(\dots)$ such that for all $k \in J_V$ it holds that

$$\sup_{t \in \mathbb{R}} \|G_k(t)\| = C(G_k), \tag{17}$$

$$\sup_{t \in \mathbb{R}} |g_k(t)| = C(g_k), \tag{18}$$

$$\sup_{z \in W, s \in [t_0, t_0+h]} \mu(D_z F(s, z)) = l \tag{19}$$

$$\sup_{z \in W, s \in [t_0, t_0+h]} \|v_k(s, z)\| = C(v_k) \tag{20}$$

$$\sup_{z, z_1 \in W, s \in [t_0, t_0+h]} \|(D_z F(s, z)) v_k(s, z_1)\| = C\left(D_z F v_k\right) \tag{21}$$

$$\sup_{z \in W, s \in [t_0, t_0+h]} \left\| \frac{\partial v_k}{\partial t}(s, z) \right\| = C\left(\frac{\partial v_k}{\partial t}\right) \tag{22}$$

$$\sup_{z \in W, s \in [t_0, t_0+h]} \left\| (D_z v_k(s, z)) \tilde{F}(s, z) \right\| = C(D_z v_k \tilde{F}) \tag{23}$$

Assume that

$$\sum_{k \in J_V} C(G_k) C(v_k) < \infty \tag{24}$$

$$\sum_{k \in J_V} C(G_k) C\left(D_z F v_k\right) < \infty \tag{25}$$

$$\sum_{k \in J_V} C(G_k) C\left(\frac{\partial v_k}{\partial t}\right) < \infty \tag{26}$$

$$\sum_{k \in J_V} C(G_k) C\left(D_z v_k \tilde{F}\right) < \infty. \tag{27}$$

Then for $t \in [0,h]$ it holds that

$$\|x(t_0+t) - y(t_0+t)\| \leq \sum_{k \in J_V} \frac{1}{|\omega_k|} b_k(t) \tag{28}$$

where continuous functions $b_k : [0,h] \to \mathbb{R}_+$ depend on constants l, $C(g_i)$, $C(G_i)$, $C(v_i)$, $C(D_z F v_i)$, $C\left(\frac{\partial v_i}{\partial t}\right)$ and $C\left(D_z v_i \tilde{F}\right)$ as follows

$$\begin{aligned} b_k(t) &= C(v_k)C(G_k)(1+e^{lt}) + C(D_z F v_k)\, C(G_k)(e^{lt}-1)/l + \\ &\quad C(G_k)\left(C\left(\frac{\partial v_k}{\partial t}\right) + C\left(D_z v_k \tilde{F}\right)\right)(e^{lt}-1)/l \end{aligned} \tag{29}$$

If $\inf_{k \in J_V} |\omega_k| > 0$, the sum in (28) is convergent.

Proof: Let $z(t) = x(t) - y(t)$. We have

$$z'(t) = F(x(t)) - F(y(t)) + \sum_{k \in J_V} g_k(\omega_k t) v_k(t,x) =$$

$$\left(\int_0^1 D_x F(t, s(x(t)-y(t)) + y(t)) ds\right) \cdot z(t) + \sum_{k \in J_V} g_k(\omega_k t) v_k(t, x(t)).$$

Therefore

$$z'(t) = A(t)z(t) + \sum_{k \in J_V} g_k(\omega_k t) v_k(t, x(t)), \tag{30}$$

where

$$A(t) = \left(\int_0^1 D_x F(t, s(x(t)-y(t)) + y(t)) ds\right). \tag{31}$$

Let $M(t_1, t_0)$ is the fundamental matrix of solutions for $x' = A(t)x$.

Since $z(t_0) = 0$, then from (12) and (13) it follows that

$$z(t_0 + t) = \sum_{k \in J_V} I_k(t + t_0) \tag{32}$$

where

$$\begin{aligned} I_k(t+t_0) = &\frac{1}{\omega_k}(G_k(\omega_k(t_0+t)) v_k(t_0+t, x(t+t_0)) - G_k(\omega_k t_0) M(t_0+t, t_0) v_k(t_0, x(t_0)) + \\ &\frac{1}{\omega_k}\int_0^t G_k(\omega_k(t_0+s)) M(t_0+t, t_0+s) A(t_0+s) v_k(t_0+s, x(t_0+s)) ds + \\ &-\frac{1}{\omega_k}\int_0^t G_k(\omega_k(t_0+s)) M(t_0+t, t_0+s)\left(\frac{d}{ds} v_k(t_0+s, x(t_0+s))\right) ds \end{aligned}$$

From the standard estimate for the logarithmic norms (see for example Lemma 4.1 in [KZ]) we know that for $t \geq 0$ it holds that

$$\|M(t+t_0, t_0)\| \leq \exp(lt).$$

Hence, we obtain the following estimate of $I_k(t)$, for $t \in [0,h]$ and $k \in J_V$

$$|\omega_k| \cdot \|I_k(t+t_0)\| \leq C(v_k)C(G_k)(1+e^{lt}) + C(D_zFv_k)C(G_k)\int_0^t e^{l(t-s)}ds +$$
$$C(G_k)\left(C\left(\frac{\partial v_k}{\partial t}\right) + \sup_{s\in[0,h]}\left\|\frac{\partial v_k}{\partial z}(t_0+s, x(t_0+s))x'(t_0+s)\right\|\right)\int_0^t e^{l(t-s)}ds \leq$$
$$C(v_k)C(G_k)(1+e^{lt}) + C(D_zFv_k)C(G_k)(e^{lt}-1)/l +$$
$$C(G_k)\left(C\left(\frac{\partial v_k}{\partial t}\right) + C\left(D_z v_k \tilde{F}\right)\right)(e^{lt}-1)/l$$

This proves (29).

To finish the proof observe that if $|\omega_k| > \epsilon > 0$ for all $k \in J_V$, then assumptions (24–27) together with the formula (29) imply the convergence of the sum in (28). ∎

2.3 Estimates for higher order variational equations and persistence of normally hyperbolic objects

The goal of this subsection it to state without much of precision what are the dynamical consequences of the just proved estimates.

The differentiation of (15) with respect to initial conditions gives ODEs describing the evolution of the partial derivatives of the local process φ. These equations have the same structure as the original ODEs, i.e. consist from the part originating from (16) and the rapidly oscillating one. It is clear that under suitable conditions on higher derivatives of v_k added to the assumptions of Lemma 2.1 we can obtain the bounds of the following type

$$\left\|\frac{\partial^{|\alpha|}\varphi}{\partial x^\alpha}(t_0,t,x) - \frac{\partial^{|\alpha|}\varphi_a}{\partial x^\alpha}(t_0,t,x)\right\| \leq \frac{B_{|\alpha|}(t)}{\omega} \tag{33}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\frac{\partial^{|\alpha|}\varphi}{\partial x^\alpha}$ is the partial derivative $\frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}$ and $B_{|\alpha|} : [0,h] \to \mathbb{R}$ is continuous function.

In particular from these conditions it will follow that the time shift map by h along the trajectories for φ and φ_a are C^k-close.

Assume that there exists, M, a compact normally hyperbolic invariant manifold (NHIM) for φ_a (see [HPS]). When seen in the extended phase space it gives rise to $\tilde{M}$ another NHIM, which is non compact but various important estimates are uniform on $\tilde{M}$. For example, a hyperbolic periodic orbit for φ_a, which is topologically a circle, in the extended phase space becomes an infinite cylinder. Now we pass from the ODEs to the time shifts viewed in the extended phase space. As we said before these maps are C^k-close and one of them has NHIM $\tilde{M}$. Despite the lack of compactness, due to uniform estimates (see for exmaple methods from [CaZ]), one can show that HHIM $\tilde{M}$ exists also for the time shift by h for the process φ. Moreover, if the forcing is time periodic, then we obtain a true NHIM in the extended phase space, which is periodic in time.

These observations are essentially contained in the works discussing the implications of averaging for asymptotic dynamics, see for example [Hale, SVM, GH].

2.4 Estimate for the variational problem

In the context of the assumptions of Lemma 2.1 we will be interested in an explicit estimate for $\frac{\partial\varphi}{\partial x}(t_0,t,x) - \frac{\partial\varphi_a}{\partial x}(t_0,t,x)$. We will use these bounds in Section 3.

Lemma 2.2. *The same assumptions and notation as in Lemma 2.1. Additionally we assume that for all $k \in J_v$ holds*

$$\begin{aligned}
\sup_{s\in[t_0,t_0+t],z\in W} \left\| \frac{\partial^2 F}{\partial z^2}(s,z) \right\| &= C(D_{zz}F) < \infty, \\
\sup_{z,z_1\in W,s\in[t_0,t_0+h]} \left\| D_zF(s,z)\frac{v_k}{\partial z}(s,z_1) \right\| &= C\left(D_zF\frac{\partial v_k}{\partial z}\right) < \infty \\
\sup_{z\in W,s\in[t_0,t_0+h]} \|D_zF(s,z)\| &= C(D_zF) < \infty \\
\sup_{s\in[t_0,t_0+t],z\in W} \left\| \frac{\partial^2 v_k}{\partial t\partial z}(s,z) \right\| &= C\left(\frac{\partial^2 v_k}{\partial t\partial z}\right) < \infty \\
\sup_{s\in[t_0,t_0+t],z\in W} \left\| \frac{\partial^2 v_k}{\partial z^2}(s,z) \right\| &= C\left(\frac{\partial^2 v_k}{\partial z^2}\right) < \infty.
\end{aligned}$$

Let

$$\begin{aligned}
l_2 &= l + \sum_{k\in J_v} C(g_k)C\left(\frac{\partial v_k}{\partial z}\right) \\
\omega &= \inf_{k\in J_v} |\omega_k|,
\end{aligned}$$

and let $\tilde{b}(t) = \sum_{k\in J_v} b_k(t)$ be as in the assertion of Lemma 2.1.

Then for $t \in [0,h]$ holds

$$\left\| \frac{\partial\varphi}{\partial x}(t_0,t,x) - \frac{\partial\varphi_a}{\partial x}(t_0,t,x) \right\| \leq \frac{B(t)}{\omega} \tag{34}$$

where $B : [0,h] \to \mathbb{R}$ is continuous function depending on $C(\dots)$ and constants l, h given by

$$B(t) \leq C(D_{zz}F)\tilde{b}(t)\left(\frac{e^{l_2t} - e^{lt}}{l_2 - l}\right) + \sum_{k\in J_v} B_k(t),$$

where

$$B_k(t) = C(G_k)\left(C\left(\frac{\partial v_k}{\partial z}\right)(e^{lt}+e^{l_2 t})\right) +$$

$$C(G_k)\frac{e^{l_2 t}-e^{lt}}{l_2-l}\cdot\left(C\left(D_z F\frac{\partial v_k}{\partial z}\right)+C\left(\frac{\partial v_k}{\partial z}\right)\left(C(D_z F)+\sum_{k\in J_v}C(g_i)C\left(\frac{\partial v_i}{\partial z}\right)\right)+\right.$$

$$\left.+\left(C\left(\frac{\partial^2 v_k}{\partial t\partial z}\right)+C\left(\frac{\partial^2 v_k}{\partial z^2}\right)\cdot\left(C(F)+\sum_{i\in J_v}C(g_i)C(v_i)\right)\right)\right)$$

Proof: Let us denote by $V_a(t,z) = \frac{\partial \varphi_a}{\partial x}(t_0, t-t_0, z)$ and $V(t,z) = \frac{\partial \varphi}{\partial x}(t_0, t-t_0, z)$. Often in the remainder of the proof, we write just $V_a(t)$ and $V(t)$ for $V_a(t, x(t_0))$ and $V(t, x(t_0))$. Observe that in this notation holds

$$V_a(t_0) = V(t_0) = \mathrm{Id}.$$

From (16) and (15) it follows that

$$\frac{dV_a}{dt}(t) = \frac{\partial F}{\partial y}(t, y(t))V_a(t) \tag{35}$$

and

$$\frac{dV}{dt}(t) = \frac{\partial F}{\partial y}(t, x(t))V(t) + \left(\sum_{k\in J_v} g_k(\omega_k t)\frac{\partial v_k}{\partial x}(t, x(t))\right)V(t). \tag{36}$$

We rewrite (36) as follows

$$\frac{dV}{dt}(t) = \frac{\partial F}{\partial y}(t, y(t))V(t) + \left(\frac{\partial F}{\partial y}(t, x(t)) - \frac{\partial F}{\partial y}(t, y(t))\right)V(t) +$$
$$\left(\sum_{k\in J_v} g_k(\omega_k t)\frac{\partial v_k}{\partial x}(t, x(t))\right)V(t).$$

We obtain the following equation for $\Delta(t) = V(t) - V_a(t)$

$$\frac{d\Delta}{dt} = \frac{\partial F}{\partial y}(t, y(t))\cdot\Delta + \left(\frac{\partial F}{\partial y}(t, x(t)) - \frac{\partial F}{\partial y}(t, y(t))\right)V(t) +$$
$$\left(\sum_{k\in J_v} g_k(\omega_k t)\frac{\partial v_k}{\partial x}(t, x(t))\right)V(t).$$

Let $M(t,t_0)$ be the fundamental matrix for (35). Observe that in our notation we have $M(t+t_0, t_0) = V_a(t_0+t) = V_a(t+t_0, x(t_0))$.

We have

$$\Delta(t+t_0) = S(t) + \sum_{k\in J_v} I_k(t),$$

where

$$S(t) = \int_0^t M(t_0+t, t_0+s)\left(\frac{\partial F}{\partial y}(t_0+s, x(t_0+s)) - \frac{\partial F}{\partial y}(t_0+s, y(t_0+s))\right)V(t_0+s)ds$$

$$I_k(t) = \int_0^t g_k(\omega_k(t_0+s))M(t_0+t, t_0+s)\frac{\partial v_k}{\partial x}(t_0+s, x(t_0+s))V(t_0+s)ds$$

To provide bounds for $S(t)$ and $I_k(t)$ we need estimates for $V(t_0+s)$ and $M(t_0+t, t_0+s)$ for $s \in [0,t]$. This can be obtained using the logarithmic norm for (15) and (16), respectively (compare the proof of Lemma 2.2). We have

$$\|V(t_0+s)\| \le \exp(l_2 s)$$

where

$$\sup_{s\in[t_0,t_0+t], t_1\in\mathbb{R}, x\in W}\left(\mu(D_x f(s,x)) + \sum_{k\in J_v}|g_k(t_1)|\left\|\frac{\partial v_k}{\partial x}(s,x)\right\|\right)$$
$$\le l + \sum_{k\in J_v} C(g_k)C\left(\frac{\partial v_k}{\partial z}\right) =: l_2$$

In the above estimate obtained from (15) we applied the logarithmic norm to $F(t,x)$ and the standard norm to the integral. This can be justified as follows.

We have $\mu(A+B) = 2\mu((A+B)/2) \le \mu(A) + \mu(B)$, the last inequality follows from the convexity of the logarithmic norm.

Since $\mu(B) \le \|B\|$, so we obtain $\mu(A+B) \le \mu(A) + \|B\|$.

The bound for $M(t_0+t, t_0+s)$ is given by

$$\|M(t_0+t, t_0+s)\| \le e^{l(t-s)}, \quad s \in [0,t].$$

We are now ready to estimate $S(t)$.

From Lemma 2.1 it follows that for $\tilde{b}(t) = \max_{s\in[0,t]}\sum_{k\in J_v} b_k(s)$ holds

$$\|S(t)\| \le \int_0^t e^{l(t-s)}C(D_{zz}F)\|x(t_0+s) - y(t_0+s)\|e^{l_2 s}ds \le$$
$$\frac{e^{lt}C(D_{zz}F)\tilde{b}(t)}{\omega}\int_0^t e^{(l_2-l)s}ds = \frac{C(D_{zz}F)\tilde{b}(t)}{\omega}\left(\frac{e^{l_2 t} - e^{lt}}{l_2 - l}\right)$$

To estimate $I_k(t)$ we will integrate by parts.

We have

$$I_k = \left.\left(\frac{G_k(\omega_k(t_0+s))}{\omega_k}M(t_0+t, t_0+s)\frac{\partial v_k}{\partial x}(t_0+s, x(t_0+s))V(t_0+s)\right)\right|_{s=0}^{s=t} +$$
$$-\frac{1}{\omega_k}\int_0^t G_k(\omega_k(t_0+s))\frac{\partial}{\partial s}\left(M(t_0+t, t_0+s)\frac{\partial v_k}{\partial x}(t_0+s, x(t_0+s))V(t_0+s)\right)ds$$

To estimate the second term we use the following identities (we use (11))

$$\frac{\partial}{\partial s}\left(M(t_0+t,t_0+s)\frac{\partial v_k}{\partial x}(t_0+s,x(t_0+s))V(t_0+s)\right) =$$
$$-M(t_0+t,t_0+s)\frac{\partial F}{\partial y}(t_0+s,y(t_0+s))\frac{\partial v_k}{\partial x}(t_0+s,x(t_0+s))V(t_0+s)+$$
$$M(t_0+t,t_0+s)\left(\frac{\partial^2 v_k}{\partial t\partial x}(t_0+s,x(t_0+s))+\frac{\partial^2 v_k}{\partial x^2}(t_0+s,x(t_0+s))x'(t_0+s)\right)V(t_0+s)+$$
$$M(t_0+t,t_0+s)\frac{\partial v_k}{\partial x}(t_0+s,x(t_0+s))V'(t_0+s).$$

Therefore we obtain for $s\in[0,t]$

$$\left\|\frac{\partial}{\partial s}\left(M(t_0+t,t_0+s)\frac{\partial v_k}{\partial x}(t_0+s,x(t_0+s))V(t_0+s)\right)\right\| \leq$$
$$e^{l(t-s)}C\left(D_zF\frac{\partial v_k}{\partial z}\right)e^{l_2 s}+$$
$$e^{l(t-s)}\left(C\left(\frac{\partial^2 v_k}{\partial t\partial z}\right)+C\left(\frac{\partial^2 v_k}{\partial z^2}\right)\cdot\left(C(F)+\sum_{i\in J_v}C(g_i)C(v_i)\right)\right)e^{l_2 s}+$$
$$e^{l(t-s)}C\left(\frac{\partial v_k}{\partial z}\right)\left(C(D_zF)+\sum_{i\in J_v}C(g_i)C\left(\frac{\partial v_i}{\partial z}\right)\right)e^{l_2 s}$$

We obtain the following estimate for $I_k(t)$.

$$|\omega_k|\cdot\|I_k(t)\| \leq C(G_k)\left(C\left(\frac{\partial v_k}{\partial z}\right)\left(e^{lt}+e^{l_2 t}\right)\right)+$$
$$C(G_k)\frac{e^{l_2 t}-e^{lt}}{l_2-l}\cdot\left(C\left(D_zF\frac{\partial v_k}{\partial z}\right)+C\left(\frac{\partial v_k}{\partial z}\right)\left(C(D_zF)+\sum_{i\in J_v}C(g_i)C\left(\frac{\partial v_i}{\partial z}\right)\right)+\right.$$
$$\left.+\left(C\left(\frac{\partial^2 v_k}{\partial t\partial z}\right)+C\left(\frac{\partial^2 v_k}{\partial z^2}\right)\cdot\left(C(F)+\sum_{i\in J_v}C(g_i)C(v_i)\right)\right)\right)$$

This finishes the proof. ∎

3 Planar non-autonomous equation with very strong expansion

Let us write a real two-dimensional version of (1)

$$\begin{aligned}\frac{dx}{dt} &= x(1+\cos(\kappa t)|z|^2)+\sin(\kappa t)|z|^2 y,\\ \frac{dy}{dt} &= -y(1+\cos(\kappa t)|z|^2)+\sin(\kappa t)|z|^2 x.\end{aligned}$$

In this section we will use quite often the following notations $c = \cos(\kappa t)$, $s = \sin(\kappa t)$. The norm on $\mathbb{C} = \mathbb{R}^2$ is the euclidian norm and it is used to define the operator norms for linear and bilinear map arising in the analysis of (1).

3.1 All solutions with large initial data go to infinity in finite time

Consider first differential inequality

$$\frac{dx}{dt} > bx^3. \tag{37}$$

Lemma 3.1. *Assume $b > 0$ and $x : [t_0, t_{max}) \to \mathbb{R}$ is C^1 function, which satisfies (37) and $x(t_0) = x_0 > 0$. Then*

$$x(t) > \frac{x_0}{\sqrt{1 - 2bx_0^2(t - t_0)}}.$$

In particular, $t_{max} - t_0 \leq \frac{1}{2bx_0}$.

Proof: Let $y(t)$ be a solution of the following Cauchy problem

$$\frac{dy}{dt} = by^3, \quad y(t_0) = y_0. \tag{38}$$

It is easy to see that

$$y(t) = \frac{y_0}{\sqrt{1 - 2by_0^2(t - t_0)}} \tag{39}$$

From the standard differential inequalities it follows that if $x_0 \geq y_0$, then $x(t) > y(t)$ for $t > t_0$. ∎

Now we will show the existence of the forward invariant cones for short times. It will turn out that in such cone a solution might explode to infinity in time shorter than the established time for the forward invariance.

We define a quadratic form $Q(x, y) = x^2 - y^2$. Let us define cones $Q^+ = \{z \mid Q(z) \geq 0\}$ and $Q^- = \{z \mid Q(z) \leq 0\}$.

Lemma 3.2. *Cone Q^+ is forward invariant as long as*

$$\cos(\kappa t) \quad > \quad 0.$$

Proof: Let $z(t) = x(t) + iy(t)$ be a nonzero solution of (1). We have

$$\frac{1}{2}\frac{d}{dt}Q(x(t), y(t)) = xx' - yy' =$$
$$x^2(1 + c|z|^2) + |z|^2 sxy + y^2(1 + c|z|^2) - |z|^2 sxy = |z|^2(1 + c|z|^2) \geq c|z|^4$$

∎

Lemma 3.3. *Assume that $z = (x+iy) \in Q^+$, $x > 0$ and $\cos(\kappa t) - |\sin(\kappa t)| > 0$, then*

$$\frac{dx}{dt} > x^3(\cos(\kappa t) - |\sin(\kappa t)|).$$

Proof: Since $|z|^2 \geq x^2$ and $|y| \leq x$ we have

$$\frac{dx}{dt} > c|z|^2 x - |s||z|^2|y| \geq |z|^2(cx - |s|x) = x|z|^2(c - |s|) \geq x^3(c - |s|)$$

∎

Lemma 3.4. *Let $t_1 > 0$ and $\epsilon > 0$ be such that the following inequality is satisfied for $t \in [0, t_1]$*

$$\cos(\kappa t) - |\sin(\kappa t)| > \epsilon > 0.$$

Assume that $z_0 = (x_0, y_0) \in Q^+$ and $x_0^2 \geq \frac{1}{2\epsilon t_1}$. Then, $z(t)$, the solution of (1) with the initial condition $z(0) = z_0$ goes to infinity in finite time.

Proof: Without any loss of the generality we can assume that $x_0 > 0$.
From Lemmas 3.2 and 3.3 it follows that for $t \in [0, t_1]$ holds

$$\frac{dx}{dt} > \epsilon x^3. \tag{40}$$

From Lemma 3.1 if follows that for $t \in [0, t_1]$ holds

$$x(t) > \frac{x_0}{\sqrt{1 - 2\epsilon x_0^2 t}}. \tag{41}$$

Observe that we have the solution blow-up before time t_1 if

$$x_0^2 \geq \frac{1}{2\epsilon t_1}. \tag{42}$$

∎

Theorem 3.5. *Let $\delta \in (0, \pi/4)$. If $|z_0|^2 \geq \frac{\kappa}{\sqrt{2}(\pi/4-\delta)\sin\delta}$, then for any t_0 the solution of (1) with initial condition $z(t_0) = z_0 = (x_0, y_0)$ goes to infinity in finite time backwards or forwards in time.*

Proof: Let us fix $t_0 = 0$. Let $t_1 = \frac{\pi/4-\delta}{\kappa}$ Then for $t \in [0, t_1]$ holds

$$\begin{aligned} \cos(\kappa t) - |\sin(\kappa t)| = \cos(\kappa t_1) - \sin(\kappa t_1) \geq \\ \cos(\pi/4 - \delta) - \sin(\pi/4 - \delta) = \sqrt{2}\sin\delta. \end{aligned}$$

From Lemma 3.4 with $\epsilon = \sqrt{2}\sin\delta$ we obtain that if the initial condition $z_0 = (x_0, y_0)$ satisfies

$$x_0^2 \geq \frac{1}{2\epsilon t_1} = \frac{\kappa}{2\sqrt{2}(\pi/4 - \delta)\sin\delta} \tag{43}$$

then $|x(t)| \to \infty$ in finite time.

Observe that by inverting time we obtain that for initial conditions in Q_1^- with $y_0^2 \geq \frac{\kappa}{2\sqrt{2}(\pi/4-\delta)\sin\delta}$ the solutions will explode backward in time. Therefore we see that if $x_0^2+y_0^2 \geq \frac{\kappa}{\sqrt{2}(\pi/4-\delta)\sin\delta}$, then we have $|x(t)| \to \infty$ forward in finite time or $|y(t)| \to \infty$ backward in finite time.

In the above reasoning $t_0 = 0$ was singled out. But in fact there is nothing special about this initial time. The transformation

$$z_1 = \exp(-i\kappa t_0/2)z \tag{44}$$

transforms (1) into

$$z_1' = \overline{z}_1(1 + |z_1|^2 \exp(i\kappa(t - t_0))). \tag{45}$$

∎

3.2 A priori estimates for small time step

Lemma 3.6. *Let φ be the local process induced by (1) and $R_0 \geq 1$. For any $t_0 \in \mathbb{R}$ if $|z_0| \leq R_0$ and $h = \frac{1}{8R_0^2}$, then $\varphi(t_0, t, z_0)$ is defined for $t \in [0, h]$ and*

$$|\varphi(t_0, t, z_0)| \leq \sqrt{2}R_0. \tag{46}$$

Proof: It is easy to see that for $|z| \geq 1$ holds

$$\frac{d|z|}{dt} \leq |z|(1 + |z|^2) \leq 2|z|^3. \tag{47}$$

From Lemma 3.1 with $b = 2$ we obtain the following estimate for $|z_0| \leq R_0$ and $t \in [0, h]$

$$|\varphi(t_0, t, z_0)| \leq \frac{R_0}{\sqrt{1 - 2bR_0^2 t}} \leq \sqrt{2}R_0 \tag{48}$$

∎

3.3 C^0 estimates for the averaging

We would like to write down explicitly the estimates from Lemma 2.1 for (1).

Let us first rewrite equation (1) in the form (15). We have

$$z' = f(z) + g_1(\omega_1 t)v_1(z) + g_2(\omega_2 t)v_2(z), \tag{49}$$

where

$$\begin{aligned} f(z) &= \overline{z}, \\ \omega_1 &= \omega_2 = \kappa, \\ g_1(t) &= \cos t, \quad G_1(t) = \sin t \\ g_2(t) &= \sin t, \quad G_2(t) = -\cos t \\ v_1(z) &= |z|^2(x, -y)^t, \\ v_2(z) &= |z|^2(y, x)^t. \end{aligned}$$

Therefore we see that $J_v = \{1,2\}$ and on J_v the counting measure is used.

We will also need the estimates for the partial derivatives of v_k.

$$\frac{\partial v_1}{\partial z} = \begin{bmatrix} 3x^2+y^2, & 2xy \\ -2xy, & -x^2-3y^2 \end{bmatrix}, \qquad \frac{\partial v_2}{\partial z} = \begin{bmatrix} 2xy, & x^2+3y^2 \\ 3x^2+y^2, & 2xy \end{bmatrix}.$$

Lemma 3.7. *For $z \in \mathbb{R}^2$ holds*

$$\|D_z v_1(z)\| = \|D_z v_2(z)\| = 3r^2 \tag{50}$$

Proof: Let $z = (x,y) = r(\cos\alpha, \sin\alpha)$. It is easy to see that

$$\begin{aligned} Dv_1(z) &= \begin{bmatrix} 2r^2+(x^2-y^2), & 2xy \\ -2xy, & -2r^2+(x^2-y^2) \end{bmatrix} = \\ &= r^2\left(\begin{bmatrix} 2, & 0 \\ 0, & -2 \end{bmatrix} + \begin{bmatrix} \cos(2\alpha), & \sin(2\alpha) \\ -\sin(2\alpha), & \cos(2\alpha) \end{bmatrix} \right), \end{aligned}$$

and

$$\begin{aligned} Dv_2(z) &= \begin{bmatrix} 2xy, & 2r^2+(y^2-x^2) \\ 2r^2+x^2-y^2, & 2xy \end{bmatrix} = \\ &= r^2\left(\begin{bmatrix} 0, & 2 \\ 2, & 0 \end{bmatrix} + \begin{bmatrix} \sin(2\alpha), & -\cos(2\alpha) \\ \cos(2\alpha), & \sin(2\alpha) \end{bmatrix} \right), \end{aligned}$$

Therefore we have

$$Dv_i(z) = r^2(A_i + Q_i), \quad i = 1,2$$

where A_i is a symmetric matrix with eigenvalues ± 2 and Q_i is a rotation matrix. Therefore we have

$$\|Dv_i(z)\| \leq r^2(\|A_i\| + \|Q_i\|) = 3r^2. \tag{51}$$

The proof of equality is left to the reader. ∎

Therefore we have the following bounds on $W = \overline{B}(0,R)$ and $t \in [t_0, t_0+h]$, where $h = \frac{1}{4R^2}$ (compare with Lemma 3.6, there R_0 was the size of the initial condition and here $R = \sqrt{2}R_0$ is the size of the enclosure)

$$\begin{aligned} C(f) &= R \\ C(G_k) &= C(g_k) = 1, \quad k = 1,2 \\ l &= 1, \\ C(v_k) &= C(D_z f v_k) = R^3, \quad k = 1,2 \\ C\left(\frac{\partial v_k}{\partial t}\right) &= 0, \quad k = 1,2 \\ C\left(\frac{\partial v_k}{\partial z}\right) &\leq MR^2, \quad k = 1,2, \\ C\left(D_z v_k \tilde{F}\right) &= C\left(\frac{\partial v_k}{\partial z}\right)\left(C(f) + \sum_{i \in J_v} C(v_i)C(g_i)\right) \end{aligned}$$

From Lemma 3.7 it follows that $M = 3$.

Hence for $k = 1, 2$ from Lemma 2.1 we obtain

$$b_k(t) \leq C(G_k) \left(C(v_k)(1 + e^{lt}) + \right.$$
$$\left. \left(C\left(D_z f v_k\right) + C\left(\frac{\partial v_k}{\partial t}\right) + C\left(\frac{\partial v_k}{\partial z}\right)\left(C(f) + \sum_{i \in J_v} C(v_i) C(g_i)\right)\right) \frac{e^{lt} - 1}{l} \right) =$$
$$R^3(1 + e^t) + (R^3 + MR^2(R + 2R^3))(e^t - 1) = R^3((2 + M)e^t - M) + 2MR^5(e^t - 1)$$

Summarizing, we have proved

Lemma 3.8. *For $t \in \left[0, \frac{1}{4R^2}\right]$ and $|z_0| \leq \frac{R}{\sqrt{2}}$ holds*

$$|\varphi(t_0, t, z_0) - (e^t x_0, e^{-t} y_0)| \leq \frac{\tilde{b}(t)}{|\kappa|} := \frac{2}{|\kappa|} \left(R^3((2 + M)e^t - M) + 2MR^5(e^t - 1) \right).$$

3.4 C^1 estimates

We want to use Lemma 2.2. For this we need to estimate also the second derivatives of v_k.

We have

$$\frac{\partial^2 v_{1,x}}{\partial z^2} = \begin{bmatrix} 6x, & 2y \\ 2y, & 2x \end{bmatrix}, \qquad \frac{\partial^2 v_{1,y}}{\partial z^2} = \begin{bmatrix} -2y, & -2x \\ -2x, & -6y \end{bmatrix},$$
$$\frac{\partial^2 v_{2,x}}{\partial z^2} = \begin{bmatrix} 2y, & 2x \\ 2x, & 6y \end{bmatrix}, \qquad \frac{\partial^2 v_{2,y}}{\partial z^2} = \begin{bmatrix} 6x, & 2y \\ 2y, & 2x \end{bmatrix}.$$

To estimate $C\left(\frac{\partial^2 v_k}{\partial z^2}\right)$ we will use the following lemma.

Lemma 3.9. *For $z \in \mathbb{R}^2$ and $k = 1, 2$ it holds*

$$\left\| \frac{\partial^2 v_k}{\partial z^2} \right\| \leq r\sqrt{24 + 16\sqrt{2}} = \left(4 + 2\sqrt{2}\right) r, \tag{52}$$

where $r = |z|$,

Proof: Fist we will estimate

$$C_{1,x} = \left\| \frac{\partial^2 v_{1,x}}{\partial z^2} \right\| \tag{53}$$

For this it is enough to estimate from above the spectrum of the matrix $\begin{bmatrix} 6x, & 2y \\ 2y, & 2x \end{bmatrix}$. An easy computation yields the following formula for the eigenvalues

$$\lambda = 4x \pm 2r.$$

Therefore we obtain

$$C_{1,x} = 4|x| + 2r \leq 6r.$$

In order to estimate $D^2 v_1$ observe first that from symmetry arguments it follows immediately that

$$C_{1,y} = \left\| \frac{\partial^2 v_{1,y}}{\partial z^2} \right\| = 4|y| + 2r.$$

Therefore we have for any vectors $a, b \in \mathbb{R}^2$

$$\|D^2 v_1(a,b)\|^2 \leq (C_{1,x}\|a\| \cdot \|b\|)^2 + (C_{1,y}\|a\| \cdot \|b\|)^2 = \left(C_{1,x}^2 + C_{1,y}^2\right)(\|a\| \cdot \|b\|)^2$$

$$\|D^2 v_1\| \leq \left(C_{1,x}^2 + C_{1,y}^2\right)^{1/2}.$$

We have

$$C_{1,x}^2 + C_{1,y}^2 = (16x^2 + 16|x|r + 4r^2) + (16y^2 + 16|y|r + 4r^2) = \\ 24r^2 + 16(|x| + |y|)r \leq 24r^2 + 16\sqrt{2}r^2 = (24 + 16\sqrt{2})r^2.$$

∎

Let h, W be as in the previous subsection. We have

$$\begin{aligned} C(D_{zz} f) &= 0, \\ C\left(D_z f \frac{\partial v_k}{\partial z}\right) &= C\left(\frac{\partial v_k}{\partial z}\right) = MR^2, \\ C(D_z f) &= 1, \\ C\left(\frac{\partial^2 v_k}{\partial t \partial z}\right) &= 0, \\ C\left(\frac{\partial^2 v_k}{\partial z^2}\right) &= NR. \end{aligned}$$

From Lemma 3.9 it follows that $N = 4 + 2\sqrt{2}$.

We have

$$\begin{aligned} l_2 &= l + \sum_{k \in J_v} C(g_k) C\left(\frac{\partial v_k}{\partial z}\right) = 1 + 2MR^2 \\ \omega &= |\kappa|, \end{aligned}$$

and $\tilde{b}(t)$ given in Lemma 3.8.

From Lemma 2.2 we obtain (we set $k = 1$ below)

$$B(t) = 2B_1(t) = 2C(G_k)\left(C\left(\frac{\partial v_k}{\partial z}\right)(e^{lt} + e^{l_2 t})\right) +$$

$$2C(G_k)\frac{e^{l_2 t} - e^{lt}}{l_2 - l} \cdot \left(C\left(D_z f \frac{\partial v_k}{\partial z}\right) + C\left(\frac{\partial v_k}{\partial z}\right)\left(C(D_z f) + \sum_{i \in J_v} C(g_i) C\left(\frac{\partial v_i}{\partial z}\right)\right) +\right.$$

$$\left.+\left(C\left(\frac{\partial^2 v_k}{\partial t \partial z}\right) + C\left(\frac{\partial^2 v_k}{\partial z^2}\right) \cdot \left(C(f) + \sum_{i \in J_v} C(g_i) C(v_i)\right)\right)\right) = 2MR^2\left(e^t + e^{(1+2MR^2)t}\right) +$$

$$2\frac{e^{(1+2MR^2)t} - e^t}{2MR^2} \cdot \left(MR^2 + MR^2(1 + 2MR^2) + NR \cdot (R + 2R^3)\right) =$$

$$2MR^2 e^t\left(1 + e^{2MR^2 t}\right) + e^t(e^{2MR^2 t} - 1)\left(2 + \frac{N}{M} + 2R^2\left(M + \frac{N}{M}\right)\right)$$

We proved the following

Lemma 3.10. *For $t \in \left[0, \frac{1}{4R^2}\right]$ and $|z_0| \leq \frac{R}{\sqrt{2}}$ holds*

$$\left\| \frac{\partial \varphi}{\partial z_0}(t_0, t, z_0) - \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \right\| \leq \frac{1}{|\kappa|} 2MR^2 e^t \left(1 + e^{2MR^2 t}\right) +$$
$$\frac{e^t(e^{2MR^2 t} - 1)}{|\kappa|}\left(2 + \frac{N}{M} + 2R^2\left(M + \frac{N}{M}\right)\right) =: \frac{\tilde{B}(t, R)}{|\kappa|}$$

3.5 The estimates for the domain of hyperbolicity

Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

For any $h > 0$ map e^{Ah} is hyperbolic and represents the linearization of (1) at 0. Now we add to e^{Ah} the influence of oscillating terms from (1) and we would like to find possibly large R, such that in the ball of radius R the shift by h along trajectory of (1) is a hyperbolic map, in the sense discussed below.

3.5.1 Cone conditions

Let us define two cone fields, for $z \in \mathbb{R}^2$ we set $Q^+(z) = \{z + (x, y) | |x| \geq |y|\}$, $Q^-(z) = \{z + (x, y) | |x| \leq |y|\}$.

Definition 3.11. *Let $\{F_j\}_{j \in \mathbb{Z}}$ be a family of maps, $F_j : \mathbb{R}^2 \supset domF_j \to \mathbb{R}^2$ be a C^1-map, such $F_j(0) = 0$. Let $N \subset domF_j$ (for $j \in \mathbb{Z}$) be a connected open set, such that $0 \in N$. We will say that family of maps $\{F_j\}$ is hyperbolic on N iff the following conditions hold:*

- *the cone field Q^+ is forward invariant relatively to N, i.e. if $z_1, z_2 \in N$ are such that $z_2 \in Q^+(z_1)$, then for all $j \in \mathbb{Z}$ $F_j(z_2) \in Q^+(F_j(z_1))$*

- *there exist constants* $0 \leq \mu < 1 < \xi$, *such that*
  - *if* $z_1, z_2 \in N$ *and* $z_2 \in Q^-(z_1)$, *then for all* $j \in \mathbb{Z}$
$$|\pi_y(F_j(z_1) - F_j(z_2))| \leq \mu|\pi_y(z_1 - z_2)|, \tag{54}$$
  - *if* $z_1, z_2 \in N$ *and* $z_2 \in Q^+(z_1)$, *then for all* $j \in \mathbb{Z}$
$$|\pi_x(F_j(z_1) - F_(z_2))| \geq \xi|\pi_x(z_1 - z_2)|. \tag{55}$$

We have the following simple theorem, which is an adaptation of results from [CaZ], where normally hyperbolic invariant manifolds are discussed, (see also [ZCC])

Theorem 3.12. *Assume that family of maps* $\{F_j\}$ *is a hyperbolic on a convex set* N *in the sense of Definition 3.11 and assume that for all* $j \in \mathbb{Z}$ $F_j^{-1}(0) = \{0\}$. *Then*

- *if* $z \in N \setminus \{0\}$, $l \in \mathbb{Z}$ *and* $z \in Q^+(0)$, *then there exists* $n \geq 1$, *such that*
$$\begin{array}{lcl} F_{k+l} \circ F_{k-1+l} \circ \cdots \circ F_l(z) & \in & Q^+(0) \cap N, \quad \text{for} \quad k = 1, \ldots, n-1, \\ F_{n+l} \circ F_{n-1+l} \circ \cdots \circ F_l(z) & \notin & N. \end{array}$$
- *if* $z \in N \setminus \{0\}$, $l \in \mathbb{Z}$, *and* $z \in Q^-(0)$, *then one of the two following conditions is satisfied*
  - *there exists* $n \geq 1$ *such that*
$$\begin{array}{lcl} F_{k+l} \circ F_{k-1+l} \circ \cdots \circ F_l(z) & \in & N \cap Q^-(0), \ \text{for } k = 1, \ldots, n-1 \\ F_{n+l} \circ F_{n-1+l} \circ \cdots \circ F_l(z) & \in & Q^+(0). \end{array}$$
  -
$$\begin{array}{rcl} F_{k+l} \circ F_{k-1+l} \circ \cdots \circ F_l(z) & \in & N \cap Q^-(0), \quad \forall k \in \mathbb{N} \\ \lim_{k \to \infty} F_{k+l} \circ F_{k-1+l} \circ \cdots \circ F_l(z) & = & 0 \end{array}$$

In particular, if N *is bounded, then for every point* $z \neq 0$ *every maximal orbit of* $\{F_j\}$ *through* z *leaves* N *either forward or backward in time.*

Below we give the criterion for the hyperbolicity of family $\{F_j\}_{j \in \mathbb{Z}}$.

Lemma 3.13. *Let for* $j \in \mathbb{Z}$ $F_j : \mathbb{R}^2 \supset domF_j \to \mathbb{R}^2$ *be a* C^1*-map, such* $F_j(0) = 0$. *Let* $\overline{N} \subset domF_j$ *for all* $j \in \mathbb{Z}$ *and* N *be a convex bounded open set, such that* $0 \in N$.

Let us set
$$\xi = \inf_{j \in \mathbb{Z}} \left(\inf_{z \in N} \left| \frac{\partial F_x}{\partial x}(z) \right| - \sup_{z \in N} \left| \frac{\partial F_x}{\partial y} \right| \right), \tag{56}$$
$$\mu = \sup_{j \in \mathbb{Z}} \left(\sup_{z \in N} \left| \frac{\partial F_y}{\partial y}(z) \right| + \sup_{z \in N} \left| \frac{\partial F_y}{\partial x} \right| \right). \tag{57}$$

If $\xi > 1 > \mu$, *then* $\{F_j\}_{j \in \mathbb{Z}}$ *is hyperbolic on* N.

Proof: We will present the proof for $F_j = F$. The generalization to a family of maps is trivial.

Let $z_1 = (x_1, y_1)$ and $z_0 = (x_0, y_0)$. Then we have

$$\begin{aligned} F_x(z_1) - F_x(z_0) &= \int_0^1 \frac{\partial F_x}{\partial x}(z_0 + t(z_1 - z_0))dt \cdot (x_1 - x_0) + \\ & \int_0^1 \frac{\partial F_x}{\partial y}(z_0 + t(z_1 - z_0))dt \cdot (y_1 - y_0), \quad (58) \\ F_y(z_1) - F_y(z_0) &= \int_0^1 \frac{\partial F_y}{\partial x}(z_0 + t(z_1 - z_0))dt \cdot (x_1 - x_0) + \\ & \int_0^1 \frac{\partial F_y}{\partial y}(z_0 + t(z_1 - z_0))dt \cdot (y_1 - y_0). \quad (59) \end{aligned}$$

From conditions (58), (59) and the convexity of N it follows that

$$\begin{aligned} |F_x(z_1) - F_x(z_0)| &\geq \inf_{z \in N} \left| \frac{\partial F_x}{\partial x}(z) \right| \cdot |x_1 - x_0| - \sup_{z \in N} \left| \frac{\partial F_x}{\partial y}(z) \right| \cdot |y_1 - y_0| \quad (60) \\ |F_y(z_1) - F_y(z_0)| &\leq \sup_{z \in N} \left| \frac{\partial F_y}{\partial x}(z) \right| \cdot |x_1 - x_0| + \sup_{z \in N} \left| \frac{\partial F_y}{\partial y}(z) \right| \cdot |y_1 - y_0| \quad (61) \end{aligned}$$

Therefore for $z_1 \in Q^+(z_0)$ (recall that $|x_1 - x_0| \geq |y_1 - y_0|$) we obtain

$$\begin{aligned} |F_x(z_1) - F_x(z_0)| &\geq \xi |x_1 - x_0| \\ |F_y(z_1) - F_y(z_0)| &\leq \mu |x_1 - x_0|. \end{aligned}$$

We see that in this case

$$|F_x(z_1) - F_x(z_0)| \geq \xi |x_1 - x_0| \geq \mu |x_1 - x_0| \geq |F_y(z_1) - F_y(z_0)|. \quad (62)$$

Hence $F(z_1) \in Q^+(F(z_0))$. This establishes the forward cone invariance of the cone field Q^+ and expansion condition (55) in positive cone.

It remains to show (54) for $z_1 \in Q^-(z_0)$. From (61) it follows that

$$|F_y(z_1) - F_y(z_0)| \leq \mu |y_1 - y_0|.$$

∎

3.5.2 Estimation of cone conditions for (1)

Let us fix any $t_0 \in \mathbb{R}$. We will show that the family of maps $\{z \mapsto \varphi(t_0 + jh, h, z)\}_{j \in \mathbb{Z}}$ is hyperbolic on some ball $B(R)$.

Let us set

$$\Delta(t_0, h, z) = \frac{\partial \varphi}{\partial z}(t_0, h, z) - e^{Ah} \quad (63)$$

Observe that from Lemma 3.10 we have the following bounds, which do not depend on t_0, on $\|\Delta(t_0,h,z)\|$ for $|z| \leq \frac{R}{\sqrt{2}}$ and $h \in \left(0, \frac{1}{4R^2}\right]$

$$\|\Delta(t_0,h,z)\| \leq \frac{\tilde{B}(h,R)}{|\kappa|}. \tag{64}$$

Let us now compute ξ and μ from Lemma 3.13. We have

$$\xi \geq e^h - \frac{2\tilde{B}(h,R)}{|\kappa|},$$
$$\mu \leq e^{-h} + \frac{2\tilde{B}(h,R)}{|\kappa|}.$$

Therefore for the hyperbolicity we need to satisfy the following conditions

$$\frac{2\tilde{B}(h,R)}{|\kappa|} < e^h - 1 \tag{65}$$
$$\frac{2\tilde{B}(h,R)}{|\kappa|} < 1 - e^{-h}. \tag{66}$$

Since for $h > 0$ holds

$$e^h - 1 > 1 - e^{-h},$$

hence we are left with the following condition

$$\frac{2\tilde{B}(h,R)}{|\kappa|} < 1 - e^{-h}, \tag{67}$$

hence

$$|\kappa| > \frac{2\tilde{B}(h,R)}{1 - e^{-h}}. \tag{68}$$

In Table 1 we list several values of κ depending on $R_0 = \frac{R}{\sqrt{2}}$

R_0	κ
1	3655
10	$2.24 \cdot 10^7$
100	$2.23 \cdot 10^{11}$

Table 1: The values of κ, such that in $B(0,R_0)$ the behavior of (1) is hyperbolic. κ is computed from (68) with $R = \sqrt{2}R_0$ from Lemma 3.10. We used $M = 3$, $N = 4 + 2\sqrt{2}$, $h = \frac{1}{4R^2}$.

Therefore we have proved the following result.

Theorem 3.14. *For any $R > 0$ let $R_0 = \frac{R}{\sqrt{2}}$, $h = \frac{1}{4R_0^2}$ and assume that*

$$|\kappa| > \frac{2\tilde{B}(h, R)}{1 - e^{-h}}. \tag{69}$$

Then for any t_0 the family of maps $\{z \mapsto \varphi(t_0 + jh, h, z)\}_{j\in\mathbb{Z}}$, is hyperbolic on $B(0, R_0)$.

Let us estimate now the growth of κ, such that we have the hyperbolic behavior on $B(0, R_0)$. It is easy to see that for $h = \frac{1}{8R_0^2}$

$$\begin{aligned} \tilde{B}(h(R_0), R_0\sqrt{2}) &= O(R_0^2) \\ |\kappa| &\geq O(R_0^2)/(1 - e^{-h}) = \frac{O(R_0^2)}{\frac{1}{8R_0^2}} = O(R_0^4) \end{aligned}$$

Corrolary 3.15. *There exists $K > 0$, such that if $z(t)$ is bounded orbit for (1), then either $z(t) \equiv 0$ or there exists t_1, such that $|z(t_1)| \geq K\kappa^{1/4}$.*

3.6 Bounds on the nontrivial bounded solutions

In the investigations of (1) in [SW, WZ, OW] the chaotic behavior was obtained by the constructions of suitably matched isolating segments, called (U, U^-) and (W, W^-) in [SW, WZ], which satisfy $U \subset W$. Segment W exists for any $\kappa \neq 0$ and its existence implies the existence of nonzero periodic orbit in it. The shape of W in the extended phasespace is given by a square $[-R, R]^2$ rotating with angular velocity $\kappa/2$ (see Section 5 in [WZ]), where R satisfies the following inequality [WZ, Lem. 13]

$$R^2(R^2 - 1) > (1/2 + \kappa/4)^2.$$

Therefore $2\pi/\kappa$ periodic orbit related to this isolating segment exists for $R = O(\kappa^{1/2})$. This is in fact the bound for any bounded orbit in this segment.

Combining this with Cor. 3.15 we obtain these periodic have to satisfy

$$\begin{gathered} \forall t\ |z(t)| \leq O(\kappa^{1/2}) \\ \exists t_1\ |z(t_1)| \geq O(\kappa^{1/4}). \end{gathered}$$

Observe that from Theorem 3.5 we obtain also that all orbits such that $|z(t_0)| > O(\kappa^{1/2})$ escape to infinity forward or backward in time. Therefore there is not point in taking substantially larger R for the isolating block.

From the above observations it follows that any bounded nonzero orbit of (1) has to satisfy the following conditions

$$\begin{aligned} &\forall t\ |z(t)| \leq O(\kappa^{1/2}) \\ \exists \{t_n\}_{n\in\mathbb{Z}}, \lim_{n\to-\infty} t_n = -\infty, \lim_{n\to\infty} t_n = \infty\ \forall n\quad &|z(t_n)| \geq O(\kappa^{1/4}). \end{aligned}$$

References

[AKN97] V. Arnold, V. Kozlov, and A. Neishtadt. *Mathematical Aspects of Classical and Celestial Mechanics.* Springer Verlag, Berlin, 1997.

[BM] N.N. Bogolyubov and Y. A. Mitropol'skii, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, 2nd ed Gordon & Breach: New York, 1961

[BZ] N.N. Bogolyubov and D. N. Zubarev, *An Asymptotic Approximation Method for a System with Rotating Phases and its Application to the Motion of a Charged Particle in a Magnetic Field* Ukrain. Math. Zh. 7 (1955)

[BL] A. Buica and J. Llibre, Averaging methods for finding periodic orbits via Brouwer degree, *Bull. Sci. math.* 128 (2004) 7–22

[CaZ] M.J. Capiński and P. Zgliczyński, Geometric proof for normally hyperbolic invariant manifolds, *J. Diff. Eq.*, 259(2015) 6215–6286

[CMTZ] J. Cyranka, P. B. Mucha, E. S. Titi, and P Zgliczyński, *Stabilizing the Long-time Behavior of the Navier-Stokes Equations and Damped Euler Systems by Fast Oscillating Forces*, preprint arXiv:1601.04612 (2016)

[CyZ] J. Cyranka and P. Zgliczyński, *Stabilizing effect of large average initial velocity in forced dissipative PDEs invariant with respect to Galilean transformations*, preprint, arXiv:1407.1712 (2014)

[GH] J. Guckenheimer and P. Holmes. Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields. Springer, New York, 1983.

[HNW] E. Hairer, S.P. Nørsett and G. Wanner, *Solving Ordinary Differential Equations I, Nonstiff Problems*, Springer-Verlag, Berlin Heidelberg 1987.

[Hale] J.K. Hale, Ordinary Differential Equations, Wiley, New York, 1969

[He] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, in: Lecture Notes in Mathematics, vol. 840, Springer 1981

[HPS] M.W. Hirsch, C.C. Pugh and M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics vol. 583, 1977

[I] A. Iserles, *Three stories of high oscillation.*, Eur. Math. Soc. Newsl. No. 87 (2013),

[JS] A. Jorba, C. Simo, *Effective Stability for Periodically Pertubed Hamiltonian Systems*, in 'Hamiltonian Mechanics NATO ASI Series. Integrability and Chaotic Behavior' Volume 331, 1994, 245–252

[KZ] T. Kapela and P. Zgliczyński, *A Lohner-type algorithm for control systems and ordinary differential inclusions*, Discrete Cont. Dyn. Sys. B, vol. 11(2009), 365-385.

[L] J. Llibre, The Averaging Theory for Computing Periodic Orbits Llibre, in *In Central Configurations, Periodic Orbits, and Hamiltonian Systems*, Birkhäuser Basel, CRM Barcelona, 2015.

[MM] K. Mischaikow, M. Mrozek, *Conley index.* Chapter 9 in Handbook of Dynamical Systems, vol 2, pp 393–460, Elsevier 2002

[MS] M. Mrozek, R. Srzednicki, *Topological Approach to Rigorous Numerics of Chaotic Dynamical Systems with Strong Expansion of Error Bounds*, Found. Comput. Math. 10 (2010) 191–220

[Nei84] A. Neishtadt, *The separation of motions in systems with rapidly rotating phase*, J. Appl. Math. Mech. 48 (1984), 133-139.

[OW] P. Oprocha, P. Wilczyński, *Distributional chaos via isolating segments*, DCDS B 8 (2007), 347–356

[SVM] J. A. Sanders F. Verhulst and J. Murdock, Averaging Methods in Nonlinear Dynamical Systems, Appl. Math. Sci., vol. 59, Springer, New York, 2007

[Si] C. Simo, *Averaging under Fast Quasiperiodic Forcing*, in 'Hamiltonian Mechanics NATO ASI Series. Integrability and Chaotic Behavior' Volume 331, 1994, 13–34

[S1] R. Srzednicki, *Periodic and bounded solutions in blocks for time-periodic nonautonomuous ordinary differential equations.* Nonlin. Analysis, TMA., 1994, 22, 707–737

[SW] R. Srzednicki, K. Wójcik, *A geometric method for detecting chaotic dynamics*, J. Diff. Eq., 1997, 135, 66–82

[Wa] J. R. Ward Jr, *Homotopy and Bounded Solutions of Ordinary Differential Equations*, J. Diff. Eq.,107 (1994), 428–445

[WZ] K. Wójcik, P. Zgliczyński, *Isolating segments, fixed point index and symbolic dynamics*, J. Diff. Eq, 161, 245–288, (2000)

[ZCC] P. Zgliczyński, *Covering relations, cone conditions and the stable manifold theorem*, J. Differential Equations. 246(5), 1774-1819 (2009)