Rigorous numerics for dissipative Partial Differential Equations II. Periodic orbit for the Kuramoto-Sivashinsky PDE - a computer assisted proof.

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This is a corrected version: The bounds for period of the orbit in Theorem 9 in the printed version are wrong. I have made a mistake rewriting them from the log file of my program into the paper. This is corrected here.

Abstract

We present a method of self-consistent a-priori bounds, which allows us to study rigorously the dynamics of dissipative PDEs. As an application we present a computer assisted proof of the existence of a periodic orbit for the Kuramoto-Sivashinsky equation

$$u_t = (u^2)_x - u_{xx} - \nu u_{xxxx}, \quad u(t,x) = u(t,x+2\pi), \quad u(t,x) = -u(t,-x),$$

where $\nu = 0.127$.

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1 Introduction

The goal of this paper is to develop further the method of self-consistent a-priori bounds for dissipative PDEs. The proposed approach, introduced in [ZM], enables a rigorous computer assisted study of the dynamics of dissipative PDEs. The method consists of two basic parts: a reduction of a PDE to a finite dimensional differential inclusion and a finite dimensional part, which is based

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on standard finite dimensional tools from dynamics. While in paper [ZM] the finite dimensional part was based on the Conley index and treated primarily as a question of the existence of fixed points, in this paper we focus on Poincaré maps and finite-dimensional tools for the detection of periodic orbits.

The method of self-consistent bounds is similar in spirit to the Cesari method introduced in [C]. In fact, in the context where the Cesari method was used originally, namely for proofs of the existence of solutions to boundary value problems for second order ODEs, our method is slightly stronger, as we drop one of Cesari's conditions (see [ZM, Section 2.4] for more details). We would like to stress here that from our point of view the Cesari method is static, as one can obtain from it steady states for PDEs, only.

In our method we look for solutions of dissipative PDEs, whose Fourier expansions are converging fast enough. This means that we restrict ourselves to regular functions. This may appear to be a limitation of the method but, in fact it is not, as bounded solutions of PDEs, which are defined on an unbounded time interval, are usually very regular, see [FT, HG, K].

This paper is organized as follows. In Section 2 we present the basics of the method of self-consistent a-priori bounds from [ZM, ZAKS] (with some minor modifications), which sets up the framework for treating all high-dimensional Galerkin projections as a *single* differential inclusion. In Section 3 we discuss the notion of a Poincaré section and a Poincaré return map in the context of self-consistent bounds. In Section 4 we present, as an application of the tools developed in the preceding section, a result about the existence of a periodic orbit for the Kuramoto-Sivashinsky PDE with odd and periodic boundary conditions for $\nu = 0.127$. This result is formulated as Theorem 9. Section 5 contains a description of a computer assisted proof of this theorem. In Section 6 we present a modification of the Lohner algorithm [Lo, Lo1, ZLo], which was used in the rigorous integration of a differential inclusion in the proof of Theorem 9.

We believe that, besides providing some tools for the rigorous detection of periodic orbits for dissipative PDEs, this paper shows that rigorous numerics for these systems is possible, hence one can think, for example, about rigorous numerical shadowing algorithms similar in spirit to the ones developed in [HYC, HYC1, H] (see Subsection 3.2).

1.1 Notation

Let X be a metric space. Let $U \subset V \subset X$, by $\operatorname{int}_V U$ we will denote an interior of U relatively to V. Let X be a vector space and let $Y \subset X$, then by $\operatorname{conv}(Y)$ we will denote the smallest closed convex set containing Y. For a set X by 2^X we denote a set of all subsets of X. For $x \in \mathbb{R}$ we set $|x|_\infty = \max_{i=1,\dots,n} |x_i|$. For $S \subset \mathbb{R}^n$ by $\operatorname{IHull}(S)$ we denote an interval enclosure of a set S, i.e., the smallest set of the form $I_1 \times I_2 \times \cdots \times I_n$, where for $i = 1, \dots, n$, $I_i = [a_i, b_i]$ and $a_i \leq b_i$.

2 The method of self-consistent bounds

In this section we describe the method of self-consistent bounds. The method was introduced in [ZM](see also [ZAKS]) and was linked there to the Conley index. In this paper we will show how other finite dimensional topological tools can be used with this method.

Our method begins with the reduction of the full dynamical system to a lower-dimensional system which can be studied numerically. In particular, we begin with a nonlinear evolution equation in a Hilbert space H (L^2 in our treatment of Kuramoto-Sivashinsky) of the form

$$\frac{du}{dt} = F(u) \tag{1}$$

where the domain of F is dense in H. By a solution of (1) we understand a function $u : [0, t_{max}) \to \mathrm{dom}\,(F)$, such that u is differentiable and (1) is satisfied for all $t \in [0, t_{max})$.

We assume that $\{e_i \mid i = 0, 1, \ldots\}$ forms a complete orthogonal basis for H. In the case of the Kuramoto-Sivashinsky equation $F(u) = Lu + B(u, u)$, where L is the linear part and B is the nonlinear part, the functions $\{e_i\}$ are chosen to be eigenvalues of L.

Fix $m \in \mathbb{N}$ and let

$$P_m : H \to X_m$$

be the orthogonal projection from H onto the finite-dimensional subspace spanned by $\{e_1, e_2, \ldots, e_m\}$. Let

$$Q_m := I - P : H \to Y_m$$

be the complementary orthogonal projection. Finally, let

$$A_k : H \to \mathbb{R}$$

be the orthogonal projection onto the subspace generated by e_k.

Given $u \in H$, let $P_m u = p$ and $Q_m u = q$. Equation (1) can be rewritten as

$$\frac{dp}{dt} = P_m F(p, q) \tag{2}$$

$$\frac{dq}{dt} = Q_m F(p, q) \tag{3}$$

The strategy adopted is fairly simple: study the dynamics of the low-dimensional Galerkin projection (2); and draw conclusions about the dynamics of (1). Before turning to the precise conditions, consider the following heuristic description of our approach.

Let $W \subset X_m$, $V \subset Y_m$ and set $V_j = Q_j(V)$ for $j > m$. Furthermore, given $q_j \in V_j$ assume that $\lim_{j\to\infty} ||q_j|| = 0$. Our only knowledge concerning the higher order modes or "tails" of the solutions is that they project into V.

This implies that our knowledge of the vector field is reduced to the following differential inclusion:

$$\frac{dp}{dt} \in P_m F(p, V)$$

where $p \in W$. Numerical calculations on this equation are used to find topological invariants (the Conley index, the fixed point index) which guarantee the existence of specific dynamics, e.g., fixed points, periodic orbits, symbolic dynamics, positive entropy, etc. It is simultaneously argued that the topological invariant is the same for any j-dimensional Galerkin projection of (1) for $j \geq m$. Thus, the same dynamical object exists for each sufficiently high Galerkin approximation. Finally, it is shown that the limit of these objects leads to the desired dynamics for the full system (1).

2.1 Self-consistent Bounds

As one might expect the orthonormal basis $\{e_i\}$ and the sets W and V must be chosen with care. The first issue that needs to be dealt with is analytic in nature - solutions to the ordinary differential equations must approximate solutions of the partial differential equation. This leads to the following definition.

Definition 1 *Let H be a Hilbert space and let $\{e_1, e_2, \dots\}$ be an orthonormal basis in H. We say that $F : H \supset dom(F) \to H$ is admissible (with respect to the basis $\{e_i\}$) if the following conditions are satisfied for any $i \in \mathbb{N}$*

- $X_i \subset dom(F)$
- $P_i F : X_i \to X_i$ *is a* C^1 *function*

Definition 2 *Assume F is an admissible function.* Let $m, M \in \mathbb{N}$ with $m \leq M$. A compact set $W \subset X_m$ and a sequence of pairs $\{a_k^{\pm} \in \mathbb{R} \mid a_k^- < a_k^+,\ k \in \mathbb{N}\}$ form *self-consistent a priori bounds* for (1) if the following conditions are satisfied:

C1 For $k > M$, $a_k^- < 0 < a_k^+$.

C2 Let $\hat{a}_k := \max |a_k^{\pm}|$ and set $\hat{u} = \sum_{k=0}^{\infty} \hat{a}_k e_k$. Then, $\hat{u} \in H$ and in particular, $\hat{u}$ is bounded in the norm on H ($||\hat{u}|| < \infty$).

C3 The function $u \mapsto F(u)$ is continuous on

$$W \oplus \prod_{k=m+1}^{\infty} [a_k^-, a_k^+] \subset H.$$

Moreover, if we define $f_k = \max_{u \in W \oplus \prod_{k=m+1}^{\infty} [a_k^-, a_k^+]} |A_k F(u)|$ and set $\hat{f} = \sum f_k e_k$, then $\hat{f} \in H$. In particular, $||\hat{f}|| < \infty$.

C4a Let $u \in W \oplus \Pi_{k=m+1}^{\infty}[a_k^-, a_k+]$. Then for $k > m$:

$$A_k u = a_k^+ \quad \Rightarrow \quad A_k F(u) < 0, \tag{4}$$
$$A_k u = a_k^- \quad \Rightarrow \quad A_k F(u) > 0. \tag{5}$$

The above definition differs slightly from Def. 2.1 in [ZM], namely condition **C4a** is added. In [ZM] this condition was used as part of *topologically self-consistent bounds* (Def. 2.11), which we are not using here.

At this point the reader may wonder if it is hard to find a tail, i.e., $\{a_k^{\pm}\}$ satisfying **C1**-**C4a**. This turns out be a relatively easy task. For example, for Kuramoto-Sivashinsky PDEs or Navier-Stokes equations (see [ZM, Z]) to satisfy **C1**, **C2** and **C3** it is sufficient to take $a_k^{\pm} = \frac{\pm C}{|k|^s}$ for s large enough. It turns out that with a careful choice of $C = C(W, s)$ condition **C4a** is also satisfied. This choice of tail means that we consider sufficiently regular functions, which may appear as a limitation of our approach. But in fact it is not, as bounded solutions of PDEs, which are defined on an unbounded time interval, are usually very regular, see [FT, HG, K].

Given self-consistent a-priori bounds W and $\{a_k^{\pm}\}$, let

$$V := \prod_{k=m+1}^{\infty} [a_k^-, a_k^+] \subset Y_m.$$

Our goal is to numerically solve (2) on W and draw conclusions about the dynamics of (1) on the set $W \oplus V \subset H$. To do this we will make use of the following results, the first two of them are obvious:

Lemma 1 *Given self-consistent a priori bounds W and $\{a_k^{\pm}\}$, $W \oplus V$ is a compact subset of H.*

Lemma 2 *Given self-consistent a priori bounds W and $\{a_k^{\pm}\}$, $W \oplus V$, then*

$$\lim_{n\to\infty} Q_n(F(u)) = 0, \quad \textit{uniformly for } u \in W \oplus V$$

The following Proposition was proved in [ZM, Prop. 2.4]

Lemma 3 *Let W and $\{a_k^{\pm}\}$ be self-consistent bounds for (1). A function $a : [0, T] \to W \oplus V$ given by*

$$a(t) := \sum_{k=0}^{\infty} a_k(t) e_k$$

is a solution to (1), if and only if, for each $k \in \mathbb{N}$ and all $t \in [0, T]$

$$\frac{da_k}{dt} = A_k F(a). \tag{6}$$

Lemma 4 *Let W and $\{a_k^{\pm}\}$ be self-consistent bounds for (1). Let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}\cup\{\infty\}$ be any sequence. Assume that for all k $x_k : [t_1,t_2] \to W\oplus V$ is a solution of*

$$\frac{dp}{dt} = P_{n_k}(F(p)), \qquad p(t)\in X_{n_k}. \tag{7}$$

Then the family of functions $\{x_k\}$ is relatively compact i.e., every sequence contains a convergent subsequence.

Proof: From Lemma 1 it follows that set $W\oplus V$ is compact, hence by the Ascoli-Arzela lemma it is enough to show that the functions x_k are equicontinuous.

For this end observe that $|x_k'|$ is bounded from above by $\sup_{x\in W\oplus V}|F(x)|$, which is bounded due to the compactness of $W\oplus V$ and condition **C3**. ∎

Lemma 5 *Let W and $\{a_k^{\pm}\}$ be self-consistent bounds for (1). Let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ be a sequence, such that $\lim_{k\to\infty} n_k = \infty$. Assume that, for all k, $x_k : [t_1,t_2] \to W\oplus V$ is a solution of*

$$\frac{dp}{dt} = P_{n_k}(F(p)), \qquad p(t)\in X_{n_k}. \tag{8}$$

Then there exists a convergent subsequence
$\lim_{l\to\infty} x_{k_l} = x^*$, where $x^* : [t_1,t_2] \to W\oplus V$ and the convergence is uniform on $[t_1,t_2]$. Moreover, x^* satisfies (1).

Proof: The existence of a convergent subsequence follows from Lemma 4. Without any loss of generality we can assume that the whole sequence $\{x_k\}$ converges uniformly to x^*. Obviously x^* is continuous.

We will show that, for all $i\in\mathbb{N}$ and $t\in[t_1,t_2]$,

$$P_i x^*(t) = P_i x^*(t_1) + \int_{t_1}^{t} P_i F(x^*(s))\, ds. \tag{9}$$

Let us fix an i. Observe that for k big enough (such that $n_k > i$, hence $P_i P_{n_k} = P_i$) we have

$$P_i x_k(t) = P_i x_k(t_1) + \int_{t_1}^{t} P_i F(x_k(s))\, ds. \tag{10}$$

Observe that $P_i x_k$ converges uniformly to $P_i x^*$, hence it remains to show that $\int_{t_1}^{t} P_i(F(x_k(s)))\, ds$ converges uniformly with respect to $t\in[t_1,t_2]$ to
$\int_{t_1}^{t} P_i(F(x^*(s)))\, ds$. This follows immediately from the uniform continuity of F on $W\oplus V$, because by **C3** F is continuous on $W\oplus V$, hence also uniformly continuous on this set.

From differentiation of (9) we obtain

$$\frac{d}{dt} P_i x^*(t) = P_i F(x^*(t)), \tag{11}$$

hence by Lemma 3 it follows that x^* is a solution of (1). ∎

3 Poincaré sections and maps

In this section we assume a **Standing Hypothesis:**
F is admissible, W and $\{a_k^{\pm}\}$ are self-consistent a priori bounds for (1).

We set

$$V := \Pi_{k=m+1}^{\infty}[a_k^-, a_k^+]$$

Definition 3 *Let $\alpha : W \supset dom(\alpha) \to \mathbb{R}$ be a C^1-function defined on some open (with respect to X_m) set. We say $\theta \subset \{x \in (\text{int}_{X_m} W) \oplus V \mid \alpha(P_m x) = 0\}$ is a* section *for (1) if*

- *$P_m(\theta)$ is an $(m-1)$-dimensional manifold*
- *there exists a set U, such that $U \subset W$, $U = \text{int}_{X_m} U$, $\overline{P_m(\theta)} \subset U$ and a real number $\beta > 0$ such that for all $x \in U$ and $q \in V$*

$$\left| \sum_{k=1}^{m} \frac{\partial \alpha}{\partial x_k} A_k F(x+q) \right| > \beta$$

Let us make a few comments regarding the above definition:

- the section θ is defined in terms of m-first modes, hence $P_n(\theta)$ is a section for the n-th Galerkin projection of (1) for $n \geq m$
- the notion of section depends on the self-consistent a priori bounds $W \oplus V$.

The n-dimensional Galerkin projection of (1), given by

$$\frac{dx}{dt} = P_n F(x), \qquad x \in X_n, \tag{12}$$

due to the admissibility of F, induces a local flow $\varphi_n(t, x_0)$, where $\varphi_n(\cdot, x_0)$ is the unique solution to (12) with an initial condition $x(0) = x_0 \in X_n$.

Definition 4 *Let θ be a section for (1). Then $P_n(\theta)$ is a section for (12) and we define* a first return time to the section θ *function by*

$$t_{n,\theta}(x) = \inf\{t > 0 \mid \varphi_n(t, x) \in P_n(\theta)\}, \qquad x \in X_n \tag{13}$$

Definition 5 *Consider sections θ_1 and θ_2. We assume that either $\theta_1 = \theta_2$ or $\theta_1 \cap \theta_2 = \emptyset$. Let*

$$D_n = \{x \in P_n(\theta_1) \mid t_{n,\theta_2}(x) < \infty\}. \tag{14}$$

We define a Poincaré map, $G_{n,\theta_1 \to \theta_2} : D_n \to P_n(\theta_2)$, between sections θ_1 and θ_2 by

$$G_{n,\theta_1 \to \theta_2}(x) = \varphi_n(t_{n,\theta_2}(x), x). \tag{15}$$

It is well known, that $G_{n,\theta_1 \to \theta_2}$ is continuous.

Now we would like to define a notion of the Poincaré map and of the first return time for (1). Since we did not assume the local uniqueness for (1) we will only define these notions with respect to a given solution of (1).

Definition 6 *Let $x : [0, t_{max}) \to H$ be a solution (1) such that $x(0) = x_0$. Let θ_1, θ_2 be sections, such that either $\theta_1 = \theta_2$ or $\theta_1 \cap \theta_2 = \emptyset$.*

- $t_{\theta_2}(x_0) = \inf\{t > 0 \mid x(t) \in \theta_2\}$
- *if $t_{\theta_2}(x_0) < \infty$ and $x_0 \in \theta_1$, then $G_{\theta_1 \to \theta_2}(x_0) = x(t_{\theta_2}(x_0))$*

Observe that the map Poincaré G defined above is in principle a multivalued map (it will in fact be multivalued in the case of nonuniqueness).

Definition 7 *Let $\theta_0, \theta_1, \ldots, \theta_r$ be sections, such that either $\theta_{i-1} = \theta_i$ or $\theta_{i-1} \cap \theta_i = \emptyset$ for all $i = 1, \ldots, r$. We say that $G_{\theta_{r-1} \to \theta_r} \circ G_{\theta_{r-2} \to \theta_{r-1}} \circ \cdots \circ G_{\theta_0 \to \theta_1}(y_0) = y_r$ iff there exists a solution, $x : [0, t_{max}) \to H$, of (1) and a sequence of points $y_1, \ldots, y_{r-1}$ such that*

- *$y_i \in \theta_i$ for $i = 0, \ldots, r$*
- *there exists a sequence of real numbers $0 = t_0 < t_1 < \cdots < t_r < t_{max}$, such that $x(t_i) = y_i$ and for all $t \in (t_{i-1}, t_i)$ $x(t) \notin \theta_i$.*

Lemma 6 *Let θ be a section for (1) and let for $n \geq m$ $G_n = G_{n,\theta \to \theta}$.*

Let $\{k_n\} \subset \mathbb{N} \setminus \{1, \ldots, m-1\}$, such that $\lim_{n\to\infty} k_n = \infty$. Assume that for any n there exists a function $x_{k_n} : [0, \infty) \to W \oplus P_{k_n}(V)$ a solution of the k_n-th Galerkin projection of (1) (i.e. (12) with $n = k_n$), such that $G_{k_n}(x_{k_n}(0)) = x_{k_n}(0)$.

Moreover, we assume that there exists a constant $T > 0$, such that

$$T_n = t_{k_n,\theta}(x_{k_n}(0)) \leq T, \qquad \text{for all } n. \tag{16}$$

Then there exists a solution $x^* : [0, \infty) \to W \oplus V$ of (1), such that $x^*(0) \in \theta$ and $G(x^*(0)) = x^*(0)$. In particular x^* is a nonconstant periodic solution of (1).

Proof: Without any loss of generality we can assume that $k_n = n$ (and we consider only $n > M$).

Observe that T_n is equal to the period of x_n. Without any loss of generality we can assume $T_n \to T^* \leq T$. Observe that from Def. 3 it follows that there exists $\epsilon > 0$, such that for all n, $T_n \geq \epsilon$, hence also $T^* \geq \epsilon$.

From Lemma 5 it follows that there exists a subsequence x_{n_k} converging uniformly on compact time intervals (we assume that the whole sequence converges) to $x^* : [0, \infty) \to W \oplus V$, which is a solution of (1). Since $x^*(0) \in \theta$, then x^* is not a constant function. It is also easy to see T^* is a first return time to θ (see Def. 6). It remains to show that $x^*(T^*) = x^*(0)$.

We have

$$\|x^*(0) - x^*(T^*)\| \le \|x^*(0) - x_n(0)\| + \|x_n(0) - x_n(T_n)\| + \|x_n(T_n) - x^*(T_n)\| + \|x^*(T_n) - x^*(T^*)\| = \|x^*(0) - x_n(0)\| + \|x_n(T_n) - x^*(T_n)\| + \|x^*(T_n) - x^*(T^*)\|.$$

The first two terms are arbitrarily small as $n \to \infty$ due to the uniform convergence and the last term tends to zero due to the continuity of x^*. ∎

3.1 Basic Differential Inclusion and computation Poincaré maps for all Galerkin projections

In this subsection we discuss how we can obtain information about Poincaré maps for the n-th Galerkin projection for $n > M$.

Consider a differential inclusion

$$\frac{dx}{dt} \in P_m(F(x)) + E, \qquad x(0) = x_0 \in X_m \tag{17}$$

where $E = \text{conv}\,(\{P_m F(x+q) - P_m F(x) \mid x \in W,\ q \in V\})$ and $x \in C^1$.

We will refer to (17) as a *Basic Differential Inclusion for (1) and self-consistent a priori bounds* $W \oplus V$.

Definition 8 *Given a multivalued map $\mathcal{F} : D \to 2^Y$. A map $f : D' \to Y$ is a selector for $\mathcal{F}$ iff $D' \subset D$ and for all $x \in D'$ $f(x) \in \mathcal{F}(x)$.*

Definition 9 *Consider sections θ_1 and θ_2, such that either $\theta_1 = \theta_2$ or $\theta_1 \cap \theta_2 = \emptyset$. We define* a Poincaré map $\mathcal{G}_{\theta_1 \to \theta_2} : P_m(\theta_1) \supset D \to 2^{P_m(\theta_2)}$ *and* the set of first return times $t_{\mathcal{G}_{\theta_1 \to \theta_2}} : D \to 2^{\mathbb{R}_+}$ *by*

- *for any $x_0 \in \theta_1$ let $S(x_0)$ be a set of right full solutions of (17), i.e. defined for $t \in [0, t_{max})$, so that either $t_{max} = \infty$ or $x(t)$ cannot be extended to $t > t_{max}$*
- *$x_0 \in D = dom\,(\mathcal{G}_{\theta_1 \to \theta_2})$ iff for all $x \in S(x_0)$ there exists $t_1 > 0$ such $x(t_1) \in \theta_2$ and*
$$x([0, t_1]) \subset \text{int}_{X_m} W. \tag{18}$$
- *Let $x_0 \in dom\,(\mathcal{G}_{\theta_1 \to \theta_2})$, for $x \in S(x_0)$ and let $t_x > 0$ be a smallest positive number such that $x(t_x) \in \theta_2$. We set*
$$\begin{aligned} \mathcal{G}_{\theta_1 \to \theta_2}(x_0) &= \{x(t_x) \mid x \in S(x_0)\}, \\ t_{\mathcal{G}_{\theta_1 \to \theta_2}}(x_0) &= \{t_x \mid x \in S(x_0)\} \end{aligned}$$

The following statement telling that, $P_m G_{n,\theta_1 \to \theta_n}$, the m-dimensional projection of the Poincaré map computed for the n-th Galerkin projection is a selector for $\mathcal{G}_{\theta_1 \to \theta_2}$ is rather obvious, but is of fundamental importance in our approach, hence we alleviate it to the theorem status.

Theorem 7 *Consider sections θ_1 and θ_2, such that either $\theta_1 = \theta_2$ or $\theta_1 \cap \theta_2 = \emptyset$. Let $\mathcal{G} = \mathcal{G}_{\theta_1 \to \theta_2}$ be a Poincaré map for (17). Let $G_n = G_{n,\theta_1 \to \theta_2}$.*

Then for all $n > M$

- *$dom\,\mathcal{G} \oplus Q_m P_n V \subset dom\, G_n$*
- *for all $p + q \in dom\,\mathcal{G} \oplus P_n V$ holds*

$$\varphi_n([0, t_{n,\theta}(p+q)], p+q) \in W \oplus Q_m P_n V$$
$$G_n(p+q) \in \mathcal{G}(p) \oplus \mathrm{int}_{Q_m X_n} Q_m P_n V$$

Proof: Let us consider the n-th Galerkin projection

$$\frac{dp}{dt} = P_m F(p+q) \tag{19}$$
$$\frac{dq}{dt} = Q_m P_n F(p+q), \tag{20}$$

where $p \in X_m$ and $q \in Q_m X_n$.

Let $p_0 + q_0 \in \mathrm{dom}\,\mathcal{G} \oplus Q_m P_n V$. Let $x(t) = (p(t) + q(t))$ for $t \in [0, t_{max})$ be a solution of system (19-20) with an initial condition $p(0) = p_0$ and $q(0) = q_0$ extended to the right to the maximum existence interval. Observe that $p_0 \in \mathrm{int}_{X_m} W$. From condition **C4a** it follows that $q(t) \in \mathrm{int}_{Q_m X_n} V$ for $t \in (0, h)$ for h sufficiently small.

Observe that as long as $p(t) + q(t) \in \mathrm{int}_{X_m} W \oplus Q_m P_n V$ for all $t \in [0, t_1)$, then $p(t)+q(t)$ for $t \in [0, t_1)$ is a solution of the differential inclusion (17). From this observation the assertion follows immediately. ∎

From Theorem 7 it follows that a computation of $\mathcal{G}$ gives bounds for G_n for all $n > M$. The question of how to actually compute $\mathcal{G}$ rigorously is treated in [ZPLo], a shorter description can be found in Section 6.

The next theorem, which uses the Brouwer fixed point theorem as a finite dimensional tool, illustrates how the information about differential inclusion (17) can be used to obtain a periodic orbit in the case of an apparently attracting periodic orbit.

Theorem 8 *Assume that we have self-consistent bounds for (1) and a section θ. Let $\mathcal{G}$ be a Poincaré map on θ for (17).*

Assume that there exists a set $B \subset P_m(\theta)$, such that

- *B is homeomorphic with an $(m-1)$-dimensional closed ball,*
- *$B \subset dom\,\mathcal{G}$,*
- *$\mathcal{G}(B) \subset B$,*
- *there exists a constant $T > 0$ such that, for any $x_0 \in B$ and any $t \in t_{\mathcal{G}_{\theta_1 \to \theta_2}}(x_0)$ we have $t \le T$.*

Then there exists a function $u : [0, \infty) \to W \oplus V$ a solution to (1), such that $u(0) \in \theta$ and $G(u(0)) = u(0)$. In particular u is T^*-periodic for some $T^* > 0$.

Proof: Consider the n-th Galerkin projection of (1). Let $G_n = G_{n,\theta\to\theta}$. From Theorem 7 it follows that for all $n > M$ we have

$$\begin{aligned} G_n(B \oplus Q_m P_n V) &\subset B \oplus Q_m P_n V & (21)\\ \varphi_n([0, t_{n,\theta}(x_0)], x_0) &\subset W \oplus Q_m P_n V, \quad \text{for } x_0 \in B \oplus Q_m P_n V & (22) \end{aligned}$$

From the Brouwer theorem [DG] it follows that for all $n > M$ there exists a fixed point, x_n, of G_n. Now the assertion follows easily from Lemma 6. ∎

3.2 How to prove the existence of unstable periodic orbit and/or of symbolic dynamics.

Theorem 8 is an example, which shows us how to link the method of self-consistent bounds with finite dimensional tools (the Brouwer theorem) to obtain some results about the existence of nontrivial periodic orbits for dissipative PDEs, in this case an apparently stable one. To obtain the existence of an unstable periodic orbit and/or symbolic dynamics (horseshoe-like chaotic behavior) one can use the method of covering relations [GiZ, Z0, Z1, Z2] as the finite dimensional tool. This has been previously successfully tested in the context of ODEs: for example in papers [Z2] and [GaZ] it was used to establish the existence of symbolic dynamics for Rössler and Lorenz equations, respectively.

Moreover, this approach (covering relations) can be used to obtain rigorous numerical shadowing algorithms similar in spirit to the ones developed in [HYC, HYC1, H] in finite dimension.

4 The existence of a periodic orbit for the Kuramoto-Sivashinsky PDE with $\nu = 0.127$

In this section we show an application of the method of self-consistent a priori bounds to the proof of the existence of a periodic orbit for the Kuramoto-Sivashinsky partial differential equation (we will use the shorthand "KS-equation" in the sequel).

The KS-equation [KT, S] introduced in the context of a wave front propagation is given by

$$u_t = -\nu u_{xxxx} - u_{xx} + (u^2)_x \qquad (t, x) \in [0, \infty) \times (-\pi, \pi), \quad \nu > 0. \tag{23}$$

Assuming odd and periodic boundary conditions

$$u(t, x) = u(t, x + 2\pi), \quad u(t, x) = -u(t, -x). \tag{24}$$

The existence, uniqueness, and regularity properties for solutions of problem (23) and (24) in suitable Sobolev spaces have been established by several authors

Table 1: Coordinates of u_0 - an approximation of an initial condition for periodic orbit in Theorem 9.

$a_1 = 2.012106e-01$	$a_2 = 1.289980$
$a_3 = 2.012109e-01$	$a_4 = -3.778662e-01$
$a_5 = -4.230950e-02$	$a_6 = 4.316159e-02$
$a_7 = 6.940217e-03$	$a_8 = -4.156484e-03$
$a_9 = -7.944907e-04$	$a_{10} = 3.316061e-04$
$a_{11} = 7.939456e-05$	$a_{12} = -2.390962e-05$
$a_{13} = -7.087251e-06$	$a_{14} = 1.568377e-06$

(see [CEES, FT] and references given there). Nevertheless, we would like to stress that we are not using these results in our investigations.

The KS-equation can be reduced (see [ZM]) to the following infinite system of ordinary differential equations

$$\dot{a}_k = k^2(1-\nu k^2)a_k - k\sum_{n=1}^{k-1} a_n a_{k-n} + 2k\sum_{n=1}^{\infty} a_n a_{n+k} \quad k = 1, 2, \dots \quad (25)$$

where

$$u(t,x) = \sum_{k=1}^{\infty} -2a_k(t)\sin(kx).$$

The above representation of u in terms of a_k has its origin in the following easy observation: if $u(t,x) = \sum_{k=-\infty}^{\infty} b_k(t)e^{ikx}$ satisfies boundary conditions (24), then $b_k = ia_k$ and $a_k = -a_{-k}$.

We will refer to coordinates a_k as *modes*. In this paper we focus on $\nu = 0.127$. For this parameter value numerical simulations suggest the existence of an attracting limit cycle, whose projection onto the (a_1, a_3)-plane is an ellipse-shaped curve, on which the point moves in the clockwise direction.

Theorem 9 *Let $u_0(x) = \sum_{k=1}^{14} -2a_k \sin(kx)$, where a_k are given in Table 1. There exists a function $u^*(t,x)$ a classical solution of (23 - 24) for $\nu = 0.127$, such that*

$$\|u_0 - u^*(0,\cdot)\|_{L_2} < 5\cdot 10^{-5}, \qquad \|u_0 - u^*(0,\cdot)\|_{C^0} < 7\cdot 10^{-5} \quad (26)$$

and u^* is periodic with respect to t, with period ***this was in the printed version*** $T \in (2.44296, 2.4438)$, ***correct value is:*** $T \in 2\cdot(1.12214, 1.12219)$

5 Proof of Theorem 9

We choose $H = l_2$ as our Hilbert space. We will adopt the following convention. By x, u, x_0, u_0 we will denote the points from H, by a_i we will always denote coordinates of points in H.

Let R be a map which leaves even modes and changes the sign of odd modes: $a_{2k} \to a_{2k}$ and $a_{2k+1} \to -a_{2k+1}$. It is easy to see that R leaves the system (25) and any Galerkin projection of it invariant.

To define a Poincaré section we use linear functions as follows: the section $\theta = \theta_\eta$ is defined by the following conditions

$$\eta(x) = \eta_0 + \sum_{i=1}^{m} \eta_i a_i = 0, \quad \eta'(x) > 0. \tag{27}$$

By the condition $\eta'(x) > 0$ we mean that the following inequality is satisfied for $x \in \theta_\eta$

$$\sum_{k=1}^{m} \frac{\partial \eta}{\partial x_k} A_k F(x) > 0. \tag{28}$$

The above condition means that trajectories of the Basic Differential Inclusion intersect the hyperplane defined by $\eta = 0$, so that $\eta(x(t))$ is an increasing function of time.

Let $\theta_1 = \theta_\eta$ be a linear section as described above. We define a symmetric section, θ_2, by $\theta_2 = R\theta_1$. Hence θ_2 is defined by the following conditions

$$\beta(x) = \eta_0 + \sum_{i=1}^{m} (-1)^i \eta_i a_i = 0, \quad \beta' > 0. \tag{29}$$

By the symmetry we have for any Galerkin projection

$$G_{n,\theta_2 \to \theta_1} = R G_{n,\theta_1 \to \theta_2} R \tag{30}$$

For the full Poincaré map on section θ_1, G_n, we obtain that

$$G_n = G_{n,\theta_2 \to \theta_1} G_{n,\theta_1 \to \theta_2} = R G_{n,\theta_1 \to \theta_2} R G_{n,\theta_1 \to \theta_2} = (R G_{n,\theta_1 \to \theta_2})^2. \tag{31}$$

Hence any fixed point for $RG_{n,\theta_1 \to \theta_2}$ is a fixed point for G_n.

Let $\mathcal{G}$ be a Poincaré map for the Basic Differential Inclusion (17) for the transition $\theta_1 \to R\theta_1$.

The proof is computer assisted and consists of the following steps:

1. an initialization: setting up the parameters: dimensions m and M, finding an approximate periodic orbit, choosing the section θ_1, finding suitable coordinates on θ_1

2. a construction of self-consistent bounds

3. a construction of a set $N \subset W$, such that $R\mathcal{G}(N) \subset \operatorname{int} N$

4. a conclusion of the proof, an application of Theorem 8.

Table 2: Approximate coordinates for the starting point on the periodic orbit for $\nu = 0.127$.

$a_1 = 0.201211$	$a_2 = 1.28998$
$a_3 = 0.201211$	$a_4 = -0.377866$
$a_5 = -0.0423095$	$a_6 = 0.0431616$
$a_7 = 0.00694021$	$a_8 = -0.00415648$
$a_9 = -0.00079449$	$a_{10} = 0.000331606$
$a_{11} = 7.93945e-05$	$a_{12} = -2.39096e-05$
$a_{13} = -7.08724e-06$	$a_{14} = 1.56839e-06$

5.1 Part 1 - an initialization

In the proof we used $m = 14$ and $M = 3m = 42$.

We define a point x_0 as in Table 2. This point is an approximation to a true periodic point for (23) and was found as follows. Let us define a section σ by $a_1 - a_3 = 0$, $(a_1 - a_3)' > 0$. Consider the map $f = RG_{m,\sigma\to R\sigma}$. Since we are looking for an attractive fixed point we just iterated a map f for some initial value until $|f(x) - x|_\infty < 10^{-6}$. A fourth order Runge-Kutta method, with a time step $h = \frac{1}{|2m^2(1-\nu m^2)|} = 0.000106773$ was used in these computations.

We define the section θ_1 as a section perpendicular to $P_m F(x_0)$ at x_0, namely we set

$$\alpha(x) = (P_m F(x_0)|x) - (P_m F(x_0)|x_0), \quad \alpha' > 0. \tag{32}$$

Let $\theta_2 = R\theta_1$. We define *section coordinates* on θ_1, which will be used later in the proof as follows.

The map $g = RG_{m,\theta_1\to\theta_2} : \theta_1 \supset U \to \theta_1$, where $x_0 \in U$, has x_0 as an approximate fixed point. Next we compute nonrigorously an approximate Jacobian matrix $Dg(x_0)$ using a fourth order Runge-Kutta method to compute the trajectory of x_0 and a second order Taylor method for the variational part, with the time step $h = \frac{1}{|2m^2(1-\nu m^2)|} = 0.000106773$. The matrix $Dg(x_0) \in \mathbb{R}^{(m-1)\times(m-1)}$ is expressed in coordinates $\widetilde{a}_{i=1,\dots,13}$, defined as follows. Let i_0 be such that $|A_i F(x_0)|$ achieves the maximum value for $i = i_0$ (for our periodic orbit $i_0 = 3$). Then we set

$$\widetilde{a}_i = \begin{cases} a_i & \text{if } i < i_0, \\ a_{i+1} & \text{if } i \geq i_0. \end{cases} \tag{33}$$

Let $r = 8$. From the matrix $Dg(x_0)$ we extract an $(r \times r)$-square matrix in an upper left corner to define $\widetilde{D} \in \mathbb{R}^{r\times r}$ by

$$\widetilde{D}_{ij} = Dg(x_0)_{ij}, \qquad \text{for } i, j = 1, \dots, r. \tag{34}$$

Next we apply to $\widetilde{D}$ a diagonalization procedure based on the QR-decomposition algorithm [R] to obtain the approximate eigenvectors $v_1, v_2, \dots, v_r$ corresponding to approximate eigenvalues $\lambda_1, \dots, \lambda_r$. We assume, additionally, that we

Table 3: Approximate eigenvalues for $\widetilde{D}$

1	0.5258
2	0.090375
3	3.5019e-08
4	1.6503e-08
5	-3.7784e-09
6	-4.0167e-11
7	-8.9431e-10
8	-6.6955e-11

Table 4: Four leading approximate eigenvectors for $\widetilde{D}$

i	v_1	v_2	v_3	v_4
1	0.31728	-0.070268	0.046791	0.088001
2	-0.81436	0.8341	-0.33709	0.23897
3	0.47674	-0.53787	0.92972	-0.45521
4	0.033933	-0.024953	-0.017497	-0.82473
5	-0.086464	0.095695	-0.13781	0.16446
6	-0.0095825	0.0093288	-0.0094954	0.14042
7	0.01127	-0.012669	0.019939	-0.019339
8	0.0016374	-0.0016399	0.0017784	-0.02442

have

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r|. \tag{35}$$

Some of the diagonalization data are contained in Tables 3 and 4. It is clear from these tables that eigenvalues are decaying rapidly to zero and the high modes are strongly damped.

To an ordered collection of eigenvectors $\{v_1, \ldots, v_r\}$ we apply an *interval* Gramm-Schmidt orthogonalization procedure, to obtain a new orthonormal set of vectors $\{w_1, \ldots, w_r\}$. These vectors define a new coordinate frame on $\mathbb{R}^r$ and together with coordinates $\widetilde{a}_{r+1}, \ldots, \widetilde{a}_m$ define the new coordinates on θ_1, such that $x_0 = 0$. We will denote these coordinates by c_i and we will refer to them as *section coordinates*.

It is essential here that the Gramm-Schmidt diagonalization procedure is performed using an interval arithmetic, because in this way we obtain a rigorous orthogonal transformation from the cartesian coordinates, $\widetilde{a}$, on section θ_1 to the *section coordinates* and its rigorous inverse. Both transformations are needed later in the proof, because the integration of the Basic Differential Inclusion for (23) is done in the cartesian coordinates, but all the important proof sets are defined using the section coordinates.

5.2 Part 2 - a construction of self-consistent a priori bounds.

We have to define a set $W \subset X_m$ and $\{a_k^{\pm}\}_{k>m}$. In principle it is enough to take any $W \subset X_m$ such that $P_m\gamma \subset W$, where γ is an approximate periodic orbit for an m_1-dimensional Galerkin projection of (23) ($m_1 \geq m$). For W, we construct $\{a_k^{\pm}\}$ using the algorithm described in Section 3.3 of [ZM]. But it is obvious that, to obtain the proof with a relatively small dimension m (hence in short computation time), it essential that we choose W as a small neighborhood of γ. It is also very important how we let $A_k(W)$ decay.

The set W was constructed as follows

1. we generate an approximate periodic trajectory for an m_1-dimensional Galerkin projection, with $m_1 > m$. In our proof $m_1 = 16$. A fourth order Runge-Kutta method with a fixed time step $h = \frac{1}{2|m_1^2(1-\nu m_1^2)|}$ was used for this purpose. As a result of this procedure we obtain a finite ordered set of points $Z \subset X_{m_1}$, which apparently is very close to the m_1-dimensional projection of the periodic orbit we are after.

2. we define an auxiliary set $\widetilde{W} \subset X_m$ as follows. On the plane (a_1, a_3) we introduce polar coordinates (r, ϕ). For $i = 0, \ldots, p-1$ ($p = 70$ in the proof) we define sets S_i by

$$S_i = \left\{ (a_1, a_3) \mid (a_1, a_3) \neq (0,0), \quad \phi(a_1, a_3) \in \left[\frac{2\pi i}{p}, \frac{2\pi(i+1)}{p} \right] \right\} \tag{36}$$

Next we define sets $D_i(n, Z) \subset X_m$ for $i = 0, 1, \ldots, p-1$ by

$$\begin{aligned} D_i(n, Z) \quad = \quad & \mathrm{IHull}\{P_n z \mid z \in Z, \\ & (A_1(z), A_3(z)) \in S_{(i-1) \bmod p} \cup S_i \cup S_{(i+1) \bmod p}\}, \end{aligned}$$

where, for a set $Y \subset \mathbb{R}^k$, by $\mathrm{IHull}(Y)$ we denote the smallest product of intervals containing Y.

For a stretching factor parameter, e ($e = 1.1$ in the proof) and any interval $I = [a, b]$ we define a new interval, $\mathrm{stretch}(I, e)$ by

$$\mathrm{stretch}(I, e) = (a+b)/2 + \left[-e\frac{b-a}{2}, e\frac{b-a}{2} \right]. \tag{37}$$

For an interval set $X = \Pi_{i=1}^k I_i$, where I_i is an interval we set

$$\mathrm{stretch}(X, e) = \Pi_{i=1}^k \mathrm{stretch}(I_i, e) \tag{38}$$

We introduce another parameter n_{iso} - the number of coordinates for which we force an isolation, but still we include them in computations (in the proof $n_{iso} = 3$).

We define, for $i = 0, \ldots, p-1$,

$$\widetilde{W}_i = \mathrm{stretch}(D_i(m - n_{iso}), e) \tag{39}$$

Table 5: The interval enclosure of W from the proof of Theorem 9. The tail is given in Table 6. Columns: c-coordinate index, bounds for the c-th coordinate

c	bounds
1	[-0.427501,0.427482]
2	[1.06609,1.33357]
3	[-0.866305,0.866316]
4	[-0.391295,-0.129789]
5	[-0.18118,0.181172]
6	[-0.0271086,0.0474366]
7	[-0.0201397,0.0201411]
8	[-0.00485958,0.00629999]
9	[-0.00182595,0.00182582]
10	[-0.000718572,0.000425729]
11	[-0.000150202,0.000150379]
12	[-3.49005e-05,7.58671e-05]
13	[-1.81432e-05,1.81445e-05]
14	[-6.24288e-06,2.6964e-06]

We set

$$\widetilde{W} = \sum_{i=0}^{p-1} \widetilde{W}_i. \tag{40}$$

Obviously $\widetilde{W} \subset X_{m-n_{iso}}$. For this set and M we compute the self-consistent bounds to obtain a collection of pairs $\{\widetilde{a}_k^{\pm}\}_{k>m-n_{iso}}$.

We set, for $i = 0, \ldots, p-1$,

$$W_i = \widetilde{W}_i \times \Pi_{k=m-n_{iso}+1}^{m} [\widetilde{a}_k^-, \widetilde{a}_k^+] \tag{41}$$

We set $W = \bigcup_{i=0}^{p-1} W_i$. Table 5 contains the interval enclosure of W i.e. the smallest product of intervals containing W.

Finally, for W and M we compute self-consistent a priori bounds $\{a_k^{\pm}\}_{k>m}$ using the algorithm outlined in Section 3.3 in [ZM]. Table 6 contains these bounds together with the initial bounds used to start the algorithm (see Section 5 in [ZM] for more details).

Let

$$E_n = \text{conv}\left(\{A_n F(x) - A_n(F(P_n(x))) \mid x \in W \oplus \Pi_{k>m}[a_k^-, a_k^+]\}\right) \tag{42}$$

The interval E_n measures the influence of the tail on the n-th coordinate of the vector field and we will call the interval vector E *the Galerkin projection error*. Having self-consistent bounds we can compute E in the Basic Differential Inclusion (17). Table 7 contains the Galerkin projection errors from the proof.

Table 6: The self-consistent a priori bounds from the proof of Theorem 9. $m = 14$, $M = 3m = 42$, the initial far tail (for $k > M$) is given by $a_k^+ = -a_k^- = 780.898/k^4$. The final far tail (for $k > M$) is given by $a_k^+ = -a_k^- = 2.67593e + 10/k^{12}$. Columns from left to right, c-coordinate index, initial bounds for the c-th coordinate, final bounds for the c-th coordinate (with an isolation for $c > m$), the ratio of diameters of the final and initial bounds.

c	initial	end	ratio
15	[-0.0159165,0.0159165]	[-1.98015e-06,1.97985e-06]	0.0001244
16	[-0.0122413,0.0122413]	[-3.16413e-07,5.00703e-07]	3.33754e-05
17	[-0.00957066,0.00957066]	[-1.80681e-07,1.80742e-07]	1.88819e-05
18	[-0.00759164,0.00759164]	[-4.12298e-08,4.03705e-08]	5.37436e-06
19	[-0.00609963,0.00609963]	[-1.48467e-08,1.48436e-08]	2.43379e-06
20	[-0.00495741,0.00495741]	[-4.13123e-09,3.34701e-09]	7.54249e-07
21	[-0.00407088,0.00407088]	[-1.21175e-09,1.212e-09]	2.97694e-07
22	[-0.00337422,0.00337422]	[-2.87756e-10,3.57275e-10]	9.55824e-08
23	[-0.0028206,0.0028206]	[-9.93966e-11,9.93814e-11]	3.52368e-08
24	[-0.00237615,0.00237615]	[-3.01295e-11,2.22132e-11]	1.10142e-08
25	[-0.00201598,0.00201598]	[-8.19686e-12,8.19757e-12]	4.06612e-09
26	[-0.00172161,0.00172161]	[-2.01848e-12,2.23854e-12]	1.23635e-09
27	[-0.00147912,0.00147912]	[-6.76212e-13,6.76347e-13]	4.5722e-10
28	[-0.00127789,0.00127789]	[-1.8801e-13,1.65843e-13]	1.38453e-10
29	[-0.00110977,0.00110977]	[-5.57064e-14,5.5693e-14]	5.01902e-11
30	[-0.000968438,0.000968438]	[-1.46681e-14,1.36154e-14]	1.46027e-11
31	[-0.00084892,0.00084892]	[-4.42188e-15,4.42286e-15]	5.20941e-12
32	[-0.000747297,0.000747297]	[-1.2231e-15,1.23527e-15]	1.64485e-12
33	[-0.000660445,0.000660445]	[-6.11705e-16,6.11627e-16]	9.26143e-13
34	[-0.000585859,0.000585859]	[-7.91392e-16,7.77735e-16]	1.33917e-12
35	[-0.000521518,0.000521518]	[-2.03049e-15,2.0304e-15]	3.89334e-12
36	[-0.000465776,0.000465776]	[-6.03481e-15,6.0351e-15]	1.29568e-11
37	[-0.000417291,0.000417291]	[-1.20708e-14,1.20708e-14]	2.89266e-11
38	[-0.000374957,0.000374957]	[-4.19424e-14,4.19423e-14]	1.1186e-10
39	[-0.00033786,0.00033786]	[-8.8779e-14,8.8779e-14]	2.62769e-10
40	[-0.000305241,0.000305241]	[-1.94964e-13,1.94964e-13]	6.38721e-10
41	[-0.000276468,0.000276468]	[-3.35639e-13,3.35639e-13]	1.21403e-09
42	[-0.000251007,0.000251007]	[-2.9855e-13,2.9855e-13]	1.18941e-09

Table 7: The Galerkin errors from the proof of a periodic orbit for $\nu = 0.127$. Computed for $m = 14$, $M = 3m = 42$. The tail is given in Table 6.

k	E_n
1	[-2.69e-11, 2.69036e-11]
2	[-1.57744e-10, 1.5313e-10]
3	[-9.63186e-10, 9.63049e-10]
4	[-2.60273e-09, 2.71426e-09]
5	[-1.49885e-08, 1.49925e-08]
6	[-4.793e-08, 4.63349e-08]
7	[-1.86867e-07, 1.86826e-07]
8	[-6.74122e-07, 6.92668e-07]
9	[-1.75323e-06, 1.75239e-06]
10	[-7.48998e-06, 7.41082e-06]
11	[-1.77666e-05, 1.77637e-05]
12	[-4.47796e-05, 4.37507e-05]
13	[-7.86937e-05, 7.8697e-05]
14	[-3.9583e-05, 4.62362e-05]

5.3 Part 3 - Basic Differential Inclusion and the construction of N

Consider the Basic Differential Inclusion (17) for (23). Let us remind the reader that by $\mathcal{G}$ we denote a Poincaré map between sections θ_1 and $R\theta_1$. Our goal is to construct a set $N \subset \theta_1 \cap W$, such that

$$N \subset \operatorname{dom}(\mathcal{G}) \tag{43}$$

$$R\mathcal{G}(N) \subset \operatorname{int} N \tag{44}$$

Observe that (see condition (18) in Def. 9) condition (43) requires that any solution of (17) starting from $x \in N$ stays in W for positive t, which is less than or equal to the first return time to $R\theta_1$. In Section 6 we present an algorithm, which allows us to compute a rigorous enclosure for $\mathcal{G}(N)$ for any $N \subset \theta_1$.

We constructed N as a result of the following simple algorithm. We would like to stress that in its description we use *the section coordinates* (introduced in Subsection 5.1).

Algorithm

1. *Initialization.* We set the parameters for the computation of $\mathcal{G}$: the time step $h = \frac{1}{2|m^2(1-\nu m^2)|} = 0.000106773$ and the order $r = 3$ of the numerical method.

We set $\delta = 2 \cdot 10^{-5}$. We initialize N as follows

$$N = [-\delta, \delta]^{m-1}. \tag{45}$$

2. *Computation of a Poincaré map.* We compute $RGN = R \circ \mathcal{G}(N)$ without checking condition (18), i.e., if the trajectory of N belongs to W. If the computation was terminated successfully then we go to step **3**, otherwise the execution of the algorithm is interrupted and **fail** is returned.

3. If

$$N \subset RGN, \tag{46}$$

then the execution of the algorithm is interrupted and **fail** is returned.

If

$$RGN \subset N, \tag{47}$$

then we go to step **4**.

If neither (46) nor (47) is satisfied then we set $N = RGN \cap N$ and jump back to step **2**

4. *Final check.* We recompute $RGN = R \circ \mathcal{G}(N)$, but this time checking condition (18). If this is the case we either return **success** or if we want more tight bounds we jump to step **5** , otherwise **fail** is returned.

5. *Further improvement.* We compute several times: $N = R \circ \mathcal{G}(N)$ and return **success**.

End of algorithm

With computer assistance we proved the following

Lemma 10 *Let $\nu = 0.127$. There exists $N \subset W \cap \theta_1$, N is a product of interval sets in the section coordinates, such that $N \subset dom\,\mathcal{G}$, $t_{\mathcal{G}_{\theta_1 \to R\theta_1}}(x) \leq T$ and $R \circ \mathcal{G}(N) \subset \operatorname{int} N$. Moreover, for any $x \in N$* ***In the printed version we had this wrong statement given below***

$$t_{\mathcal{G}_{\theta_1 \to R\theta_1}}(x) \subset (2.44296, 2.4438)$$

the correct bounds are:

$$t_{\mathcal{G}_{\theta_1 \to R\theta_1}}(x) \subset (1.12214, 1.12219)$$

About the proof: The main loop was executed three times. A Pentium III, 450 MHz computer was used. A computation of $R\mathcal{G}(N)$ took around 751 seconds.

Table 8 describes the set N, as follows: $N = \Pi_{i=1}^{m-1} N_i$, the interval N_i is given by the i-th row as $N_i = x + r$. For example $N_1 = -3.880465e-07 + [-1.838038e-05, 1.838038e-05]$.

After two iterates we had already an inclusion $R\mathcal{G}(N) \subset N$ (all ratios of diameters in the second column in Table 10 are less than 1). Hence the third iterate was used only to improve estimates (see ratios in the third column in Table 10).

Table 9 contains $R\mathcal{G}(N)$. In Table 10 we illustrate how the set N was changing during the execution of the algorithm. Instead of displaying the actual coordinates we present the ratios between the size of the image and the set N in each direction.

Table 8: The input data from the third iterate of the algorithm from the proof of Theorem 9. $N = x + r$, where r is an interval vector. Columns from left to right are the coordinate index, c, x and r. The section coordinates are used for x and r. Output data are in Table 9.

c	x	r
1	-3.880465e-07	[-1.838038e-05,1.838038e-05]
2	-2.115301e-09	[-5.817347e-07,5.817347e-07]
3	4.895745e-10	[-2.728741e-08,2.728741e-08]
4	-2.659308e-10	[-1.082108e-08,1.082108e-08]
5	-2.060927e-10	[-7.368567e-09,7.368567e-09]
6	4.658323e-11	[-6.102844e-09,6.102844e-09]
7	-2.881488e-11	[-5.661919e-09,5.661919e-09]
8	9.706697e-12	[-6.403534e-09,6.403534e-09]
9	4.434868e-10	[-3.128193e-08,3.128193e-08]
10	7.738122e-11	[-1.738133e-08,1.738133e-08]
11	-4.550985e-11	[-2.210128e-08,2.210128e-08]
12	-6.382979e-12	[-2.505782e-08,2.505782e-08]
13	-5.991690e-12	[-1.147062e-08,1.147062e-08]

Table 9: The output data from the third iterate of the algorithm from the proof of Theorem 9. $N = x + r$, where r is an interval vector. Columns from left to right are the coordinate index, c, $R\mathcal{G}(x + r)$ and diam $(R\mathcal{G}(x + r)$. The section coordinates are used for $R\mathcal{G}(x + r)$. The input data i.e. x and r are given in Table 8.

c	$R\mathcal{G}(x + r)$	diam $(R\mathcal{G}(x + r))$
1	[-1.283714e-05,1.167644e-05]	2.451359e-05
2	[-2.800816e-07,2.784039e-07]	5.584855e-07
3	[-2.350128e-08,2.397283e-08]	4.747412e-08
4	[-9.619064e-09,9.372982e-09]	1.899205e-08
5	[-6.762973e-09,6.567153e-09]	1.333013e-08
6	[-5.891352e-09,5.935270e-09]	1.182663e-08
7	[-5.384946e-09,5.358340e-09]	1.074329e-08
8	[-6.297353e-09,6.307526e-09]	1.260488e-08
9	[-2.359918e-08,2.494372e-08]	4.854291e-08
10	[-1.584270e-08,1.608458e-08]	3.192728e-08
11	[-2.133511e-08,2.120401e-08]	4.253913e-08
12	[-2.457033e-08,2.454699e-08]	4.911733e-08
13	[-1.121413e-08,1.120587e-08]	2.242001e-08

Table 10: Ratios: $\mathrm{diam}\,(A_c R\mathcal{G}(x+r))/\mathrm{diam}\,(A_c r)$ from the proof. Columns from left to right are the coordinate index, c, the ratios in the first, second and third iteration, i, of the main loop in the algorithm.

c	ratios $i=1$	ratios $i=2$	ratios $i=3$
1	2.759255e+00	9.190188e-01	6.668413e-01
2	1.847428e-01	1.574446e-01	4.800173e-01
3	3.566016e-03	3.826037e-01	8.698905e-01
4	1.376623e-03	3.930301e-01	8.775484e-01
5	8.089586e-04	4.554355e-01	9.045263e-01
6	4.380839e-04	6.965385e-01	9.689436e-01
7	4.862820e-04	5.821643e-01	9.487318e-01
8	4.017088e-04	7.970368e-01	9.842128e-01
9	3.662924e-03	4.270078e-01	7.758937e-01
10	1.312009e-03	6.623943e-01	9.184358e-01
11	1.361729e-03	8.115161e-01	9.623678e-01
12	1.406283e-03	8.909244e-01	9.800796e-01
13	6.549903e-04	8.756328e-01	9.772798e-01

5.4 Conclusion of the proof.

From Lemma 10 and an obvious modification of Theorem 8 it follows that there exists a solution, $u^* : [0, T/2] \times (-\pi, \pi) \to \mathbb{R}$ of (23-24) for $\nu = 0.127$ such that

$$R\mathcal{G}_{\theta_1 \to R\theta_1}(u^*(0,\cdot)) = R(u^*(T/2,\cdot)) = u^*(0,\cdot). \tag{48}$$

The domain of definition of u^* can now be extended to $[0,\infty) \times (-\pi,\pi)$ using the symmetry R. Observe that the decay rate of the tail of u^* (see Table 6) guarantees that $u^*(t,\cdot)$ is at least a C^{10} function, hence it defines a classical solution of (23).

6 A Lohner-type algorithm for an integration of differential inclusions

In this section we present a Lohner-type algorithm for a rigorous integration of ODEs with controlled perturbations. This part depends heavily on [ZLo], as the proposed algorithm is just a modification running on top of the C^0-Lohner algorithm for ODEs described (after [Lo, Lo1]) there.

We study the following ODEs

$$\begin{aligned} x'(t) &= f(x(t), y(t)) \\ y'(t) &= g(x(t), y(t)) \end{aligned} \tag{49}$$

where $x \in \mathbb{R}^{n_1}$ and $y(t) \in \mathbb{R}^{n_2}$ (we allow for $n_2 = \infty$). Assume that we have some knowledge of $y(t)$, for example, $|y(t)| < \epsilon$ for $0 \le t \le T$. We would like to find a rigorous enclosure for $x(t)$.

In the context of the method self-consistent bounds, x in (49) represents a point in X_m ($m = n_1$) and y represents the tail and we know that as long as $x(t) \in W$, then $y(t) \in V$.

6.1 Basic notation

We will use the same conventions as in [ZLo]. In the sequel, by arabic letters we denote single valued objects like vectors, real numbers, matrices. Quite often we will use square brackets, for example $[r]$, to denote sets. Usually this will be some set constructed in the algorithm. Sets will also be denoted by single letters, for example S, when it is clear from the context that it represents a set.

For a set $[S]$ by $[S]_I$ we denote the interval hull of $[S]$, i.e. the smallest product of intervals containing $[S]$. The symbol hull$(x_1, \ldots, x_k)$ will denote the interval hull of intervals $x_1, \ldots, x_k$. For any interval set $[S] = [S]_I$ by m$([S])$ we will denote a center point of $[S]_I$. For any interval $[a, b]$ we define a diameter by diam$([a, b]) = b - a$. For an interval vector or an interval matrix $[S] = [S]_I$ by diam $([S])$ we will denote the maximum of diameters of its components. For an interval $[x^-, x+]$ we set $right([x^-, x^+]) = x^+$ and $left([x^-, x^+]) = x^-$.

For a set $X \subset \mathbb{R}^d$ by int X we denote an interior of X.

For $v, w \in \mathbb{R}^n$ and $A, B \in \mathbb{R}^{n \times n}$ ($n = 1, \ldots, \infty$) we say that

$$\begin{array}{lll} v \le w & \text{iff} & \forall i \quad v_i \le w_i, \\ A \le B & \text{iff} & \forall ij \quad A_{ij} \le B_{ij}. \end{array}$$

6.2 A fundamental estimate

For a fixed $y_c \in \mathbb{R}^{n_2}$ we compare the solutions of two ODEs

$$x' = f(x, y_c), \quad x(t_0) = x_0 \tag{50}$$

$$x' = f(x, y_c) + (f(x, y(t)) - f(x, y_c)), \quad x(t_0) = x_0 \tag{51}$$

where $y(t)$ is a given function.

Let $x_1(t)$ be a solution of (50) and let $x_2(t)$ be a solution of (51). We assume that a convex set $[W_y] \subset \mathbb{R}^{n_2}$ is an enclosure for $y([t_0, t_0 + h])$.

Let $[W_1] \subset [W_2] \subset \mathbb{R}^{n_1}$ be convex and compact. We assume that for $s \in [t_0, t_0 + h]$ $x_1(s) \in [W_1] \subset \mathbb{R}^{n_1}$ and $x_2(s) \in [W_2] \subset \mathbb{R}^{n_1}$ for any continuous function $y : [t_0, t_0 + h] \to [W_y]$.

The following lemma is a particular case of Theorem 1 in Section 13 in [W](see subsection IV 'The Lipschitz condition'), a self-contained proof (with precisely specified assumptions) can also be found in [ZPLo].

Lemma 11 *The following inequality holds for $t \in [t_0, t_0 + h]$ and for $i = 1, \ldots, n_1$*

$$|x_{1,i}(t) - x_{2,i}(t)| \le \left(\int_{t_0}^{t} e^{J(t-s)} C \, ds \right)_i, \tag{52}$$

where

$$\begin{aligned}
[\delta] &= \{f(x,y_c) - f(x,y) \mid x \in [W_1], y \in [W_y]\},\\
C_i &\geq \sup|[\delta_i]|, \quad i = 1,\ldots,n_1\\
J_{ij} &\geq \begin{cases} \sup \frac{\partial f_i}{\partial x_j}([W_2],[W_y]) & \text{if } i = j,\\ \sup\left|\frac{\partial f_i}{\partial x_j}([W_2],[W_y])\right| & \text{if } i \neq j. \end{cases}
\end{aligned}$$

6.3 One step of the algorithm

Let $\varphi(t,x_0,y_0)$ denote a solution of equations (49) with initial conditions $x(0) = x_0$ and $y(0) = y_0$. Let $\overline{\varphi}(t,x_0,y_0)$ be a solution of the system

$$x' = f(x,y), \qquad y' = 0 \tag{53}$$

with the same initial conditions $x(0) = x_0$ and $y(0) = y_0$. Observe that for system (53) $y = \text{const}$.

Let $\pi_x : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ be a projection onto $\mathbb{R}^{n_1}$, i.e. $\pi_x(x,y) = x$.

We are interested in finding rigorous bounds for $\pi_x\phi(t,x,y)$ for $x \in [x_0]$ and $y \in [y_0]$. To this end we propose a modification of the original Lohner algorithm [Lo, Lo1]. Our presentation and notation follow a description of the C^0-Lohner algorithm presented in [ZLo].

In the description below the objects with an index k refer to the current values and those with an index $k+1$ are the values after the next time step.

One step of the Lohner algorithm is a shift along the trajectory of system (49) with the following input and output data:

Input data:

- t_k - a current time,
- h_k - a time step,
- $[x_k] \subset \mathbb{R}^{n_1}$, such that $\pi_x\varphi(t_k,[x_0],[y_0]) \subset [x_k]$,
- $[y_k]$ - bounds for $y(t_k)$.

Output data:

- $t_{k+1} = t_k + h_k$ - a new current time,
- $[x_{k+1}] \subset \mathbb{R}^{n_1}$, such that $\pi_x\varphi(t_{k+1},[x_0],[y_0]) \subset [x_{k+1}]$,
- $[y_{k+1}]$ - bounds for $y(t_{k+1})$.

We do not specify here a form (a representation) of sets $[x_k]$. They can be interval sets, balls, doubletons etc. (see [MZ, ZLo]). This issue is very important in handling the wrapping effect and is discussed in detail in [Lo, Lo1] (see also Section 3 in [ZLo]).

One step of the algorithm consists of the following parts:

1. Generation of a priori bounds for φ. We find a convex and compact set $[W_2] \subset \mathbb{R}^{n_1}$ and a convex set $[W_y] \subset \mathbb{R}^{n_2}$, such that

$$\varphi([0, h_k], [x_k], [y_k]) \subset [W_2] \times [W_y]. \tag{54}$$

2. We fix $y_c \in [W_y]$.

3. Computation of an unperturbed x-projection. We apply one step of the C^0-Lohner algorithm to (53) with a time step h_k and an initial condition given by $[x_k] \times \{y_c\}$. Since $y = \text{const}$ for $\overline{\varphi}$, this is a computation of an ODE in $\mathbb{R}^{n_1}$.

As a result we obtain $[\overline{x}_{k+1}] \subset \mathbb{R}^{n_1}$ and a convex and compact set $[W_1] \subset \mathbb{R}^{n_1}$, such that

$$\begin{aligned} \pi_x \overline{\varphi}(h_k, [x_k], y_c) &\subset [\overline{x}_{k+1}] \\ \pi_x \overline{\varphi}([0, h_k], [x_k], y_c) &\subset [W_1] \end{aligned}$$

4. Computation of perturbation. Using Lemma 11 we find a set $[\Delta] \subset \mathbb{R}^{n_1}$, such that

$$\pi_x \varphi(t_{k+1}, [x_0], [y_0]) \subset \pi_x \overline{\varphi}(h_k, [x_k], y_c) + [\Delta]. \tag{55}$$

Hence

$$\pi_x \varphi(t_{k+1}, [x_0], [y_0]) \subset [x_{k+1}] = [\overline{x}_{k+1}] + [\Delta] \tag{56}$$

5. Computation of $[y_{k+1}]$. This part is not necessary if the bounds for y are known and fixed in advance. (This is the case for self-consistent a priori bounds.)

6.4 Part 1 - comments

In the context of a dissipative PDE and self-consistent a priori bounds $W \oplus V$, we set

$$[W_y] = V, \tag{57}$$

and we have to satisfy the following

$$[W_2] \subset W. \tag{58}$$

The last condition is a consistency condition required by the Basic Differential Inclusion, namely E is computed under this assumption. In the proof of Theorem 9 in the construction of set N (see Subsection 5.3) in Step **2** we ignore (58), but in the final check (Step **4**) we need to verify it.

6.5 Part 4 - details

1. We set

$$\begin{aligned}
[\delta] &= [\{f(x,y_c) - f(x,y) \mid x \in [W_1], y \in [W_y]\}]_I \\
C_i &= \text{right}\left(|[\delta_i]|\right), \quad i = 1,\dots,n_1 \\
J_{ij} &= \begin{cases} \text{right}\left(\frac{\partial f_i}{\partial x_i}([W_2],[W_y])\right) & \text{if } i = j, \\ \text{right}\left(\left|\frac{\partial f_i}{\partial x_j}([W_2],[W_y])\right|\right). & \text{if } i \neq j. \end{cases}
\end{aligned}$$

   In the context of self-consistent a priori bounds $[\delta]_i = E_i$, where E_i is the Galerkin projection error defined by (42).

2. $D = \int_0^h e^{J(h-s)} C \, ds$

3. $[\Delta_i] = [-D_i, D_i]$, for $i = 1,\dots,n_1$

It remains to explain how we compute $\int_0^t e^{A(t-s)} C \, ds$. First observe that

$$\int_0^t e^{A(t-s)} C \, ds = t \left(\sum_{n=0}^{\infty} \frac{(At)^n}{(n+1)!} \right) \cdot C. \tag{59}$$

We fix any norm $\|\cdot\|$, preferably the L^∞-norm, i.e. $\|x\|_\infty = \max_i |x_i|$. Let us set

$$\tilde{A} = At, \qquad A_n = \frac{\tilde{A}^n}{(n+1)!}.$$

In this notation

$$\sum_{n=0}^{\infty} \frac{(At)^n}{(n+1)!} = \sum_{n=0}^{\infty} A_n, \quad A_0 = \text{Id}, \qquad A_{n+1} = A_n \cdot \frac{\tilde{A}}{n+2}$$

For the remainder term we use the following estimate

$$\|A_{N+k}\| \leq \|A_N\| \cdot \left\| \frac{\tilde{A}}{N+2} \right\|^k$$

Hence if $\left\| \frac{\tilde{A}}{N+2} \right\| < 1$, then

$$\left\| \sum_{n>N} A_n \right\| \leq \|A_N\| \cdot \left\| \frac{\tilde{A}}{N+2} \right\| \cdot \left(1 - \left\| \frac{\tilde{A}}{N+2} \right\| \right)^{-1} \tag{60}$$

6.6 Rearrangement

The rearrangement is an essential ingredient in the Lohner algorithm, designed to reduce the wrapping effect [Lo, Lo1, Mo]. We will not discuss this issue here, but we will only include the necessary formulas (see [ZLo] for more details and motivation).

Evaluations 2 and 3. In this representation

$$[x_k] = x_k + [B_k][\tilde{r}_k]. \tag{61}$$

In the context of our algorithm in Part **3** we obtain

$$[\overline{x}_{k+1}] = \overline{x}_{k+1} + [B_{k+1}][\overline{r}_{k+1}]. \tag{62}$$

Now we have to take into account (56). We set

$$\begin{aligned} x_{k+1} &= \mathrm{m}(\overline{x}_{k+1} + [\Delta]) && (63)\\ [\tilde{r}_{k+1}] &= [\overline{r}_{k+1}] + [B_{k+1}^{-1}]\,(\overline{x}_{k+1} + [\Delta] - x_{k+1})\,. && (64) \end{aligned}$$

Evaluation 4. In this representation

$$[x_k] = x_k + C_k[r_0] + [B_k][\tilde{r}_k]. \tag{65}$$

In the context of our algorithm in part 3 we obtain

$$[\overline{x}_{k+1}] = \overline{x}_{k+1} + C_{k+1}[r_0] + [B_{k+1}][\overline{r}_{k+1}]. \tag{66}$$

Equation (56) is taken into account exactly in the same way as in previous evaluations, i.e., we use (63) and (64).

The fourth evaluation was used in the proof of Theorem 9.

6.7 Computation of the Poincaré map

If as in [ZLo] we assume that the section is given by $\alpha(x) = 0$, $\alpha' > 0$ then an algorithm discussed in Section 5 in [ZLo] also applies in the present context provided that we have a procedure which gives a rigorous estimate between time steps for the x-variable in (49). This procedure is described below.

Input parameters:

- h_k - a time step,
- $[x_k] \subset \mathbb{R}^{n_1}$, such that $\pi_x\varphi(t_k, [x_0], [y_0]) \subset [x_k]$,
- $[x_{k+1}] \subset \mathbb{R}^{n_1}$, such that $\pi_x\varphi(t_k + h_k, [x_0], [y_0]) \subset [x_{k+1}]$,
- a convex and compact set $[W_2] \subset \mathbb{R}^{n_1}$ and a convex set $[W_y] \subset \mathbb{R}^{n_2}$, such that
$$\varphi([t_k, t_k + h_k], [x_0], [y_0]) \subset [W_2] \times [W_y], \tag{67}$$

- $y_c \in [W_y]$,
- $[\overline{x}_{k+1}] \subset \mathbb{R}^{n_1}$, such that $\pi_x \overline{\varphi}(h_k, [x_k], y_c) \subset [\overline{x}_{k+1}]$,
- $[W_1] \subset \mathbb{R}^{n_1}$ compact and convex, such that $\pi_x \overline{\varphi}([0, h_k], [x_k], y_c) \subset [W_1]$.

Output:
We compute $[E_k] \subset \mathbb{R}^{n_1}$ such that

$$\pi_x \varphi(t_k + [0, h_k], [x_0], [y_0]) \subset [E_k],$$

Algorithm:

- if $0 \notin f_i([W_2], [W_y])_i$, then the i-th coordinate is strictly monotone on $[W_2] \times [W_y]$, hence we set

$$[E_k]_i = \text{hull}([x_k]_i, [x_{k+1}]_i)$$

- if $0 \in f_i([W_2], [W_y])$, then we compute $[\overline{E}_k] \subset \mathbb{R}^{n_1}$, such that

$$\pi_x \overline{\varphi}([0, h_k], [x_k], y_c) \subset [\overline{E}_k] \tag{68}$$

  using a procedure for an ODE described in [ZLo]. This procedure requires as input data: h_k, $[x_k]$, $[\overline{x}_{k+1}]$ and $[W_1]$.

  We have

$$\pi_x \varphi(t_k + [0, h_k], [x_0], [y_0])_i \subset [E_k]_i = [\overline{E}_k]_i + [\Delta]_i. \tag{69}$$

References

[C] L. Cesari, Functional analysis and Galerkin's method, *Mich. Math. Jour.* **11** (1964) 383-414.

[CEES] P. Collet, J.-P. Eckmann, H. Epstein, J. Stubbe *Analyticity for the Kuramoto–Sivashinsky equation*, Physica D 67, (1993), 321–326

[DG] J. Dugundji and A. Granas, *Fixed Point Theory*, Monografie Matematyczne 61, PWN, Warszawa 1982

[FT] C. Foias and R. Temam, Gevrey Class Regularity for the Solutions of the Navier-Stokes Equations, *Journal of Functional Analysis*, Vol. 87, No. 2, 1989, 359–369,

[GaZ] Z. Galias and P. Zgliczyński, Computer assisted proof of chaos in the Lorenz system, *Physica D*, 115, 1998,165–188

[GiZ] M. Gidea and P. Zgliczyński, Covering relations for multidimensional dynamical systems, submitted, *http://www.im.uj.edu.pl/~zgliczyn*

[HG] J.K. Hale and G. Raugel, Galerkin methods and regularity. *Differential equations and dynamical systems (Lisbon, 2000),* 173–188, Fields Inst. Commun., 31, Amer. Math. Soc., Providence, RI, 2002.

[HYC] S. M. Hammel, J.A. Yorke and C. Crebogi, Do Numerical Orbits of Chaotic Dynamical Processes Represent True Orbits? *Journal of Complexity* 3, 136–145, 1987

[HYC1] S. M. Hammel, J.A. Yorke and C. Crebogi, Numerical Orbits of Chaotic Dynamical Processes Represent True Orbits, *Bull Am. Math. Soc* 19, 465–470, 1988

[H] W. B. Hayes, Rigorous Shadowing of Numerical Solutions of Ordinary Differential Equations by Containment, Ph.D. Thesis, Computer Science Dept., Univ. of Toronto, 2001, *http://www.cs.toronto.edu/NA/reports.html#hayes-01-phd*

[K] H.-O. Kreiss, Fourier expansions of the solutions of the Navier-Stokes equations and their exponential decay rate, in *Analyse mathmatique et applications,* 245–262, Gauthier-Villars, Montrouge, 1988.

[KT] Y. Kuramoto, T. Tsuzuki, Persistent propagation of concentration waves in dissipative media far from thermal equilibrium, *Prog. Theor. Phys.*, **55**,(1976), 365

[Lo] R.J. Lohner, *Computation of Guaranteed Enclosures for the Solutions of Ordinary Initial and Boundary Value Problems*, in: Computational Ordinary Differential Equations, J.R. Cash, I. Gladwell Eds., Clarendon Press, Oxford, 1992.

[Lo1] R.J. Lohner, *Einschliessung der Lösung gewonhnlicher Anfangs- and Randwertaufgaben und Anwendungen* , Universität Karlsruhe (TH), these 1988

[Mo] R.E. Moore, *Methods and Applications of Interval Analysis,* SIAM, Philadelphia, 1979

[MZ] M. Mrozek, P. Zgliczyński, Set arithmetic and the enclosing problem in dynamics, *Annales Pol. Math.*, 2000, 237–259

[R] A. Ralston, *A First Course in Numerical Analysis*, 1965 McGraw-Hill, Inc

[S] G.I. Sivashinsky, Nonlinear analysis of hydrodynamical instability in laminar flames – 1. Derivation of basic equations, *Acta Astron.* 4 (1977), no. 11-12, 1177–1206

[W] W. Walter, *Differential and integral inequalities,* Springer-Verlag Berlin Heidelberg New York, 1970

[Z0] P. Zgliczyński, Fixed point index for iterations of maps, topological horseshoe and chaos, *Topological Methods in Nonlinear Analysis* **8**, (1996), 169–177

[Z1] P. Zgliczyński, Sharkovskii's Theorem for multidimensional perturbations of 1-dim maps, *Ergodic Theory and Dynamical Systems*, (1999), 19, 1655–1684

[Z2] P. Zgliczyński, Computer assisted proof of chaos in the Rössler equations and the Hénon map, *Nonlinearity* **10** *(1997), 243-252.*

[Z] P. Zgliczyński, Trapping regions and an ODE-type proof of an existence and uniqueness for Navier-Stokes equations with periodic boundary conditions on the plane, *Univ. Iag. Acta Math.* to appear, *http://arXiv.org/abs/math/0103053*

[ZAKS] P. Zgliczyński, Attracting fixed points for the Kuramoto-Sivashinsky equation - a computer assisted proof, *SIAM Journal on Applied Dynamical Systems*, (2002) Volume 1, Number 2 pp. 215-235, *http://epubs.siam.org/sam-bin/dbq/article/40176*

[ZLo] P. Zgliczyński, C^1-Lohner algorithm, *Foundations of Computational Mathematics,* (2002) 2:429–465

[ZPLo] P. Zgliczyński, Lohner Algorithm for ODEs with a controlled perturbation, *http://www.im.uj.edu.pl/~zgliczyn*

[ZM] P. Zgliczyński and K. Mischaikow, Rigorous Numerics for Partial Differential Equations: the Kuramoto-Sivashinsky equation, *Foundations of Computational Mathematics,* (2001) 1:255-288