Multivalued Discrete Dynamical System Framework for Surface Modeling

Part I: Mathematical model

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Outline

1. Part I: Mathematical model
   - Why discrete model?
   - Background from the classical Morse theory
   - Critical components
   - Dynamical systems
   - Stable, unstable manifolds, and Morse connections graph

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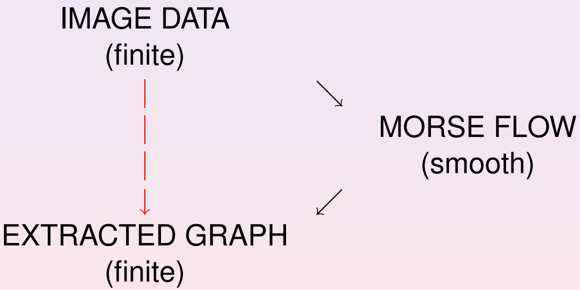
Classical CS Applications of Morse Theory



Outline

Classical Morse Theory

Let M be a smooth manifold and $f : M \to \mathbb{R}$ a smooth function.

definition

- f Morse function $\Leftrightarrow$ all of its critical points are *non degenerate*.
- The index $\lambda(p)$ is the number of negative eigenvalues of the determinant of the Hessian $H_f(p)$.

$$\lambda(p) = \dim W^s(p) = \dim \left\{ q \in M \mid \lim_{m \to \infty} \varphi(t, q) = p \right\},$$

- $W^u(p) = \{q \in M \mid \lim_{m \to -\infty} \varphi(t, q) = p\}$, $\dim W^u(p) = \dim(M) - \lambda(p)$.

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  $\dim W^u(p) = \dim(M) - \lambda(p)$.

Covering properties

- For $p \neq q$:

$$W^u(p) \cap W^u(q) = \emptyset \text{ and } W^s(p) \cap W^s(q) = \emptyset. \tag{1}$$

- If $\{p_1, \dots, p_k\}$ represents the set of all critical points of a f:

$$\bigcup_{i=1}^{k} W^s(p_i) = M \text{ and} \tag{2}$$

$$\bigcup_{i=1}^{k} W^u(p_i) = M \tag{3}$$

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Smooth Morse Connection Graph

Definition

The smooth Morse Connections Graph is a graph whose nodes V are critical points of f and edges E are defined as follows:

$$E = \{(p,q) \in V \times V \mid \exists \text{ trajectory connecting } p \text{ to } q\}$$

Equivalently, (p,q) is an edge of the graph if

$$W^u(p) \cap W^s(q) \neq \emptyset \text{ or } W^u(q) \cap W^s(p) \neq \emptyset.$$

Outline

The values of f are only known on a set of pixels, denoted by X. Thus, $f : X \longrightarrow \mathbb{R}$ is a discrete function.
We interpret pixels $x \in X$ as unit squares of the form $x = [k, k+1] \times [l, l+1]$, k, l integers, for a chosen grid scale.
Given any $A \subset X$ the support of A is the set $|A| \in \mathbb{R}^2$ given by

$$|A| = \bigcup A.$$

Thus the association $A \mapsto |A|$ provides the passage from combinatorics to geometry.

Definition

Two pixels x and y in X are

- 0-connected, denoted by $x0y$, if $\exists$ a sequence $x_1 = x, x_2, \ldots x_n = y$ such that $x_i \cap x_{i+1}$ contains a vertex for all i,
- 1-connected, denoted by $x1y$, if $\exists$ a sequence $x_1 = x, x_2, \ldots x_n = y$ such that $x_i \cap x_{i+1}$ contains an edge.

This relation is an equivalence relation and the 1-connectedness implies the 0-connectedness.

x y
z t

Figure: $x1y$, $x1z$, $y1t$, $z1t$, $x0t$, $y0z$.

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This relation is an equivalence relation and the 1-connectedness implies the 0-connectedness.

x y
z t

Figure: $x1y,\ x1z,\ y1t,\ z1t,\ x0t,\ y0z$.

Definition

A component $\mathcal{X}$ is a maximal set of 0-connected pixels with the same value of f.

1	-1	1	1	1
1	0	0	1	2
1	1	1	0	1
1	1	1	2	1

Figure: Pixels with the 0 value form a component.

Definition

The distance between two adjacents pixels $x, y \in X$ is

$$dist(x, y) = \begin{cases} \sqrt{2} & \text{if } x \cap y \text{ is a vertex,} \\ 1 & \text{if } x \cap y \text{ is an edge.} \end{cases}$$

Definition

The directional derivative of f at x in the direction of y is

$$\frac{\partial f}{\partial y}(x) = \frac{f(y) - f(x)}{dist(x, y)}.$$

By convention, we define $\frac{\partial f}{\partial x}(x) = 0$.

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Definition

$$\overline{wrap}\,(x) := \left\{ y \in bd(x) \mid \frac{\partial f}{\partial y}(x) > 0 \right\},$$

$$\underline{wrap}\,(x) := \left\{ y \in bd(x) \mid \frac{\partial f}{\partial y}(x) < 0 \right\}.$$

$$bd(x) = wrap(x) \setminus \{x\}.$$

If A is subset of X:

$$\overline{wrap}\,(A) := \{ y \in bd(A) \mid f(y) > f(x) \text{ for all } x \in A \cap bd(y) \},$$

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Definition

Given a function $f : X \to \mathbb{R}$, a component $\mathcal{X} \subset X$ is called a

$$\begin{aligned}
\text{maximum} \quad &\Longleftrightarrow \quad \frac{\partial f}{\partial y}(x) < 0, \;\; \forall x \in \mathcal{X}, \forall y \in bd(x) \cap bd(\mathcal{X}) \\
&\Longleftrightarrow \quad f(y) < f(x), \;\; \forall x \in \mathcal{X}, \forall y \in bd(x) \cap bd(\mathcal{X}) \\
\text{minimum} \quad &\Longleftrightarrow \quad \frac{\partial f}{\partial y}(x) > 0, \;\; \forall x \in \mathcal{X}, \forall y \in bd(x) \cap bd(\mathcal{X}) \\
&\Longleftrightarrow \quad f(y) > f(x), \;\; \forall x \in \mathcal{X}, \forall y \in bd(x) \cap bd(\mathcal{X}).
\end{aligned}$$

2	0	4	0
2	1	3	4
0	4	0	4

Figure: Adjacent center pixels with values 1 and 3 are both saddles and they form a component which is not a level set of f but it has a property of a 4-saddle.

Definition

A *k*-saddle component is a maximal connected set of saddle pixels such that $\overline{wrap}\,(\mathcal{X}) \neq \emptyset$ and its support is 1-disconnected with $(k+1)$ 1-connected components, $k \geq 1$.

Definition

A set $\mathcal{X}$ is a critical component if it is either a minimum, maximum or a k-saddle.

Outline

Definition

- A map $F : X \times \mathbb{Z} \rightrightarrows X$ is called a discrete multivalued dynamical system (dmds) on X if:
  1. For all $x \in X$, $F(x, 0) = \{x\}$;
  2. For all $n, m \in \mathbb{Z}$ with $nm > 0$ and all $x \in X$, $F(F(x, n), m) = F(x, n + m)$.
- *A map $F : X \times \mathbb{N} \rightrightarrows X$ is called a discrete multivalued semidynamical system (dmss) if (1) holds, and if (2) is satisfied for all $n, m \in \mathbb{N}$.*

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We propose several experimental approaches to defining a dynamics of a function $f : X \to \mathbb{R}$ on a set of pixels X. For abbreviation, given $x \in X$, let

$$max = \max_{y \in wrap(x)} \frac{\partial f}{\partial y}(x) \quad \text{and} \quad min = \min_{y \in wrap(x)} \frac{\partial f}{\partial y}(x).$$

Exact steepest trajectory

$$\mathcal{F}_+(x) = \left\{ y \in wrap(x) \mid \frac{\partial f}{\partial y}(x) = max \right\},$$

$$\mathcal{F}_-(x) = \left\{ y \in wrap(x) \mid \frac{\partial f}{\partial y}(x) = min \right\}.$$

Admissible error bound approach

$$\mathcal{F}_+(x) := \left\{ y \in wrap(x) \mid \frac{\partial f}{\partial y}(x) \in [(1-\epsilon)max, max] \right\}$$

$$\mathcal{F}_-(x) := \left\{ y \in wrap(x) \mid \frac{\partial f}{\partial y}(x) \in [min, (1-\epsilon)min] \right\}.$$

‘Permissive’ approach

$$\mathcal{F}_+(x) := \left\{ y \in wrap(x) \mid \frac{\partial f}{\partial y}(x) \geq 0 \right\},$$

$$\mathcal{F}_-(x) := \left\{ y \in wrap(x) \mid \frac{\partial f}{\partial y}(x) \leq 0 \right\}.$$

Outline

Definition

Let $\mathcal{F} : X \rightrightarrows X$ be a generator of a dmss. The stable and unstable manifolds of a point $x \in X$ relatively to $\mathcal{F}$ are

$$\begin{aligned} W^u(x, \mathcal{F}) &:= \bigcup_{n \geq 1} \mathcal{F}^n(x); \\ W^s(x, \mathcal{F}) &:= \bigcup_{n \geq 1} \mathcal{F}^{-n}(x). \end{aligned}$$

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The stable and unstable manifolds of a critical component P are defined by

$$\begin{aligned} W^u(P, \mathcal{F}) &:= \bigcup_{x \in P} W^u(x); \\ W^s(P, \mathcal{F}) &:= \bigcup_{x \in P} W^s(x). \end{aligned}$$

Proposition

If p and q are pixels such that $W^u(p) \cap W^s(q) \neq \emptyset$, then there exists a trajectory from p to q.

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Corollary

Le P and Q be critical components such that $W^u(P) \cap W^s(Q) \neq \emptyset$. Then there exists a trajectory connecting P to Q, in the sense, that it connects a point in P to a point in Q.

Definition

Let P and Q be two critical components of $f : X \to \mathbb{R}$. There is

- an upward connection from P to Q, denoted $P \nearrow Q$, if

$$W^u(P, \mathcal{F}+) \cap W^s(Q, \mathcal{F}_+) \neq \emptyset;$$

- a downward connection from P to Q, denoted $P \searrow Q$, if

$$W^u(P, \mathcal{F}_-) \cap W^s(Q, \mathcal{F}_-) \neq \emptyset.$$

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Covering property

Theorem

Let $\{P_1, \ldots ; P_k\}$ *be the set of all critical components of* $f : X \to \mathbb{R}$*. Then*

$$\bigcup_{i=1}^{k} W^s(P_i, \mathcal{F}_+) = X = \bigcup_{i=1}^{k} W^s(P_i, \mathcal{F}_-).$$

Morse Connection Graph

Definition

The Morse Connections Graph $MCG_f = (V_f, E_f)$ is a graph whose nodes V_f and edges E_f are defined as follows:

$$V_f = \{\text{critical components of } f\};$$

$$E_f = \left\{(P_i, P_j) \in V_f \times V_f \mid P_i \nearrow P_j \text{ or } P_i \searrow P_j\right\}$$

Equivalently, (P_i, P_j) is an edge of the graph if

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