

On the Petras algorithm for verified integration of piecewise analytic functions

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Abstract

We consider the algorithm for verified integration of piecewise analytic functions presented in Petras' paper [6]. The analysis of the algorithm contained in that paper is limited to a narrow class of functions and gives upper bounds only. We present an estimate of the cost (measured by a number of evaluations of the integrand) of the algorithm, both upper and lower bounds, for a wider class of functions. We show examples of functions with cost $\Theta(|\ln \varepsilon|/\varepsilon^{p-1})$, for any $p > 1$, where ε is the desired accuracy of the computed integral.

Keywords: verified integration, piecewise analytic function

1 Introduction

In [6] K. Petras described an algorithm for verified integration of piecewise analytic functions. By verified integration we understand a computer program which returns the result with a guaranteed error bound. As a motivation for developing such an algorithm, in [7] Petras gave several examples showing how QUADPACK [8], the standard package for numerical integration, can be fooled even for the analytic functions. The Petras algorithm uses interval arithmetic and formulas for rigorous error bounds, which require estimates for the function in the complex neighbourhood of the integration interval.

In our paper we investigate the cost of the Petras algorithm. We want to emphasize that the cost is measured by the number of evaluations of an integrand for a particular input (a function f and a parameter ε describing the required accuracy). Therefore, we omit the cost of subroutines in the algorithm, which the full cost analysis should take into account. We should mention the proper treatment of these issues will require a definition of computable analytic functions. Such definitions exist in the literature (see for example [2, 3] and the references given there), however in the present work we choose not to focus on those issues and concentrate on the geometric aspects of the problem.

Our task is, given a piecewise analytic function f on $[a, b]$ and $\varepsilon > 0$, to find a number \mathcal{I} , such that

$$\left| \int_a^b f(x) dx - \mathcal{I} \right| \leq \varepsilon.$$

If f is analytic, then there are algorithms for integration (based for example on Gauss-Legendre quadrature) with the apparent cost $\mathcal{O}(|\ln \varepsilon|)$. Many authors neglect the problem that constants in those estimates in fact depend on f , mainly on the shape of its domain of analyticity; see [4] for a discussion of optimal quadratures depending on the domain of analyticity.

However, a disturbance of analyticity at some points makes the convergence deteriorate. For the sake of discussion, let us call them breakpoints or singularities. Knowing the location of singularities is not enough; we cannot simply partition the interval at those points and apply a standard algorithm for analytic functions to smaller intervals. The problem is that algorithms require the function to be analytic on the interval of integration *and* its neighbourhood (disk or ellipse containing the interval in the complex plane).

Our case is (seemingly) even more difficult: we know neither the number nor the location of the breakpoints; we might not even know whether they exist.

We consider the Petras algorithm for verified integration of piecewise analytic functions. The analysis of the algorithm contained in [6] is not sufficient, because the class of functions considered is unnaturally restricted and only an upper bound for the cost for this class is given.

The main result of the present paper is a more sophisticated estimate of the cost of the algorithm for a wider class of functions. Our results still do not cover the whole range of piecewise analytic functions, but only those that satisfy Petras-type conditions.

The main idea explored in our paper is that the cost depends mainly on the region of analyticity of the integrand. The difference between a “simple” and a “difficult” function is not that the former is analytic, while the latter is not. Even an analytic function might be hard to integrate, if it has singularities very close to the real axis. On the other hand, a piecewise analytic function might be relatively simple, if the region of analyticity around breakpoints is wide ($p \leq 1$ in the Definition 2). Notice that the cost analysis focuses on the phenomena occurring in the small neighbourhood of singular points, since the most significant increase in the number of evaluations of the integrand takes place there. The number of evaluations generated in the last step of the algorithm (i.e., near breakpoints) is comparable to the number of evaluations generated in all previous steps.

We show that the cost indeed depends on the order p of PPC and NPC conditions that the integrated function satisfies, i.e., if $p > 1$, then the cost is $\Theta(|\ln \varepsilon|/\varepsilon^{p-1})$, while for $p \leq 1$ the cost is $\Theta(\ln^2 \varepsilon)$, as shown by Petras. Moreover, we show examples of functions (see Section 8) for which the cost scales as $|\ln \varepsilon|/\varepsilon^{p-1}$ for any $p > 1$.

The paper is organized in the following way. Section 2 contains basic definitions and notation. In Section 3 we present the Petras algorithm. In Section 4 we introduce main tools for its analysis; in particular the Petras-type conditions PPC and NPC are given. In Sections 5 and 6 we show lower and upper bounds for the cost of the algorithm. Section 7 contains the main theorem of the paper.

2 Notation, core definitions and general assumptions

As usual, by \mathbb{N} , \mathbb{Z} , \mathbb{R} , \mathbb{C} we denote the sets of natural numbers (including 0), integers, real numbers and complex numbers, respectively. By \mathbb{N}_+ and \mathbb{R}_+ we denote the set of positive natural and real numbers, respectively. We use \overline{A} to denote the closure of a set A . We will use $|z|$ to denote the absolute value of $z \in \mathbb{C}$ or $z \in \mathbb{R}$. For a point $x \in \mathbb{C}$ and a set $Z \subset \mathbb{C}$ we define

$$\text{dist}(x, Z) = \inf_{y \in Z} |x - y|.$$

Definition 1 *Assume that $a, b \in \mathbb{R}$ and $a < b$.*

We say that a function $f : \mathbb{C} \supset \text{dom } f \rightarrow \mathbb{C}$ is a piecewise analytic function on $[a, b]$ if there exist a family $\{\mathcal{D}_j\}_{j \in \mathcal{J}}$, where $\mathcal{J} \subset \mathbb{N}$, of open, pairwise disjoint sets contained in \mathbb{C} , such that

1. f is analytic on each \mathcal{D}_j and

$$\text{dom } f \cap \{x + iy : a \leq x \leq b\} \subset \overline{\bigcup_{j \in \mathcal{J}} \mathcal{D}_j},$$

2. for $(a_j, b_j) := \mathcal{D}_j \cap [a, b]$ we have

$$[a, b] = \overline{\bigcup_{j \in \mathcal{J}} (a_j, b_j)}.$$

We define the domain of analyticity of f by

$$\text{doa } f = \bigcup_{j \in \mathcal{J}} \mathcal{D}_j.$$

Example 1 Some examples of piecewise analytic functions on $[-1, 1]$:

- all analytic functions whose domain contains $[-1, 1]$,
- the function $f(z) = \exp(-1/z^2)$,
- the function $f(z) = |\sin(1/z)|$.

Definition 2 For a given $\gamma, p > 0$ and a closed set $S \subset [-1, 1]$ define the region (see Figure 1)

$$\mathcal{D}_{\gamma, S}^p := \{x + iy : |y| \leq \gamma \cdot \text{dist}(x, S)^p\}.$$

When S is a singleton and its only element is clear from the context, we omit the subscript S and write \mathcal{D}_{γ}^p . Notice that $\mathcal{V}_{\gamma, S}$ used in [6] is a special case of $\mathcal{D}_{\gamma, S}^p$, namely we have

$$\mathcal{V}_{\gamma, S} = \mathcal{D}_{\gamma, S}^1. \quad (1)$$

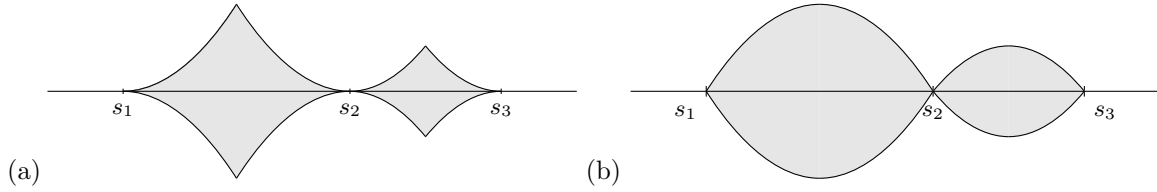


Figure 1: Region $\mathcal{D}_{\gamma, S}^p$ between s_1 and s_3 , where $S = \{s_1, s_2, s_3\}$ (a) with $p > 1$, (b) with $p < 1$.

As usual we use $\mathcal{Q}_{n, [\alpha, \beta]}[f]$ to denote an n -point quadrature formula for the evaluation of the integral of f over $[\alpha, \beta]$. $R_{n, [\alpha, \beta]}[f]$ is the error of the quadrature, i.e.,

$$R_{n, [\alpha, \beta]}[f] = \left| \int_{\alpha}^{\beta} f(x) dx - \mathcal{Q}_{n, [\alpha, \beta]}[f] \right|.$$

Definition 3 For fixed $A > 1, B > 0$, let the rectangle $\varrho(\alpha, \beta, A, B)$ in the complex plane (see Figure 2) be defined as

$$\varrho(\alpha, \beta, A, B) = \left\{ x + iy : \left| x - \frac{\beta + \alpha}{2} \right| \leq A \frac{\beta - \alpha}{2}, |y| \leq B \frac{\beta - \alpha}{2} \right\}.$$

When the parameters A, B are clear from the context, we omit them and write $\varrho(\alpha, \beta)$.

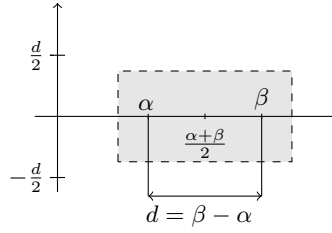


Figure 2: Rectangle $\varrho(\alpha, \beta, A, B) = \{x + iy : \frac{\alpha+\beta}{2} - A\frac{d}{2} \leq x \leq \frac{\alpha+\beta}{2} + A\frac{d}{2}, -B\frac{d}{2} \leq y \leq B\frac{d}{2}\}$

Definition 4 For fixed $A > 1$, $B > 0$, we set

$$m(f; \alpha, \beta, A, B) = \sup\{|f(z)| : z \in \varrho(\alpha, \beta, A, B) \cap \text{dom } f\}.$$

When the parameters A, B are clear from the context, we omit them and write $m(f; \alpha, \beta)$.

Originally in [6] the functional m has a different meaning: it is a functional (or perhaps a subroutine) which returns a value greater or equal to

$$\sup\{|f(z)| : z \in \varrho(\alpha, \beta, A, B) \cap \text{dom } f\},$$

which in this paper is realized by a function `ComplexBound` (see the beginning of Section 3).

For the estimation of error of the quadrature we need know the size of the largest ellipse with the foci at $(\alpha, 0)$ and $(\beta, 0)$ which is contained in $\varrho(\alpha, \beta, A, B)$.

Lemma 1 Let $A > 1$. If an ellipse with foci at $F_1 = (\alpha, 0)$, $F_2 = (\beta, 0)$, major semi-axis $\mathbf{a} = A(\beta - \alpha)/2$ and minor semi-axis $\mathbf{b} = B(\beta - \alpha)/2$ is inscribed (i.e., tangent to the edges) in $\varrho(\alpha, \beta, A, B)$, then $B = \sqrt{A^2 - 1}$.

Proof. By definition of an ellipse the distance from the center of the ellipse to the focal point is $\sqrt{\mathbf{a}^2 - \mathbf{b}^2}$, thus

$$\frac{\beta - \alpha}{2} = \sqrt{\left(\frac{A(\beta - \alpha)}{2}\right)^2 - \left(\frac{B(\beta - \alpha)}{2}\right)^2}.$$

This gives

$$B = \sqrt{A^2 - 1}. \quad \blacksquare$$

By Lemma 1 the parameters A and B of ϱ are not independent if we want to inscribe an ellipse into $\varrho(\alpha, \beta, A, B)$. Therefore, in the sequel, we will only use $B = \sqrt{A^2 - 1}$.

For $[\alpha, \beta] = [-1, 1]$ we have $\mathbf{a} = A$ and $\mathbf{b} = B$. In particular for $A = \frac{5}{4}$ we have $B = \frac{3}{4}$ and the largest ellipse contained in $\varrho(-1, 1, A, B)$ has $\mathbf{c} = \mathbf{a} + \mathbf{b} = 2$.

Remark 2 Let $A = \frac{5}{4}$ and $B = \frac{3}{4}$.

1. For Gaussian quadrature we have

$$R_{n, [\alpha, \beta]}[f] \leq 2 \cdot 4^{-n} \cdot (\beta - \alpha) m(f; \alpha, \beta) \quad (2)$$

2. For Clenshaw-Curtis quadrature we have

$$R_{n, [\alpha, \beta]}[f] \leq 3 \cdot 2^{-n} \cdot (\beta - \alpha) m(f; \alpha, \beta) \quad (3)$$

for functions analytic on $\varrho(\alpha, \beta, A, B)$.

Below we give an explanation of the denominators in (2) and (3). For references regarding the constants used, see [1] and [5].

The map

$$T(z) = \frac{\alpha + \beta}{2} + \frac{\beta - \alpha}{2}z$$

is an affine isomorphism, such that $T(\varrho(-1, 1, A, B)) = \varrho(\alpha, \beta, A, B)$. The formulas for Gauss or Clenshaw-Curtis quadratures and their errors are transported from $[-1, 1]$ to $[\alpha, \beta]$ by this map.

On the normalized interval $[-1, 1]$ the error of Gauss quadrature contains the term $1/\mathfrak{c}^{2n}$, where \mathfrak{c} is the sum of major and minor semi-axes. The transformation of $[\alpha, \beta]$ onto $[-1, 1]$ multiplies this error by $\beta - \alpha$ and gives $\mathfrak{c} = 2$ (see Lemma 1). Therefore we obtain the factor $(\beta - \alpha)/2^{2n} = 4^{-n} \cdot (\beta - \alpha)$.

2.1 General assumptions

When we say ‘the constant’ we actually mean a value that does depend on some previously fixed variables but does not depend on any variables quantified afterwards.

From now on, unless stated otherwise, whenever we refer to a function f and an interval $[a, b]$, we mean “a piecewise analytic function f on $[a, b]$, where $[a, b]$ is the range of integration”.

Additionally, we assume that $A > 1$ and $c > 1$.

3 The Petras algorithm

In this section we recall an algorithm from [6] for the verified integration of piecewise analytic functions. We will call it *the Petras algorithm*. We assume that we have at our disposal the following subroutines (or oracles):

- **UpperBound**(f, a, b) which returns an upper bound M for $\|f\|_\infty = \sup_{x \in [a, b] \cap \text{dom } f} |f(x)|$,
- **IsAnalytic**(f, α, β, A) such that if it is true, then f is analytic on $\varrho(\alpha, \beta, A, \sqrt{A^2 - 1})$; we do not require that the converse is true,
- **ComplexBound**(f, α, β, A) returning a value greater than or equal to $m(f; \alpha, \beta, A, \sqrt{A^2 - 1})$ provided that f is analytic on $\varrho(\alpha, \beta, A, \sqrt{A^2 - 1})$.

In further analysis of the Petras algorithm we will formulate some conditions regarding the properties of these subroutines, however we will not include their cost in the cost estimate, even though they might be hard to compute and depend substantially on f . These issues will require the precise definition of computable analytic functions (see for example [2, 3] and the references given there), which we do not consider in this paper.

3.1 Formulation of the algorithm

We have a fixed sequence $(\mathcal{Q}_n)_{n \in \mathbb{N}}$ of quadrature formulas to be Gaussian or Clenshaw-Curtis.

The input of the algorithm consists of:

- the integrand f and the interval of integration $[a, b]$; we require that f is piecewise analytic and bounded on $[a, b]$,

- an accuracy bound $\varepsilon > 0$.

The algorithm also uses configuration constants:

- $A > 1$ is used in the definition of the area ϱ (see Definition 3) which is needed to compute `IsAnalytic` and `ComplexBound` subroutines (recall that $B = \sqrt{A^2 - 1}$).
- $c > 1$ is used in the estimation of function values.
- constants D and E (compare with (2) and (3) in Remark 2). The quadratures \mathcal{Q}_n satisfy error estimates of the form

$$R_{n, [\alpha, \beta]}[f] \leq D \cdot E^{-n} \cdot (\beta - \alpha) m(f; \alpha, \beta)$$

for functions analytic on $\varrho(\alpha, \beta, A, \sqrt{A^2 - 1})$.

The algorithm is as follows.

In: $f, [a, b], \varepsilon$

1. $M := \text{UpperBound}(f, a, b)$.
2. We define the initial partition of $[a, b]$ by setting $[a_0, a_1] = [a, b]$.
3. Assume that we have already partitioned $[a, b]$ into k intervals, i.e.,

$$[a, b] := [a_0, a_1] \cup [a_1, a_2] \cup \dots \cup [a_{k-1}, a_k]$$

and there is $J \subset \{1, \dots, k\}$ such that

$$\text{IsAnalytic}(f, a_{j-1}, a_j, A) \text{ and } \text{ComplexBound}(f, a_{j-1}, a_j, A) \leq cM, \quad \text{for } j \in J \quad (4)$$

$$\sum_{j \notin J} |a_j - a_{j-1}| > \frac{\varepsilon}{2M}. \quad (5)$$

Bisect the longest interval not belonging to J . Repeat step 3 as long as the condition (5) holds.

4. Now $\sum_{j \notin J} |a_j - a_{j-1}| \leq \varepsilon/(2M)$ (condition (5) does not hold)
 - a) for the intervals $[a_{j-1}, a_j]$ where $j \in J$ we calculate the integral using the quadrature \mathcal{Q}_n , where

$$n \geq \frac{1}{\ln E} \ln \frac{2D(b-a)cM}{\varepsilon}, \quad (6)$$

- b) for all remaining intervals we take the integral to be 0.

Out: $\sum_{j \in J} \mathcal{Q}_{n; [a_{j-1}, a_j]}[f]$

Remark 3 The requirement in (4) that f is analytic on $\varrho(a_{j-1}, a_j, A, \sqrt{A^2 - 1})$ was missing in Petras' original paper [6].

3.2 Numerical accuracy

Lemma 4 *Assume that the Petras algorithm terminates and returns q . Then*

$$\left| \int_a^b f(x) dx - q \right| < \varepsilon.$$

Proof. For each $[\alpha, \beta]$ for which $m(f; \alpha, \beta) \leq cM$ we want the error of the quadrature to be less than

$$(\beta - \alpha) \frac{\varepsilon}{2(b - a)},$$

so that (summing the error over all intervals) the global error on $[a, b]$ is less than or equal to $\varepsilon/2$. We can calculate n using the estimate

$$R_{n, [\alpha, \beta]}[f] \leq D \cdot E^{-n} \cdot (\beta - \alpha) m(f; \alpha, \beta) \leq D \cdot E^{-n} \cdot (\beta - \alpha) cM < (\beta - \alpha) \frac{\varepsilon}{2(b - a)},$$

obtaining

$$n \geq \frac{1}{\ln E} \ln \frac{2D(b - a)cM}{\varepsilon}.$$

Now we can estimate the total error of the quadrature on all the intervals belonging to J :

$$\begin{aligned} \sum_{j \in J} R_{n, [a_{j-1}, a_j]}[f] &\leq \sum_{j \in J} (a_j - a_{j-1}) \frac{\varepsilon}{2(b - a)} \leq \frac{\varepsilon}{2(b - a)} \sum_{j \in J} (a_j - a_{j-1}) \\ &\leq \frac{\varepsilon}{2(b - a)} (b - a) = \frac{\varepsilon}{2} \end{aligned}$$

Now $\|f\|_\infty \leq M$ and the total of the lengths of all intervals not belonging to J is not greater than $\varepsilon/(2M)$. Hence the error over these intervals is at most $M \cdot \varepsilon/(2M) = \varepsilon/2$.

Therefore we obtain

$$R_{n, [a, b]}[f] = \sum_{j \in J} R_{n, [a_{j-1}, a_j]}[f] + \sum_{j \notin J} R_{n, [a_{j-1}, a_j]}[f] \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \blacksquare$$

3.3 The algorithm terminates under reasonable assumptions

It is not obvious whether the algorithm terminates. If the subroutine `ComplexBound` returns values which do not “match” the bound $c \cdot \sup_{x \in [a, b] \cap \text{dom } f} |f(x)|$ (for example, it always returns 2, while $c \cdot \sup_{x \in [a, b] \cap \text{dom } f} |f(x)| \approx 1$) the algorithm might run forever. Thus we have to assure that such case will never happen. For this we need to provide certain assumptions regarding the quality of subroutines `IsAnalytic`, `ComplexBound` and `UpperBound`.

Definition 5 *We say that the subroutines `IsAnalytic`, `ComplexBound` and `UpperBound` satisfy the compatibility condition if for any $x \in [a, b]$ such that f is analytic in some neighbourhood of x , there exists an open set $U \subset \mathbb{C}$, such that $x \in U$ and*

$$\varrho(\alpha, \beta) \subset U \implies \text{IsAnalytic}(f, \alpha, \beta, A) \text{ and } \text{ComplexBound}(f, \alpha, \beta, A) \leq c \cdot \text{UpperBound}(f, \alpha, \beta)$$

Theorem 5 *Assume that f is a bounded piecewise analytic function on $[a, b]$ and the Lebesgue measure of points in $[a, b]$ in which f is not analytic is equal to zero. If `IsAnalytic`, `ComplexBound` and `UpperBound` satisfy the compatibility condition, then the Petras algorithm terminates on f , $[a, b]$ and any $\varepsilon > 0$.*

Proof. From the assumptions about `IsAnalytic`, `ComplexBound` and `UpperBound`, it follows that any interval created by the Petras algorithm containing only the points of analyticity will be subdivided (possibly after several subdivisions) into smaller intervals that will eventually be accepted.

Hence the total length of bad intervals (not in J) goes to zero and the algorithm terminates. \blacksquare

3.4 The case of analytic functions

Lemma 6 *Assume $[\alpha, \beta] \subset \text{doa } f \cap [a, b]$ and `IsAnalytic`, `ComplexBound` and `UpperBound` satisfy the compatibility condition. Then the number of intervals accepted in the Petras algorithm and covering $[\alpha, \beta]$ is finite and does not depend on ε and depends on the set U from Definition 5.*

Proof. It follows from the compactness of $[\alpha, \beta]$ and the compatibility condition that there exists $\delta = \delta(U) > 0$, such that any interval of length less than or equal to δ is accepted by the Petras algorithm. ■

4 Tools for the analysis of the algorithm

The Petras algorithm has two parts: geometrical and computational. During the geometrical part (step 3) a partition of $[a, b]$ is constructed based on a region of analyticity and boundedness \mathcal{D} . In the computational part (step 4), the integral is computed using a chosen quadrature on each of the resulting sub-intervals.

For a piecewise analytic function f and $w \in \mathbb{R}_+$ we define the region of analyticity and boundedness as

$$\mathcal{D}(f, w) = \overline{\{z \in \mathbb{C} \mid z \in \text{doa } f \wedge |f(z)| \leq w\}}.$$

Notice that in a perfect world, the actual object of concern in the geometrical part should be the set $\mathcal{D} = \mathcal{D}(f, c \cdot \sup_{x \in [a, b] \cap \text{dom } f} |f(x)|)$. However, the set \mathcal{D} is not known explicitly. Instead, the algorithm implicitly analyzes the set $\mathcal{D}(f, cM) \subset \mathcal{D}$, where $M = \text{UpperBound}(f, a, b)$.

In the cost estimates we say that an interval is “bad“ or “proper“ according to whether or not it needs to be bisected.

The number of proper intervals generated by the Petras algorithm is denoted by $\mathcal{Z}(\mathcal{D}(f, cM), \varepsilon)$. Recall that on each of the proper intervals the quadrature (Gauss-Legendre or Clenshaw-Curtis) with

$$n = \left\lceil \frac{1}{\ln E} \ln \left(2Dc \cdot \frac{M(b-a)}{\varepsilon} \right) \right\rceil = \Theta \left(\ln \frac{M}{\varepsilon} \right)$$

points (see (6) in step 4 of the algorithm) is calculated, and thus the algorithm performs

$$N(f, \varepsilon) = \Theta \left(\ln \frac{M}{\varepsilon} \right) \cdot \mathcal{Z}(\mathcal{D}(f, cM), \varepsilon)$$

evaluations of f . To complete the cost estimate we need to count the number of proper intervals generated by the algorithm.

For any $p > 1$ we give examples of functions for which

$$\mathcal{Z}(\mathcal{D}(f, cM), \varepsilon) = \Theta \left(\left(\frac{M}{\varepsilon} \right)^{p-1} \right),$$

while the analysis in Petras’ paper[6] deals with the classes of functions for which

$$\mathcal{Z}(\mathcal{D}(f, cM), \varepsilon) = \mathcal{O} \left(\ln \frac{M}{\varepsilon} \right).$$

4.1 Petras-type conditions

Let us fix A and $B = \sqrt{A^2 - 1}$. In [6] Petras proposed a condition (recall that $\mathcal{V}_{\gamma,S}$ is defined by (1))

$$\begin{aligned} \exists \gamma > 0 \forall \alpha, \beta : [\varrho(\alpha, \beta) \subset \mathcal{V}_{\gamma,S} \implies f \text{ is analytic on } \varrho(\alpha, \beta) \text{ and} \\ m(f; \alpha, \beta) \leq c \cdot \sup\{|f(x)| : x \in [-1, 1] \cap \text{dom } f\}] \end{aligned} \quad (7)$$

to estimate the cost of the algorithm. We find this requirement too strong, as it excludes a lot of functions.

Before presenting our generalization of Petras' condition we state several theorems for the function $f(x) = \sin(1/x)$ for $x \in [-1, 1]$. These results imply that this condition does not hold for said f , motivating the use of the sets $D_{\gamma,S}^p$ in further analysis.

Theorem 7 *Let us take an arbitrary $Z > 0$. Let $f(z) = \sin(1/z)$. In the neighborhood of 0 the condition*

$$\exists \gamma > 0 \forall \alpha, \beta \in [-1, 1] : \left(\varrho(\alpha, \beta) \subset \mathcal{V}_{\gamma, \{0\}} \implies \sup_{z \in \varrho(\alpha, \beta)} |f(z)| \leq Z \right)$$

does not hold for any $\gamma > 0$, i.e.,

$$\forall \gamma > 0 \exists \alpha, \beta \in [-1, 1] : \left(\varrho(\alpha, \beta) \subset \mathcal{V}_{\gamma, \{0\}} \wedge \sup_{z \in \varrho(\alpha, \beta)} |f(z)| > Z \right).$$

Proof. Assume $z = x + iy$. Let $r^2 = x^2 + y^2$. Then

$$\frac{1}{z} = \frac{x - iy}{r^2}. \quad (8)$$

Since

$$\sin \frac{1}{z} = \frac{1}{2i} \left(\exp \frac{i}{z} - \exp \left(-\frac{i}{z} \right) \right)$$

we obtain

$$\sin \frac{1}{z} = \frac{1}{2i} \left(\exp \left(i \frac{x}{r^2} \right) \exp \frac{y}{r^2} - \exp \left(-i \frac{x}{r^2} \right) \exp \left(-\frac{y}{r^2} \right) \right)$$

and

$$\left| \sin \frac{1}{z} \right| \geq \frac{1}{2} \left(\exp \frac{|y|}{r^2} - \exp \left(-\frac{|y|}{r^2} \right) \right). \quad (9)$$

For any $\gamma > 0$ consider a point $z = x + i\gamma x$ with $x > 0$, since $y = \gamma x > 0$ we have

$$\frac{|y|}{r^2} = \frac{\gamma x}{\gamma^2 x^2 + x^2} = \frac{\gamma}{x(\gamma^2 + 1)}$$

and finally by (9) we have

$$\begin{aligned} \left| \sin \frac{1}{z} \right| &\geq \frac{1}{2} \left(\exp \left(\frac{\gamma}{x(\gamma^2 + 1)} \right) - \exp \left(-\frac{\gamma}{x(\gamma^2 + 1)} \right) \right) \\ &\rightarrow \infty, \quad \text{as } x \rightarrow 0. \end{aligned}$$

■

Theorem 7 demonstrates that Petras' condition (7) for function $f(z) = \sin(1/z)$ on $[-1, 1]$ does not hold on \mathcal{V}_γ for any $\gamma > 0$. Indeed, since $\lim_{z \rightarrow 0, z \in \partial \mathcal{V}_\gamma} \exp(|y|/r^2) = \infty$, we see $|\sin(1/z)|$ is too large on the boundary of \mathcal{V}_γ . However, we can work around this problem by using the region $D_{\gamma,S}^2$ instead of \mathcal{V}_γ , so that the values of $\exp(|y|/r^2)$ will be restricted.

Theorem 8 Let $S = \{0\}$ and let $f(z) = \sin(1/z)$. Then for any $c > 1$ there exists $\gamma > 0$ such that:

$$\forall \alpha, \beta \in [-1, 1] : \left\{ \varrho(\alpha, \beta) \subset \mathcal{D}_\gamma^2 \implies f \text{ is analytic on } \varrho(\alpha, \beta) \text{ and} \right. \\ \left. \sup\{|f(z)| : z \in \varrho(\alpha, \beta)\} \leq c \cdot \sup\{|f(x)| : x \in [-1, 1] \cap \text{dom } f\} \right\}.$$

Proof. If $\varrho(\alpha, \beta) \subset \mathcal{D}_\gamma^2$ then it is obvious that f is analytic on $\varrho(\alpha, \beta)$. Thus it is enough to check the second part of the conjunction.

Assume $z = x + iy$. Let $r^2 = x^2 + y^2$. As in the proof of Theorem 7, we have

$$\sin \frac{1}{z} = \frac{1}{2i} \left(\exp\left(i \frac{x}{r^2}\right) \exp\left(\frac{y}{r^2}\right) - \exp\left(-i \frac{x}{r^2}\right) \exp\left(-\frac{y}{r^2}\right) \right),$$

so that

$$\left| \sin \frac{1}{z} \right| \leq \frac{1}{2} \left(\exp\left(\frac{|y|}{r^2}\right) + \exp\left(-\frac{|y|}{r^2}\right) \right).$$

For $\gamma > 0$, define \mathcal{W}_γ by

$$\mathcal{W}_\gamma = \left\{ x + iy : \frac{|y|}{r^2} \leq \gamma \right\}.$$

Since the function $x \mapsto x + 1/x$ is increasing for $x > 1$ we have

$$\left| \sin \frac{1}{z} \right| \leq \frac{1}{2} (\exp(\gamma) + \exp(-\gamma)), \quad z \in \mathcal{W}_\gamma.$$

Now

$$\frac{1}{2} \lim_{\gamma \rightarrow 0} (\exp(\gamma) + \exp(-\gamma)) = 1.$$

By taking γ sufficiently close to 0, we get

$$\sup \left\{ \left| \sin \frac{1}{z} \right| : z \in \mathcal{W}_\gamma \right\} < c \cdot \sup\{|\sin(1/x)| : x \in [-1, 1]\}.$$

To complete the proof it is enough to show that $\mathcal{D}_\gamma^2 \subset \mathcal{W}_\gamma$. Let $y \geq 0$ (the other case is symmetric). It is easy to see that \mathcal{W}_γ describes the complement of a disk of radius $1/(2\gamma)$ centered at $(0, 1/(2\gamma))$, namely

$$z = x + iy \in \mathcal{W}_\gamma \iff \left(y - \frac{1}{2\gamma} \right)^2 + x^2 \geq \frac{1}{4\gamma^2}.$$

Observe that \mathcal{W}_γ contains the set

$$\widetilde{\mathcal{W}}_\gamma = \left\{ x + iy : |x| \leq \frac{1}{2\gamma}; \quad |y| \leq \frac{1 - \sqrt{1 - 4\gamma^2 x^2}}{2\gamma} \right\} \cup \left\{ x + iy : |x| > \frac{1}{2\gamma} \right\}$$

and notice that

$$\frac{1 - \sqrt{1 - 4\gamma^2 x^2}}{2\gamma} > \gamma x^2, \quad \text{for } |x| \leq \frac{1}{2\gamma}.$$

This shows that $\mathcal{D}_\gamma^2 \subset \widetilde{\mathcal{W}}_\gamma \subset \mathcal{W}_\gamma$. ■

Theorem 8 says that there exists a region \mathcal{D}_γ^2 where the function $z \mapsto \sin(1/z)$ is analytic and appropriately bounded. The next theorem says the opposite: there exists a region of the same shape as before, but such that (on the boundary of this region close to the singular point) the values of the function are arbitrarily large.

Theorem 9 Let $S = \{0\}$ and let $f(z) = \sin(1/z)$. For any $c > 1$ there exists $\gamma > 0$ such that for any $\alpha \in [-1/\gamma, 1/\gamma]$ and $\beta \in (\alpha, 1]$

$$\varrho(\alpha, \beta) \not\subset \mathcal{D}_\gamma^2 \implies f \text{ is not analytic on } \varrho(\alpha, \beta) \text{ or } \sup\{|f(z)| : z \in \varrho(\alpha, \beta)\} > c. \quad (10)$$

Proof. Let us fix $c > 1$. If $\alpha < 0 < \beta$ then f is not analytic on $\varrho(\alpha, \beta)$.

Let us consider the case when f is analytic on $\varrho(\alpha, \beta)$. Notice that if $\varrho(\alpha, \beta) \not\subset \mathcal{D}_\gamma^2$ then $\varrho(\alpha, \beta) \cap \partial\mathcal{D}_\gamma^2 \neq \emptyset$, where $\partial\mathcal{D}_\gamma^2$ is the boundary of \mathcal{D}_γ^2 . Thus let us consider the region

$$\mathcal{W}_\gamma = \partial\mathcal{D}_\gamma^2 \cap \{z : \operatorname{Re}(z) \in [-1/\gamma, 1/\gamma]\} \setminus \{0\}.$$

Then

$$\sup_{z \in \varrho(\alpha, \beta)} \left| \sin \frac{1}{z} \right| \geq \inf_{z \in \mathcal{W}_\gamma} \left| \sin \frac{1}{z} \right|.$$

Assume $z = x + iy \in \mathcal{W}_\gamma$ and let $r^2 = x^2 + y^2$. Then (for $|x| \leq 1/\gamma$)

$$\frac{|y|}{r^2} = \frac{\gamma x^2}{x^2 + (\gamma x^2)^2} = \frac{\gamma}{1 + \gamma^2 x^2} \geq \frac{\gamma}{2}.$$

Since (see proof of Theorem 7)

$$\left| \sin \frac{1}{z} \right| \geq \frac{1}{2} \left(\exp\left(\frac{|y|}{r^2}\right) - \exp\left(-\frac{|y|}{r^2}\right) \right)$$

and $x \mapsto x - 1/x$ is an increasing function, we have

$$\inf_{z \in \mathcal{W}_\gamma} \left| \sin \frac{1}{z} \right| \geq \frac{1}{2} \left(\exp\left(\frac{\gamma}{2}\right) - \exp\left(-\frac{\gamma}{2}\right) \right).$$

Now, for any $c > 1$ it is enough to take $\gamma > 0$ such that $\inf_{z \in \mathcal{W}_\gamma} |\sin(1/z)| > c$, i.e.,

$$\gamma > 2 \ln(c + \sqrt{1 + c^2})$$

and the condition (10) holds. ■

The two theorems given above justify why we consider regions bounded by curves γx^p . We do realize that this choice is arbitrary and non-exhaustive. Nevertheless it allows us to show that the region of analyticity is an important parameter in the cost investigations. In Section 8 we consider the functions $f(z) = z^k \sin(1/z)$ which naturally leads us to consider sets \mathcal{D}_γ^p .

Although we have only considered the case $p \geq 1$ so far, we can in fact extend these definitions to include any $p \in \mathbb{R}_+$. We now present a modified version of Petras' condition, introducing the notion of order and having a converse implication.

Definition 6 Let us fix A, B and $c > 1$. Assume that $p > 0$, $M \geq 0$, $S = \{s_1, \dots, s_m\}$ with $a \leq s_1 < \dots < s_m \leq b$ and $\gamma > 0$. We say that a function f satisfies the positive Petras condition of order p (abbreviated as PPC(p, γ, S, M)) on $[a, b]$, if

$$\varrho(x, y) \subset \mathcal{D}_{\gamma, S}^p \implies f \text{ is analytic on } \varrho(x, y) \text{ and } m(f; x, y) \leq cM, \\ \text{for all } x, y \text{ such that } [x, y] \subset [a, b].$$

Definition 7 Let us fix A, B and $c > 1$. Assume that $p > 0$, $M = \sup_{x \in [a, b] \cap \text{dom } f} |f(x)|$, $s \in [a, b]$, $\gamma > 0$ and $\beta > 0$. We say that a function f satisfies the negative Petras condition of order p (abbreviated as NPC(p, γ, s, β)) on $[a, b]$, if of the following conditions are satisfied

$$\varrho(x, y) \notin \mathcal{D}_\gamma^p \implies \text{either } f \text{ is not analytic on } \varrho(x, y) \text{ or } m(f; x, y) > cM, \quad (11)$$

for all x, y such that $x \in [s, s + \beta]$ and $y \in (x, b]$,

$$\varrho(x, y) \notin \mathcal{D}_\gamma^p \implies \text{either } f \text{ is not analytic on } \varrho(x, y) \text{ or } m(f; x, y) > cM, \quad (12)$$

for all x, y such that $y \in (s - \beta, s]$ and $x \in [a, y]$,

We refer to (11) as NPC(p, γ, s, β)-right and (12) as NPC(p, γ, s, β)-left.

Notice that in the definition above we used $\mathcal{D}_\gamma^p = \mathcal{D}_{\gamma, \{s\}}^p$.

The idea (how we use PPC and NPC conditions to estimate the number of proper intervals) is presented in Remark 10, while the detailed treatment is contained in Sections 5 and 6. Notice that, in general, we want to point out the classes of functions for which the cost of the Petras algorithm is $\Theta(|\ln \varepsilon|/\varepsilon^{p-1})$. Therefore we consider only those functions for which NPC and PPC are satisfied. To show that the cost of the algorithm cannot be better it is enough to show one singular point of f where the number of proper intervals generated by the algorithm is $\Omega(1/\varepsilon^{p-1})$, yielding a lower bound. To show that the cost of the algorithm is not worse we have to prove that in any point of S the algorithm cannot produce more proper intervals than $\mathcal{O}(1/\varepsilon^{p-1})$, yielding an upper bound.

Remark 10 Let $M = \text{UpperBound}(f, a, b)$. Let $\mathcal{D} = \mathcal{D}(f, cM)$. Assume that a function f satisfies PPC($p, \gamma, \{s\}, M$) and NPC(p', γ', s, β) (with $\gamma' \geq \gamma$, $p' \leq p$) on the whole interval, i.e., for $x \in [a, b]$ in (11,12). Therefore $\mathcal{D}_\gamma^p \subset \mathcal{D} \subset \mathcal{D}_{\gamma'}^{p'}$. If we consider intervals $[\alpha, x]$, $[\alpha, \alpha']$ and $[\alpha, y]$ such that

$$\varrho(\alpha, x) \in \mathcal{D}_\gamma^p, \quad \varrho(\alpha, \alpha') \in \mathcal{D}, \quad \varrho(\alpha, y) \notin \text{int} \mathcal{D}_{\gamma'}^{p'}, \quad \varrho(\alpha, y) \in \mathcal{D}_{\gamma'}^{p'},$$

then

$$[\alpha, x] \subset [\alpha, \alpha'] \subset [\alpha, y].$$

Thus investigating intervals in \mathcal{D}_γ (they are the shortest and therefore their number is the largest) we are able to estimate the number of proper intervals from above. And while investigating intervals on the edge of $\mathcal{D}_{\gamma'}^{p'}$ (the longest ones) we are able to estimate the number of proper intervals from below.

In order to use PPC in the cost estimates from above, we need to impose additional requirements regarding the quality of the subroutines `UpperBound`, `IsAnalytic` and `ComplexBound`.

Definition 8 Assume the function f satisfies PPC(p, γ, S, M) on $[a, b]$. We will say that `UpperBound`, `IsAnalytic` and `ComplexBound` are PPC(p, γ, S, M)-compatible for f on $[a, b]$ if

$$\varrho(x, y) \in \mathcal{D}_{\gamma, S}^p \implies \text{IsAnalytic}(f, x, y, A) \text{ and } \text{ComplexBound}(f, x, y, A) \leq c \text{UpperBound}(f, a, b),$$

for all x, y such that $[x, y] \subset [a, b]$.

4.2 Geometric lemmas

Assume that $\Gamma : [0, \infty) \rightarrow [0, \infty)$ is continuous, $\Gamma(0) = 0$ and Γ is strictly increasing for $x > 0$. We define

$$\mathcal{D}^\Gamma = \{x + iy : |y| \leq \Gamma(x)\},$$

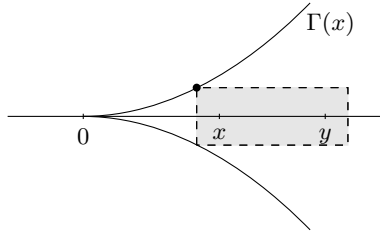


Figure 3: Points x and y are chosen so that the top left corner of $\varrho(x, y, A, B)$ belongs to the line $x \mapsto \Gamma(x)$.

The goal of this section is to find, for a fixed x or y , the largest possible $d = y - x$ such that $\varrho(x, y, A, B) \subset \mathcal{D}^\Gamma$.

By the definition of ϱ the top left corner of $\varrho(x, x + d, A, B)$ is at

$$\left(x - \frac{d}{2}(A - 1), B\frac{d}{2}\right)$$

and we want this point to lie on the line $x \mapsto \Gamma(x)$, thus (see Figure 3)

$$\Gamma\left(x - \frac{d}{2}(A - 1)\right) = B\frac{d}{2}. \quad (13)$$

This is the desired formula for $d = y - x$ parametrized by x , the left end of the interval. To obtain a formula parametrized by y we substitute $y - d$ for x in equation (13) and obtain

$$\Gamma\left(y - \frac{d}{2}(A + 1)\right) = B\frac{d}{2}. \quad (14)$$

Theorem 11 *There exist strictly increasing, continuous functions $d_L, d_R : [0, \infty) \rightarrow [0, \infty)$ solving equations (13), (14), respectively. Moreover:*

$$d_L(x) < \frac{2}{B}\Gamma(x), \quad x > 0, \quad (15)$$

$$d_R(y) < \frac{2}{B}\Gamma(y), \quad y > 0. \quad (16)$$

Proof. It is clear that d_L is strictly increasing in x . It is enough to translate $\varrho(x, y, A, B)$, with top left corner on the line $x \mapsto \Gamma(x)$, to the right. The shifted rectangle will be contained in the interior of \mathcal{D}^Γ . Similarly, d_R is increasing in y .

For the proofs of (15) and (16) observe that for $x > 0$ it holds

$$\begin{aligned} \frac{B}{2}d_L(x) &= \Gamma\left(x - \frac{A-1}{2}d_L(x)\right) < \Gamma(x) \\ \frac{B}{2}d_R(y) &= \Gamma\left(y - \frac{A+1}{2}d_R(y)\right) < \Gamma(y). \end{aligned}$$

■

We would like to obtain bounds of the following form

$$c_2\Gamma(x) \leq d_L(x), d_R(x) \leq c_1\Gamma(x),$$

for some $c_1, c_2 > 0$. The existence of c_1 follows from Theorem 11. The existence of c_2 is treated in Section 4.2.1 for $\Gamma(x) = \gamma x^p$, $p > 1$. It turns out that for $p \leq 1$ this is not true; in fact we obtain linear estimates from above and from below (see Theorem 15 and equation (23)).

For the case $\Gamma(x) = \gamma x^p$ let us set

$$h = \frac{B}{2\gamma}.$$

Then observe that equations (13) and (14) have the following form

$$(x - g \cdot d)^p = h \cdot d, \tag{17}$$

provided that we set either

$$g = g_L = \frac{A-1}{2},$$

or

$$g = g_R = \frac{A+1}{2},$$

respectively. Equation (17) defines implicitly a function $d(x)$. In the following subsections we will estimate $d(x)$.

4.2.1 The case $p > 1$

Our goal in this section is to obtain estimates for the solution of (17) for $p > 1$. To develop intuitions consider an integer p and a series expansion of $d(x) = d_0 + d_1x + d_2x^2 + \dots$. Regrouping the terms in (17) and taking $d_0 = 0$ we obtain

$$\begin{aligned} d(x) &= \frac{1}{h}x^p + d_{p+1}x^{p+1} + \dots \\ &= \frac{1}{h}x^p(1 + d_{p+1}x + \dots) \\ &= \frac{c(x)}{h} \cdot x^p, \end{aligned}$$

where $c(x) = O(1)$ for small x . These considerations lead us to a hypothesis that

$$d(x) = \frac{c(x)}{h} \cdot x^p \tag{18}$$

for any $p > 1$, such that there exist c_1, c_2 and

$$0 < c_2 \leq c(x) \leq c_1,$$

for a bounded range of x . Therefore from (17) and (18) we have the following implicit equation for $c(x)$

$$\left(1 - g \cdot \frac{c(x)}{h} \cdot x^{p-1}\right)^p = c(x).$$

For given g, h and $p > 1$ define

$$\tilde{x} = \left(\frac{h}{g}\right)^{\frac{1}{p-1}}. \tag{19}$$

Lemma 12 *Let us consider the equation*

$$F(x, c(x)) = \left(1 - c(x) \cdot \frac{g}{h} \cdot x^{p-1}\right)^p - c(x) = 0.$$

This equation has exactly one solution $c(x) \in [0, 1]$ which is continuous on $[0, \tilde{x}]$ and

$$\forall x \in [0, \tilde{x}] \exists c', c'' : (0 < c'' < c' = 1) \wedge (c'' \leq c(x) \leq c').$$

Proof. Notice that for every x , $F(x, 1) < 0$ and $F(x, 0) = 1 > 0$ thus there exists $c \in (0, 1]$ such that $F(x, c) = 0$. We now show the uniqueness of c :

$$\frac{\partial F}{\partial c} = p \left(1 - c \cdot \frac{g}{h} x^{p-1}\right)^{p-1} \left(-\frac{g}{h} x^{p-1}\right) - 1 < 0.$$

The function $c : [0, \tilde{x}] \rightarrow (0, 1]$ is continuous. Since $[0, \tilde{x}]$ is compact, there exists $x_0 \in [0, \tilde{x}]$ such that

$$\forall x \in [0, \tilde{x}] : 0 < c(x_0) \leq c(x)$$

hence $c'' = c(x_0)$ is the bound we need. ■

The following theorem gives us the desired upper and lower estimates for $d(x)$.

Theorem 13 *Let $p > 1$ and $d(x)$ be the nonnegative solution of (17). Then for any $x_0 \in \mathbb{R}_+$ there exist $0 < c_2 < c_1$ such that for $x \in [0, x_0]$*

$$c_2 \cdot x^p \leq d(x) \leq c_1 \cdot x^p.$$

Proof. By (18) we have $d(x) = x^p c(x)/h$. Let \tilde{x} be as in (19), then by Lemma 12 we immediately obtain:

$$\tilde{c}_2 \cdot x^p \leq d(x) \leq \tilde{c}_1 \cdot x^p, \quad x \in [0, \tilde{x}]$$

for some $0 < \tilde{c}_2 < \tilde{c}_1$. For $x \in [\tilde{x}, x_0]$ by Theorem 11 we have:

- $d(x)$ is positive, thus there exists $\bar{c}_2 > 0$ such that $d(x) \geq \bar{c}_2 x^p$;
- $d(x) < 2\gamma x^p/B$, thus there exists $\bar{c}_1 > \bar{c}_2$ such that $d(x) \leq \bar{c}_1 \cdot x^p$.

Taking $c_1 = \max\{\tilde{c}_1, \bar{c}_1\}$ and $c_2 = \min\{\tilde{c}_2, \bar{c}_2\}$ we obtain our assertion. ■

4.2.2 The case $p < 1$

Consider the equation (17) for $0 < p < 1$:

$$\begin{aligned} (x - g \cdot d)^p &= h \cdot d \\ x - g \cdot d &= h^{\frac{1}{p}} \cdot d^{\frac{1}{p}}. \end{aligned} \tag{20}$$

Similar considerations as before for $p > 1$ lead us to a hypothesis that

$$d(x) = \frac{x}{g} \left(1 - c(x) \cdot \frac{h^{\frac{1}{p}}}{g^{\frac{1}{p}}} \cdot x^{\frac{1}{p}-1}\right) \tag{21}$$

where $c(x)$ is a bounded positive function. Substituting (21) for d in (20) we obtain the following implicit equation for $c(x)$

$$c(x) = \left(1 - c(x) \cdot \frac{h^{\frac{1}{p}}}{g^{\frac{1}{p}}} \cdot x^{\frac{1}{p}-1}\right)^{\frac{1}{p}}.$$

As in the case $p > 1$ let us define, for given g, h and $p < 1$

$$\tilde{x} = \left(\frac{g}{h}\right)^{\frac{1}{1-p}}. \tag{22}$$

Now, for $p < 1$ we have an analogue of Lemma 12.

Lemma 14 *Let us consider the equation*

$$F(x, c(x)) = \left(1 - c(x) \cdot \frac{h^{\frac{1}{p}}}{g^{\frac{1}{p}}} \cdot x^{\frac{1}{p}-1}\right)^{\frac{1}{p}} - c(x) = 0.$$

This equation has exactly one solution $c(x) \in [0, 1]$ which is continuous on $[0, \tilde{x}]$ and

$$\forall x \in [0, \tilde{x}] \exists c', c'' : (0 < c'' < c' = 1) \wedge (c'' \leq c(x) \leq c').$$

From (21) and the above lemma we obtain the following bounds for $d(x)$.

Theorem 15 *Let $p < 1$ and $d(x)$ be the nonnegative solution of (17). Then for any $x_0 \in \mathbb{R}_+$ there exist $0 < c_2 < c_1$ such that for $x \in [0, x_0]$*

$$c_2 \cdot x \leq d(x) \leq c_1 \cdot x.$$

Proof. The proof is similar to the proof of Theorem 13. From Lemma 14 it follows that there exists $\eta > 0$ such that for $x \in (0, \eta]$ holds

$$\frac{x}{2g} \leq d(x) \leq \frac{x}{g}.$$

Since $d(x)$ is positive on $[\eta, x_0]$ we can find the desired c_2 and c_1 .

4.2.3 The case $p = 1$

The equation (17) for $p = 1$ has a form

$$x - g \cdot d = h \cdot d$$

thus we obtain

$$d(x) = \frac{x}{g+h}. \tag{23}$$

Remark 16 *In further analysis we will refer to Theorem 15 with $p = 1$ (as it holds for $p \leq 1$). Thus it is sufficient to consider two cases, $p > 1$ and $p \leq 1$.*

5 Lower bound under NPC condition

We assume

- $[a, b] = [-1, 1]$,
- $M = \sup\{|f(x)| : x \in [-1, 1] \cap \text{dom } f\}$,
- $\mathcal{D} = \mathcal{D}(f, cM)$,
- f satisfies NPC(p', γ', s, β)-right (with $s \in S$, $\gamma' \geq \gamma$, $p' \leq p$).

Therefore $\mathcal{D} \cap \{x + iy \mid x \in [s, s + \beta]\} \subset \mathcal{D}_{\gamma'}^{p'} \cap \{x + iy \mid x \in [s, s + \beta]\}$. For the sake of simplicity of calculations we assume that $s = 0$.

The case of functions satisfying NPC($p, \gamma, 0, \beta$)-left is analogous and will not be considered separately.

5.1 Estimation of the number of proper intervals from below

We are interested in the number of proper intervals generated in the segment $[s, \beta]$ by the Petras algorithm (computing the integral attaining global precision ε on $[a, b]$).

Assume that Petras' algorithm terminates and we obtain to the left of s proper intervals $[\alpha_i, \alpha_{i+1}]$ for $i = 0, \dots, n$ such that

$$s < \alpha_0 < \alpha_1 < \dots < \alpha_n, \quad \text{with} \quad \alpha_{n-1} < \beta \leq \alpha_n, \quad (24)$$

where $\alpha_0 \leq \varepsilon/(2M)$.

Since $s = 0$ thus β can be treated as a distance from the singular point.

5.1.1 Case $p > 1$

Lemma 17 *Assume that $p > 1$ and f satisfies NPC($p, \gamma, 0, \beta$)-right. Then, for sufficiently small (ε/M) , the number of proper intervals intersecting the segment $[0, \beta]$ is $\Omega\left(\left(\frac{M}{\varepsilon}\right)^{p-1}\right)$. Therefore we have*

$$\mathcal{Z}(\mathcal{D}, \varepsilon) = \Omega\left(\left(\frac{M}{\varepsilon}\right)^{p-1}\right).$$

Proof. Let α_i for $i = 0, \dots, n$ be as in (24).

Let $\varepsilon/(2M) < \beta$. From Theorem 13 (with $h = B/(2\gamma')$ and $g = g_L = (A-1)/2$) it follows that there exists $c_1 > 0$, such that

$$d_L(x) \leq c_1 \cdot x^p,$$

therefore

$$\alpha_{i+1} - \alpha_i \leq d_L(\alpha_i) \leq c_1 \cdot \alpha_i^p, \quad i = 0, \dots, n-1$$

It is easy to see that if $x(t)$ is a solution of $x' = c_1 \cdot x^p$ with an initial condition $x(0) = \varepsilon/(2M)$, then

$$x(i) \geq \alpha_i, \quad i = 0, \dots, n.$$

The solution of $x' = c_1 \cdot x^p$ is given by

$$x(t) = \frac{x(0)}{[1 - x(0)^{p-1} c_1 (p-1)t]^{\frac{1}{p-1}}},$$

hence we can calculate the exit time to the right from $[\varepsilon/(2M), \beta]$ (taking $x(t) = \beta$), i.e.,

$$n \geq t = \left(\frac{M}{\varepsilon}\right)^{p-1} \frac{2^{p-1}}{c_1(p-1)} - \frac{1}{c_1(p-1)\beta^{p-1}} = \Omega\left(\left(\frac{M}{\varepsilon}\right)^{p-1}\right).$$

■

5.1.2 Case $p \leq 1$

Lemma 18 *Assume that $p \leq 1$ and f satisfies NPC($p, \gamma, 0, \beta$)-right. Then, for sufficiently small ε/M , the number of proper intervals intersecting the segment $[0, \eta]$, where $\eta = \min\{\tilde{x}, \beta\}$, for \tilde{x} as in (22), is $\Omega(\ln(M/\varepsilon))$. Therefore we have*

$$\mathcal{Z}(\mathcal{D}, \varepsilon) = \Omega\left(\ln \frac{M}{\varepsilon}\right).$$

Proof. Let α_i for $i = 0, \dots, n$ be as in (24). Let $\eta = \min\{\tilde{x}, \beta\}$ and $\varepsilon/(2M) < \eta$. Let k be such that $\alpha_{k-1} < \eta \leq \alpha_k$. Hence k is the number of proper intervals in $[0, \eta]$.

By Theorem 15 (with $h = B/(2\gamma')$ and $g = g_L = (A - 1)/2$) it follows that there exists c_2 such that for $x \in [0, \tilde{x}]$

$$d_L(x) \leq x \left(\frac{1}{g_L} - c_2 \cdot x^{\frac{1}{p}-1} \right) \leq \frac{x}{g_L}.$$

Now, for $\alpha_i \in [0, \eta]$, we have

$$\alpha_{i+1} - \alpha_i \leq d_R(\alpha_i) \leq \frac{\alpha_i}{g_L}$$

thus

$$\alpha_{i+1} \leq \alpha_i \left(1 + \frac{1}{g_L} \right)$$

and, since $\alpha_0 \leq \varepsilon/(2M)$, we obtain

$$\alpha_i \leq \alpha_0 \left(1 + \frac{1}{g_L} \right)^i \leq \frac{\varepsilon}{2M} \left(1 + \frac{1}{g_L} \right)^i.$$

Therefore in particular

$$\eta \leq \frac{\varepsilon}{2M} \left(1 + \frac{1}{g_L} \right)^k$$

and

$$k \geq \frac{\ln \left(\frac{2M\eta}{\varepsilon} \right)}{\ln \left(1 + \frac{1}{g_L} \right)} - 1,$$

thus according to the Definition 9 we obtain our assertion. ■

6 Upper bound from PPC condition

We assume:

- $[a, b] = [-1, 1]$,
- $S = \{s_1, \dots, s_m\} \subset \mathbb{R}$, where $-1 \leq s_1 < \dots < s_m \leq 1$, be a set of points such that the function f is analytic in each open interval (s_i, s_{i+1}) for $i = 1, \dots, m - 1$,
- $M = \text{UpperBound}(f, a, b)$,
- $\mathcal{D} = \mathcal{D}(f, cM)$,
- f satisfies $\text{PPC}(p, \gamma, S, M)$, therefore $\mathcal{D}_{\gamma, S}^p \subset \mathcal{D}$,
- UpperBound , IsAnalytic and ComplexBound are $\text{PPC}(p, \gamma, S, M)$ -compatible for f on $[-1, 1]$.

Observe that the assumption about PPC compatibility of the subroutines UpperBound , IsAnalytic and ComplexBound implies that these routines also satisfy the compatibility condition (Def. 5).

6.1 The modified Petras algorithm

We consider a modified Petras' algorithm (abbreviated MPA) in which a decision if an interval is proper or bad is made using the PPC condition. More precisely, instead of checking condition (4) we use the following

$$\varrho(a_{j-1}, a_j) \subset \mathcal{D}_{\gamma, S}^p, \quad j \in J.$$

Additionally, if MPA decides that it is still running (the length of all bad intervals is greater than $\varepsilon/(2M)$), then it bisects all bad intervals in step 3 in the Petras algorithm before checking condition (5); thus in MPA all bad intervals have the same length.

Notice that any interval accepted by MPA is contained in an interval accepted by Petras' algorithm. Thus we can state:

1. the union of proper intervals generated in Petras' algorithm contains the union proper intervals generated in MPA:

$$\bigcup_{j \in J(\text{MPA})} [a_{j-1}, a_j] \subseteq \bigcup_{j \in J(\text{PA})} [a_{j-1}, a_j];$$

2. the sum of lengths of proper intervals generated in Petras' algorithm is greater than or equal to the sum of lengths of proper intervals generated in MPA:

$$\sum_{j \in J(\text{PA})} |a_j - a_{j-1}| \geq \sum_{j \in J(\text{MPA})} |a_j - a_{j-1}|,$$

where $J(\text{PA})$ and $J(\text{MPA})$ are sets of indices of proper intervals in Petras' algorithm and MPA, respectively; it is obvious that for bad intervals the converse inequality holds;

3. if MPA terminates then Petras' algorithm terminates as well, since by (ii) we have

$$\sum_{j \notin J(\text{PA})} |a_j - a_{j-1}| \leq \sum_{j \notin J(\text{MPA})} |a_j - a_{j-1}| \leq \frac{\varepsilon}{2M}.$$

Thus we have proved the following lemma.

Lemma 19 *MPA terminates. The cost of MPA provides an upper bound for the cost of Petras' algorithm.*

6.1.1 Preliminary estimates for MPA

Our goal is to find a lower bound for the distance to S of the set of proper intervals obtained in MPA.

Lemma 20 *There exists constant T_0 such that the distance to S of the set of proper intervals obtained in MPA is greater than or equal to $\varepsilon/(MT_0)$. The constant T_0 depends on S , A , c , γ and p .*

Proof. In this proof we will write x for both the coordinate x and the distance $\text{dist}(x, S)$, and the precise meaning should be clear from the context.

Consider two cases:

1. For $p > 1$ the shape of $\mathcal{D}_{\gamma, S}^p$ is as in Figure 4.

From Theorem 13 we know that:

$$\begin{aligned} \text{interval } [x, x+d] \text{ is proper, iff } d \leq d_L(x), \text{ where } d_L(x) \leq c_{1,L} \cdot x^p, \\ \text{interval } [x-d, x] \text{ is proper, iff } d \leq d_R(x), \text{ where } c_{2,R} \cdot x^p \leq d_R(x). \end{aligned}$$

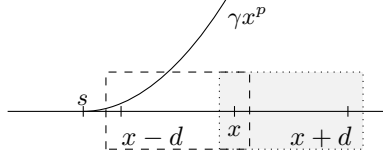


Figure 4: An interval $[x, x + d]$ is proper, but $[x - d, x]$ is bad.

Consider x (as in Figure 4). Since $[x, x + d]$ is proper and $[x - d, x]$ is bad, we have:

$$c_{2,R} \cdot x^p \leq d_R(x) < d \leq d_L(x) \leq c_{1,L} \cdot x^p. \quad (25)$$

Hence, at any stage of the algorithm, all points x which are the closest to the points from S satisfy the estimate (recall that all bad intervals have the same length):

$$\left(\frac{d}{c_{1,L}}\right)^{\frac{1}{p}} \leq x < \left(\frac{d}{c_{2,R}}\right)^{\frac{1}{p}}.$$

Thus, if x_1 and x_2 are the closest to some singular point, then

$$0 < \left(\frac{c_{2,R}}{c_{1,L}}\right)^{\frac{1}{p}} < \frac{x_1}{x_2} < \frac{\left(\frac{d}{c_{2,R}}\right)^{\frac{1}{p}}}{\left(\frac{d}{c_{1,L}}\right)^{\frac{1}{p}}} = \left(\frac{c_{1,L}}{c_{2,R}}\right)^{\frac{1}{p}} < +\infty.$$

2. For $p \leq 1$ from Theorem 15 we know that for sufficiently small x

interval $[x, x + d]$ is proper, iff $d \leq d_L(x)$, where $d_L(x) \leq x/g_L$,

interval $[x - d, x]$ is proper, iff $d \leq d_R(x)$, where there exists g_R^+ such that

$$\frac{x}{g_R^+} < x \left(\frac{1}{g_R} - c_{1,R}x^{\frac{1}{p}-1}\right) \leq d_R(x).$$

Since $[x, x + d]$ is proper and $[x - d, x]$ is bad, we have:

$$\frac{x}{g_R^+} < d \leq \frac{x}{g_L}. \quad (26)$$

Hence, at any stage of the algorithm, all points x which are the closest to the points from S satisfy the estimate (recall that all bad intervals have the same length):

$$d \cdot g_L \leq x < d \cdot g_R^+.$$

Thus if we have points x_1 and x_2 which are the closest to some singular point, then

$$0 < \frac{A-1}{A+1} \approx \frac{g_L}{g_R^+} \leq \frac{x_1}{x_2} \leq \frac{d \cdot g_R^+}{d \cdot g_L} = \frac{g_R^+}{g_L} \approx \frac{A+1}{A-1} < +\infty.$$

Hence for any $p > 0$ at any stage of the MPA, for any points x_i, x_j which are the closest to a singular point from S :

$$0 < \frac{x_i}{x_j} \leq T < +\infty, \quad (27)$$

where T is a constant depending on p, γ, S , but independent of ε and M . Because (27) holds for the minimal and maximal distance denoted by x_{\min}, x_{\max} we have

$$x_{\max} \leq T \cdot x_{\min}. \quad (28)$$

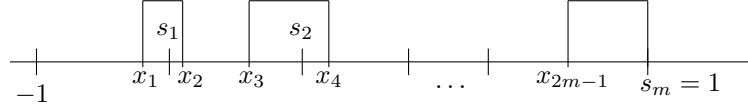


Figure 5: Singularities s_1, \dots, s_m and blocks of bad intervals between x_{2i-1} and x_{2i} for $i = 1, \dots, m$.

Bad intervals are in the neighbourhood of any singular point from S , so there exists $I \leq 2m$ such that for any $1 \leq i \leq I$ a point x_i is an end of the connected block of proper intervals (see Figure 5).

While the algorithm is running

$$\frac{\varepsilon}{2M} < \sum_{1 \leq i \leq I} x_i \leq \sum_{1 \leq i \leq I} x_{\max}$$

and from (28) we have

$$\frac{\varepsilon}{2M} < \sum_{1 \leq i \leq I} T \cdot x_{\min} \leq 2m \cdot T \cdot x_{\min}$$

obtaining

$$x_{\min} \geq \frac{\varepsilon}{4mMT}. \quad (29)$$

Let us stress that (29) holds as long as the stopping condition in MPA is not satisfied. We need to estimate how far we can go in the last stage.

Let x_1 be a point from the proper interval, which is the closest to some $s \in S$ from right or left. For simplicity we will assume that $s < x_1$, the other case is analogous. Let d be the length of bad intervals, which will be now divided by 2. Let x_2 (with $s < x_2 < x_1$) be such that $[x_2, x_1]$ is covered by proper intervals of length $d/2$. Therefore interval $[x_2, x_2 + d/2]$ is proper.

1. Case $p > 1$. By (25), (29) and Theorem 13 we have the following estimates

$$\left. \begin{array}{l} c_{2,R} \cdot x_1^p < d \\ \frac{d}{2} < d_L(x_2) \leq c_{1,L} \cdot x_2^p \end{array} \right\} \implies x_2 \geq \left(\frac{1}{2} \cdot \frac{c_{2,R}}{c_{1,L}} \right)^{1/p} x_1 \geq \left(\frac{1}{2} \cdot \frac{c_{2,R}}{c_{1,L}} \right)^{1/p} \frac{\varepsilon}{4mMT}.$$

2. Case $p \leq 1$. By (26), (29) and Theorem 15 we obtain

$$\left. \begin{array}{l} \frac{x_1}{g_R^+} < d \\ \frac{d}{2} < d_L(x_2) \leq \frac{x_2}{g_L} \end{array} \right\} \implies x_2 \geq \frac{1}{2} \cdot \frac{g_L}{g_R^+} \cdot x_1 \geq \frac{1}{2} \cdot \frac{g_L}{g_R^+} \cdot \frac{\varepsilon}{4mMT}.$$

Observe that in both above cases we have obtained

$$x_2 \geq \frac{\varepsilon}{MT_0},$$

for some constant T_0 depending on S, A, c, γ and p ■

6.2 Estimation of the number of proper intervals from above

Lemma 21

$$\mathcal{Z}(\mathcal{D}, \varepsilon) = \mathcal{O}\left(\left(\frac{M}{\varepsilon}\right)^{p-1}\right), \quad \text{if } p > 1,$$

$$\mathcal{Z}(\mathcal{D}, \varepsilon) = \mathcal{O}\left(\ln \frac{M}{\varepsilon}\right), \quad \text{if } p \leq 1.$$

Proof. For any s_i , $i = 1, \dots, m$ we will count the number of proper intervals created by MPA to right of s_i that cover a segment $[s_i^R, \beta]$, where $\beta = (s_i + s_{i+1})/2$ if $i < m$ and $\beta = 1$ for $i = m$; and s_i^R is the nearest point from the proper intervals to the right of s_i . By the symmetry the estimate will be also good for the intervals to the left of points from S .

Let us fix s_i . We can assume that $s = 0$ and $\beta \leq 2$. We consider the intervals to right of s .

Let us assume that $[\alpha_i, \alpha_{i+1}]$ is obtained during MPA by bisecting an interval $[u, v]$. Since $[u, v]$ was bisected, it was bad, thus

$$v - u > d_R(v). \quad (30)$$

From Theorem 13 for $p > 1$ and Theorem 15 for $p \leq 1$ we know that $d_R(x) \geq w(x)$, where $w(x) = c_2 x^p$ if $p > 1$ and $w(x) = c_2 x$, if $p \leq 1$, for some $c_2 > 0$ and $c_2 < 1$ if $p \leq 1$.

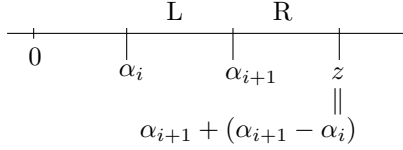


Figure 6: The case (L)

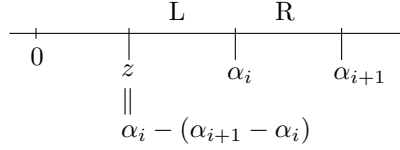


Figure 7: The case (R)

Two cases are possible as a result of the bisection:

(L) $[\alpha_i, \alpha_{i+1}]$ is the left part in $[\alpha_i, z]$ (see Figure 6), where $z = \alpha_{i+1} + (\alpha_{i+1} - \alpha_i)$, and from (30) we have

$$\begin{aligned} z - \alpha_i = 2(\alpha_{i+1} - \alpha_i) &> d_R(\alpha_{i+1} + (\alpha_{i+1} - \alpha_i)) \\ &\geq w(\alpha_{i+1} + (\alpha_{i+1} - \alpha_i)) > w(\alpha_{i+1}). \end{aligned}$$

(R) $[\alpha_i, \alpha_{i+1}]$ is the right part in $[z, \alpha_{i+1}]$ (see Figure 7) so for both $z > 0$ and $z \leq 0$ it holds (by (30))

$$2(\alpha_{i+1} - \alpha_i) = \alpha_{i+1} - z > d_R(\alpha_{i+1}) \geq w(\alpha_{i+1}).$$

Therefore in both cases the following estimate holds:

$$\alpha_{i+1} - \alpha_i > \frac{1}{2} \cdot w(\alpha_{i+1}). \quad (31)$$

Let

$$\alpha_0 < \alpha_1 < \dots < \alpha_n, \quad \alpha_n \geq \beta$$

be the end points of proper intervals obtained by MPA to the right of $s = 0$. From Lemma 20 it follows that

$$\alpha_0 \geq \frac{\varepsilon}{MT_0}. \quad (32)$$

Substituting α_{n-i} by y_i ($0 \leq i \leq n$) in (31) we obtain

$$y_{i+1} \leq y_i - \frac{1}{2} \cdot w(y_i).$$

It is easy to see that

$$y_i \leq y(i),$$

where $y(t)$ is a solution of

$$y' = -\frac{1}{2} \cdot w(y) = -ay_i^{p^*}. \quad (33)$$

with $p^* = \max(1, p)$, $a = 1/2 \cdot c_2$ and $y(0) = y_0 = \alpha_n$.

For $p > 1$ the solution of (33) is of the form

$$y(t) = \frac{y(0)}{(1 + a \cdot y(0)^{p-1}(p-1)t)^{\frac{1}{p-1}}}$$

and setting $y(0) = \beta$ and $y(t) = \varepsilon/(MT_0)$ (compare (32)) we calculate

$$\begin{aligned} t &= \frac{\left(\frac{MT_0\beta}{\varepsilon}\right)^{p-1} - 1}{a\beta^{p-1}(p-1)} \\ &= \frac{(MT_0)^{p-1}}{\varepsilon^{p-1}} \cdot \frac{1}{a(p-1)} - \frac{1}{a(p-1)\beta^{p-1}}. \end{aligned}$$

Observe that t is an upper bound for n , hence

$$n \leq \left(\frac{M}{\varepsilon}\right)^{p-1} \left(\frac{T_0^{p-1}}{a(p-1)}\right).$$

By taking into account all points in S and the proper intervals on both side we obtain

$$\mathcal{Z}(\mathcal{D}, \varepsilon) \leq \left(\frac{M}{\varepsilon}\right)^{p-1} \left(\frac{2mT_0^{p-1}}{a(p-1)}\right).$$

This finishes the proof for the case $p > 1$.

For $p \leq 1$ the solution of (33) is of the form

$$y(t) = e^{-at}y(0),$$

and setting $y(0) = \beta$ and $y(t) = \varepsilon/(MT_0)$ (compare (32)) we calculate

$$\begin{aligned} t &= \frac{1}{a} \ln \frac{\beta T_0 M}{\varepsilon} \\ &\leq \frac{1}{a} \ln \frac{2T_0 M}{\varepsilon}. \end{aligned}$$

Observe that t is an upper bound for n , hence

$$n \leq \frac{1}{a} \ln \frac{M}{\varepsilon} + \frac{1}{a} \ln(2T_0).$$

By taking into account all points in S and the proper intervals on both side we obtain

$$\mathcal{Z}(\mathcal{D}, \varepsilon) \leq \frac{m}{a} \ln \frac{M}{\varepsilon} + \frac{m}{a} \ln(2T_0).$$

This finishes the proof for the case $p \leq 1$. ■

7 The cost of Petras' algorithm

Finally we can state a theorem estimating the cost of Petras' algorithm for functions satisfying PPC and NPC conditions. As we mentioned in the introduction by the cost we understand the number of evaluations of an integrand at the nodes produced by the algorithm. Therefore this is not a comprehensive evaluation, because we neglect the cost of checking analyticity, calculating bounds and the precision of arithmetic operations.

Theorem 22 *Assume that $S = \{s_1, \dots, s_m\}$, $m \geq 1$. Let $M = \text{UpperBound}(f, a, b)$. Consider functions satisfying the following conditions*

- $\text{PPC}(p, \gamma, S, M)$
- $\text{NPC}(p, \gamma', s_0, \beta)$ for some $s_0 \in S$,
- UpperBound , IsAnalytic and ComplexBound are $\text{PPC}(p, \gamma, S, M)$ -compatible for f on $[a, b]$.

Then the cost of Petras' algorithm for such f is

$$\Theta \left(\left| \ln \frac{M}{\varepsilon} \right| \left(\frac{M}{\varepsilon} \right)^{p-1} \right), \quad \text{for } p > 1,$$

$$\Theta \left(\ln^2 \frac{M}{\varepsilon} \right), \quad \text{for } p \leq 1.$$

The constants in Θ depend S , p , γ , γ' , A and c .

Proof. First notice that by Lemmas 17 and 18 for $s_0 \in S$ the number of proper intervals created during the execution of the algorithm is

$$\Omega((M/\varepsilon)^{p-1}), \quad \text{for } p > 1,$$

$$\Omega(\ln(M/\varepsilon)), \quad \text{for } p \leq 1.$$

In Lemma 21 we obtained the upper bound for the number of proper intervals created during the execution of the algorithm:

$$\mathcal{O}((M/\varepsilon)^{p-1}), \quad \text{for } p > 1,$$

$$\mathcal{O}(\ln(M/\varepsilon)), \quad \text{for } p \leq 1.$$

Hence we obtain the assertion of the theorem. ■

8 Functions giving rise to the cost $\Theta(|\ln \varepsilon|/\varepsilon^{p-1})$

In Section 4.1 we have shown that function $f(z) = \sin(1/z)$ satisfies $\text{PPC}(2, \gamma, \{0\}, 1)$ and $\text{NPC}(2, \gamma, 0, 1)$. The cost of Petras' algorithm for such functions is $\Theta(|\ln \varepsilon|/\varepsilon)$. In this section we give examples of functions for which the cost is $\Theta(|\ln \varepsilon|/\varepsilon^{p-1})$, for any $p > 1$.

Theorem 23 *Let $S = \{0\}$ and let $f(z) = \sin(1/z^{p-1})$, where $p \in \mathbb{N}$, $p > 1$. Then for any $c > 1$ there exists $\gamma > 0$ such that for any $\alpha, \beta \in [-1, 1]$*

$$\varrho(\alpha, \beta) \subset \mathcal{D}_\gamma^p \implies f \text{ is analytic on } \varrho(\alpha, \beta) \text{ and}$$

$$\sup\{|f(z)| : z \in \varrho(\alpha, \beta)\} \leq c \cdot \sup\{|f(x)| : x \in [-1, 1] \cap \text{dom } f\}.$$

Proof. Let $\gamma \leq 1$. If $\varrho(\alpha, \beta) \subset \mathcal{D}_\gamma^p$ then it is obvious that f is analytic on $\varrho(\alpha, \beta)$. Thus it is enough to check the second part of the conjunction.

Let us set $n = p - 1$. Assume $z = x + iy \in D_\gamma^n \cap \{z : |\operatorname{Re}(z)| \leq 1\}$. Let $r^2 = x^2 + y^2$. As in the proof of Theorem 7, we have

$$\left| \sin \frac{1}{z^n} \right| \leq \frac{1}{2} \left(\exp \left| \operatorname{Im} \left(\frac{1}{z^n} \right) \right| + \exp \left(- \left| \operatorname{Im} \left(\frac{1}{z^n} \right) \right| \right) \right).$$

By (8) we have

$$\operatorname{Im} \left(\frac{1}{z^n} \right) = \frac{1}{r^{2n}} \left(\binom{n}{1} x^{n-1} y - \binom{n}{3} x^{n-3} y^3 + \dots \right)$$

and it is easy to see that for $z \in D_\gamma^n \cap \{z : |\operatorname{Re}(z)| \leq 1\}$ and for some constant $\Phi = \Phi(n)$,

$$\left| \operatorname{Im} \left(\frac{1}{z^n} \right) \right| \leq \frac{\Phi}{r^{2n}} |x^{n-1} y| \leq \Phi \frac{\gamma |x|^{n-1} |x|^p}{|x|^{2n}} = \Phi \gamma.$$

For $z \in D_\gamma^n \cap \{z : |\operatorname{Re}(z)| \leq 1\}$ we have (because $x \mapsto x + \frac{1}{x}$ is increasing for $x > 1$)

$$\left| \sin \frac{1}{z^n} \right| \leq \frac{1}{2} (\exp(\Phi \gamma) + \exp(-\Phi \gamma)) \rightarrow 1, \quad \text{as } \gamma \rightarrow 0.$$

Thus there exists $c > 1$ such that taking γ sufficiently small we get

$$\sup \left\{ \left| \sin \frac{1}{z^{p-1}} \right| : z \in D_\gamma^p \cap \{z : |\operatorname{Re}(z)| \leq 1\} \right\} < c \cdot \sup \left\{ \left| \sin \frac{1}{x^{p-1}} \right| : x \in [-1, 1] \right\}.$$

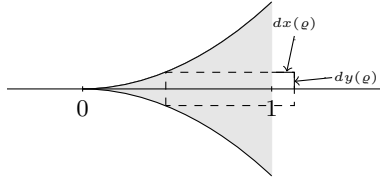


Figure 8: Rectangle $\varrho(\alpha, \beta)$ does not have to be entirely contained in $D_\gamma^p \cap \{z : |\operatorname{Re}(z)| \leq 1\}$, but for small γ the projecting part $(\varrho(\alpha, \beta) \setminus (D_\gamma^p \cap \{z : |\operatorname{Re}(z)| \leq 1\})) = 2(dx(\varrho) \times dy(\varrho))$ is small.

Note that (see Figure 8) the rectangle $\varrho(\alpha, \beta)$ does not have to be entirely contained in the region $D_\gamma^p \cap \{z : |\operatorname{Re}(z)| \leq 1\}$, but the width and height of the projecting area linearly depend on γ ($dx(\varrho) < \gamma(A-1)/B$ and $dy(\varrho) < \gamma$ — see Theorem 11) and the function is continuous in the wide neighbourhood of 1, therefore further reducing γ we obtain

$$\sup \left\{ \left| \sin \frac{1}{z^{p-1}} \right| : z \in \varrho(\alpha, \beta) \right\} \leq c \cdot \sup \left\{ \left| \sin \frac{1}{x^{p-1}} \right| : x \in [-1, 1] \right\}.$$

■

Theorem 23 says that for $z \mapsto \sin(1/z^{p-1})$ there exists a region $\mathcal{D}_\gamma^p \cap \{z : |\operatorname{Re}(z)| \leq 1\}$ where the function is analytic and appropriately bounded. The next theorem says that there exists a region of the same shape as before, but such that (on the boundary of this region close to the singular point) the values of the function are arbitrarily large.

Theorem 24 *Let $S = \{0\}$ and let $f(z) = \sin(1/z^{p-1})$, where $p \in \mathbb{N}$, $p > 1$. Then for any $c > 1$ there exist $\gamma > 0$ and $\eta > 0$ such that for any $\alpha \in [-\eta, \eta]$ and $\beta \in (\alpha, 1]$*

$$\begin{aligned} \varrho(\alpha, \beta) \not\subset \mathcal{D}_\gamma^p &\implies f \text{ is not analytic on } \varrho(\alpha, \beta) \text{ or} \\ &\sup\{|f(z)| : z \in \varrho(\alpha, \beta)\} > c \cdot \sup\{|f(x)| : x \in [-1, 1] \cap \operatorname{dom} f\}. \end{aligned}$$

Proof. Let us fix $\gamma > 0$ and consider points $z = x + iy \in \partial D_\gamma^p \cap \{z : |\operatorname{Re}(z)| \leq 1\}$. We will restrict our attention to $x > 0$ and $y = \gamma x^p > 0$. By (8), for sufficiently small x and for some constants $\Phi, \Phi' > 0$ there is

$$\begin{aligned} \operatorname{Im}\left(\frac{1}{z^{p-1}}\right) &= \frac{1}{r^{2(p-1)}} \left(\binom{p-1}{1} x^{p-2} y - \binom{p-1}{3} x^{p-4} y^3 + \dots \right) \\ &\geq \frac{\gamma \Phi x^{2(p-1)}}{r^{2(p-1)}} \\ &= \frac{\gamma \Phi x^{2(p-1)}}{x^{2(p-1)} (1 + \gamma^2 x^{2(p-1)})^{p-1}} \\ &= \frac{\gamma \Phi}{(1 + \gamma^2 x^{2(p-1)})^{p-1}} \\ &\geq \gamma \Phi'. \end{aligned}$$

Observe that (compare the proof of Theorem 9), since $x \mapsto x - 1/x$, we have

$$\begin{aligned} \inf \left\{ \left| \sin \frac{1}{z^l} \right| : z \in \mathcal{W}_\gamma \right\} &\geq \frac{1}{2} \left(\exp \left| \operatorname{Im} \left(\frac{1}{z^l} \right) \right| - \exp \left(- \left| \operatorname{Im} \left(\frac{1}{z^l} \right) \right| \right) \right) \\ &\geq \frac{1}{2} (\exp(\gamma \Phi') - \exp(-\gamma \Phi')), \end{aligned}$$

where $\mathcal{W}_\gamma = \partial D_\gamma^p \cap \{z : |\operatorname{Re}(z)| \leq \eta\} \setminus \{0\}$.

Now, for any $c > 1$ it is enough to take $\gamma > 0$ big enough to have $\inf_{z \in \mathcal{W}_\gamma} |\sin(1/z^l)| > c$. ■

From the above theorems it follows that functions $f(z) = \sin(1/z^{p-1})$ for $p \in \mathbb{N}$, $p > 1$ satisfy the assumptions of Theorem 22, thus their cost scales as $|\ln \varepsilon|/\varepsilon^{p-1}$.

9 Appendix: asymptotic rate of growth

Here we recall the notation Ω , \mathcal{O} , Θ .

Definition 9 Let $f, g : \mathbb{R} \rightarrow \mathbb{R}_+$.

1. We say that f is at least of order g , if there exist $\varepsilon_0 > 0$ and $c > 0$, such that:

$$\forall 0 < x \leq \varepsilon_0 : f(x) \geq c \cdot g(x).$$

Notation: $f(x) \in \Omega(g(x))$.

2. We say that f is at most of order g , if there exist $\varepsilon_0 > 0$ and $c > 0$, such that:

$$\forall 0 < x \leq \varepsilon_0 : f(x) \leq c \cdot g(x).$$

Notation: $f(x) \in \mathcal{O}(g(x))$.

3. We say that f is exactly of order g , if there exist $\varepsilon_0 > 0$, $c_1 > 0$ and c_2 , such that:

$$\forall 0 < x \leq \varepsilon_0 : c_1 \cdot g(x) \leq f(x) \leq c_2 \cdot g(x).$$

Notation: $f(x) \in \Theta(g(x))$.

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