Rigorous numerics to verify heteroclinic connections

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Contents:

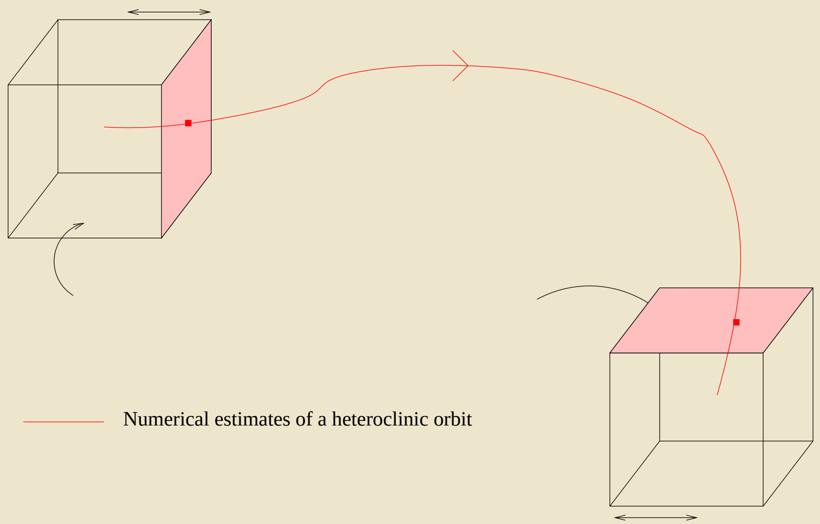
Transverse heteroclinic connections

Let $F\colon \mathbb{R}^n \to \mathbb{R}$ be smooth. We consider a gradient system

$$\dot{x} \;=\; -\nabla F(x) \;=:\; f(x)\,, \quad x \in \mathbb{R}^n \tag{1}$$

with two hyperbolic equilibria x_R and $x_A \in \mathbb{R}^n$ with Morse indices Σ^{k+1} and $\Sigma^k (0 \le k \le n-1)$, respectively. From purely computational investigations we have some numerical "evidence" that there could be a heteroclinic connection between x_R (repeller) and x_A (attraktor), which is transverse (i.e. the stable $(W^s(x_A))$ and unstable $(W^u(x_R))$ manifolds intersect transversely).

Let $\tilde{x}^R$ be the numerical approximation of x^R and $\tilde{x}^A$ be the numerical approximation of x^A. The numerical situation is as follows:



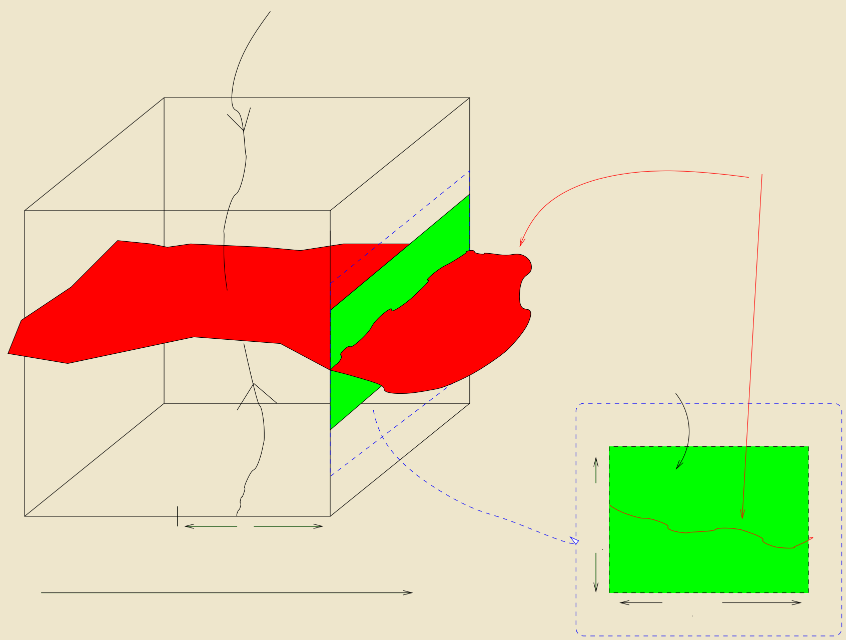
left cube: $\mathcal{C}_R$

right cube: $\mathcal{C}_A$

in pink: the Poincaré sections P_R and P_A

Here $\mathcal{C}_R = B_r^{\|\cdot\|_\infty}(\tilde{x}_R)$ and $\mathcal{C}_A = B_\delta^{\|\cdot\|_\infty}(\tilde{x}_A)$ are cubes around the approximative equilibria. P_R and P_A are faces of $\mathcal{C}_R$ and $\mathcal{C}_A$, respectively, where the numerical orbit leaves or enters the cube.

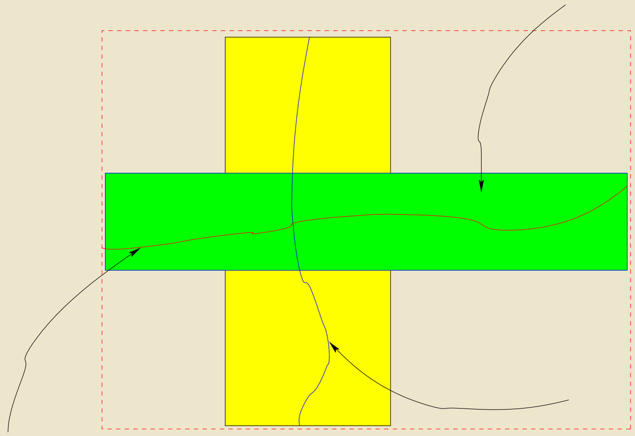
Suppose now we can control the true unstable manifold $W^u(x^R)$ in all stable directions, on the Poincaré section P_R



local flat unstable manifold intersecting P_R ($n = 3$); x^R with index Σ^2

Similarly, the stable manifold of x_A may be controlled.

The main idea, is to move the subset of the Poincaré section which "clamps" the unstable manifold of x^R by an algorithm until it intersects the subset of the Poincaré section, which "clamps" the stable manifold of x^A.



Situation on the Poincaré section P_A
x^R with index Σ^2
x^A with index Σ^1

Here both $W^u(x^R)$ and $W^s(x^A)$ are 2-dimensional.

Goal: Construction of a rigorous numerical method (combined analysis and numerical verification) that indeed proves the existence of a heteroclinic connection between x_R and x_A.

Local situation near the equilibria

Let $x_0 \in \mathbb{R}^n$ be a hyperbolic equilibrium of $\dot{x} = f(x)$, with regular and symmetric linearization $Df(x_0) \in \mathbb{R}^{n \times n}$. Then there exists some orthogonal matrix $T \in O(n)$ such that

$$T\, Df(x^0)\, T^{-1} \;=\; \Lambda_0 \;:=\; diag\,(\lambda_1, ..., \lambda_n)$$

with eigenvalues $\lambda_i \neq 0$.

Problem: All these quantities are only known as numerical approximations up to some error $\varepsilon > 0$, i.e. we can calculate some $\tilde{x}^0 \in \mathbb{R}^n$, some $\tilde{\Lambda}_0 = diag\,(\tilde{\lambda}_1, ..., \tilde{\lambda}_n)$, $\tilde{\lambda}_i \neq 0$, and some $\tilde{T} \in O(n)$ with

(ERR) $\|x^0 - \tilde{x}^0\|_2 \;\leq\; \varepsilon\, r, \quad \|\Lambda_0 - \tilde{\Lambda}_0\|_2 \;\leq\; \varepsilon\, \|\tilde{\Lambda}_0\|_2$
and $\|\tilde{T} - T\|_2 \;\leq\; \varepsilon \;, \; \|\tilde{T}^{-1} - T^{-1}\|_2 \;\leq\; \varepsilon$

Lemma 1: Under the assumption (ERR) and using the linear map $x \to \tilde{T}(x - \tilde{x}^0)$, the differential equation $\dot{x} = f(x)$ in $B_{2r}(x^0)$ is equivalent to the ODE

$$\dot{y} = \tilde{\Lambda}_0 \, y + \tilde{h}(y) + C_\varepsilon(y) , \qquad \text{for all} \quad y \in B_{2r}(0) , \qquad (2)$$

where $\tilde{h} \colon \mathbb{R}^n \to \mathbb{R}^n$ and $C_\varepsilon \colon \mathbb{R}^n \to \mathbb{R}^n$ satisfy

$$\|\tilde{h}(y)\|_2 \leq const \cdot r^2 \quad \text{and} \quad \|C_\varepsilon(y)\|_2 \leq const \cdot r \, \varepsilon \, .$$

Observe that the fixed point x^0 of (1) is mapped to a fixed point $y^0 = \tilde{T}(x^0 - \tilde{x}^0)$ of (2) near zero.

Obviously it is much easier to make calculation with (2). In particular, we obtain from (2) precise estimates for points in the stable/unstable manifolds. For instance:

Lemma 2: (flat unstable manifold)

For $y^* = (y_1^*, ..., y_n^*) \in W^u(y^0) \cap B_{2r}(0)$ with backward orbit lying in $B_{2r}(0)$ we obtain

$|y_i^*| \leq const \cdot (\varepsilon \cdot r + r^2)$ for all i with $\tilde{\lambda}_i < 0$ (*stable directions*!).

(cf. Figure p. 4) A similar lemma holds for the "flat stable manifold".

Remark: The numerical conditions to be assumed may be strengthend (spectral gap assumptions) in order to guarantee that $W^u(y^0) \cap B_{2r}(0)$ is in fact a graph over the linear subspace of unstable eigenfunctions.

Transport of the unstable manifold of the repeller

Clearly the estimates obtained for (2) can be used for (1) near the equilibria x_R (repeller) and x_A (attractor). In order to bring that information together we somehow have to transport the unstable manifold (or at least the relevant part of it) into a neighborhood of x_A.

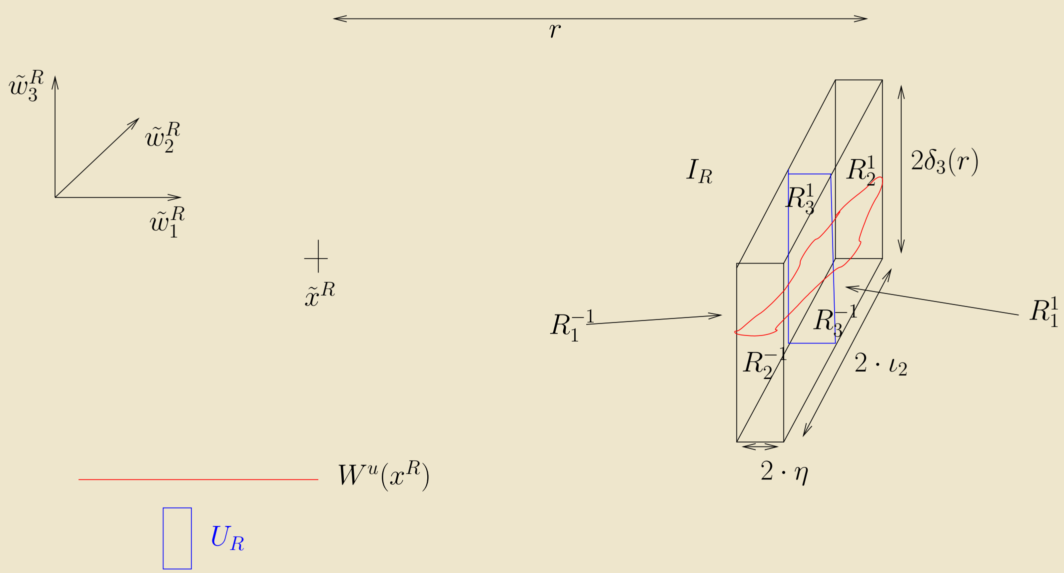
Let $\tau > 0$ and $\Phi_\tau \colon \mathbb{R}^n \to \mathbb{R}^n$ be the flow corresponding to $\dot{x} = f(x)$. Then an enclosing algorithm $\mathcal{A}_\tau \colon G^n \to G^n$ has the property

$$\Phi_\tau(S) \subset \mathcal{A}_\tau(S) \quad \text{for all} \quad S \in G^n \ ,$$

where G^n contains ”generalized cubes” in $\mathbb{R}^n$.

I.e. we deal with numerical approximations which in a certain sense enclose the exact quantities (our implementation uses the CAPD–library of Zglicszynski & Wilczak).

The next figure illustrates the part I_R of $B_r^{\|\cdot\|_\infty}(\tilde{x}^R)$ which contains the relevant part of $W^u(x^R)$ and therefore has to be transported by $\mathcal{A}_\tau$ into a neighborhood of x^A.

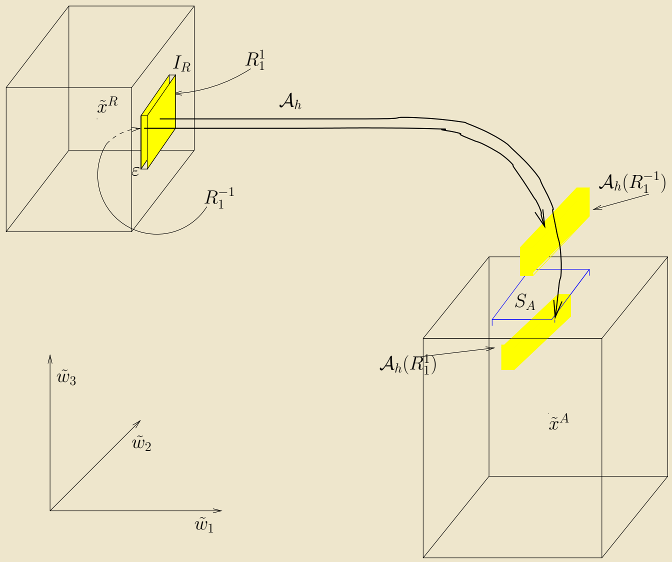


Neighborhood of $\tilde{x}^R$
x^R with index Σ^2
($n = 3$ and $k = 2$)

I_R corresponds to the Poincaré section P_R, but is small orthogonal to $W^u(x^R)$ and thickened in the direction of the heteroclinic.

With $\mathcal{A}_\tau$ we transport

- The whole set I_R
- Faces of I_R
- In particular those faces of I_R which contain parts of $W^u(x^R)$.



I_R is transported to a neighborhood of $\tilde{x}^A$ (near P_A)

Intersection of $W^u(x_R)$ and $W^s(x_A)$

Goal:

Formulate conditions on the images of $\mathcal{A}_\tau$ which guarantee $W^u(x^R) \cap W^s(x^A) \neq \emptyset$, and therefore the existence of a heteroclinic connection.

Assumptions:

(A1) $\mathcal{A}_\tau$ is an enclosing algorithm for $\dot{x} = f(x), \quad x \in \mathbb{R}^n$

(A2) $\tilde{W}^u(x^R) := W^u(x^R) \cap I_R$ is "clamped" in I_R from boundary to boundary, that is between faces which correspond to unstable directions (cf. Figure p. 11)

(A3) $\tilde{W}^s(x^A) := W^s(x^A) \cap P_A$ is similarly "clamped" between boundaries in the Poincaré section of the attractor. (for $k = 0$; i.e. x^A stable, we assume $B_r^{\|\cdot\|_\infty}(x^A) \subset W^s(x^A)$.)

Theorem 1: $(\boldsymbol{\Sigma^1 \to \Sigma^0})$

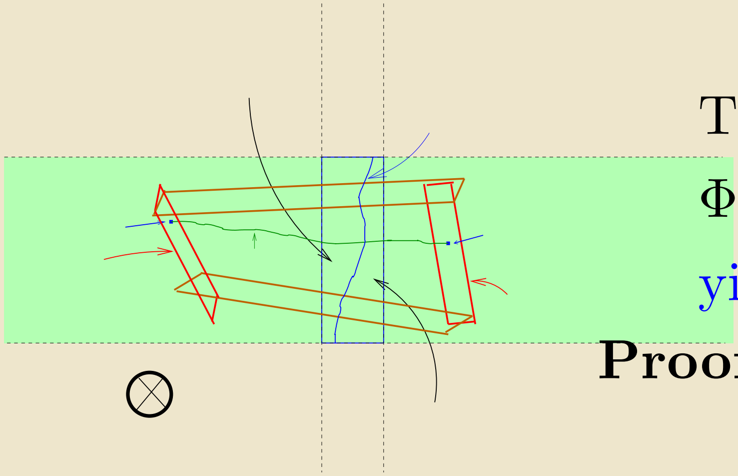
Assume x^R and x^A have Morse index Σ^1 and Σ^0, respectively. We assume besides **(A1)** – **(A3)** that

(A4) For sufficiently large $\tau > 0$ we have $\mathcal{A}_\tau(I_R) \subset B_r(\tilde{x}^A)$

Then $\Phi_\tau\left(\tilde{W}^u(x^R)\right) \cap W^s(x^A) \neq \emptyset$, i.e. there is a connecting orbit between x^R and x^A.

Proof: The enclosing algorithm guarantees that at least one point of $W^u(x^R)$ gets transported into a small neighborhood of x^A which is part of the basin of attraction of x^A.

Theorem 2 $(\boldsymbol{\Sigma^2 \rightarrow \Sigma^1})$
Assume x^R and x^A have Morse index Σ^2 and Σ^1, respectively. We assume **(A1)** – **(A3)**. Then $\tilde{W}^s(x^A) = W^s(x^A) \cap P_A$ is a $n-2$ dimensional curve and separates P_A locally into two parts. On the other hand the intersection of $\Phi_\cdot\left(\tilde{W}^u(x^R)\right)$ with P_A is a one–dimensional curve. Clearly one can formulate conditions on $\mathcal{A}_\tau$ of the faces of I_R that guarantee that these two curves lie "orthogonal" like in the following figure:



Then again
$\Phi_\bullet\left(\tilde{W}^u(x^R)\right) \cap W^s(x^A) \neq \emptyset,$
yielding a heteroclinic connection.

Proof: Intermediate value theorem.

Outlook

Besides the cases $\Sigma^1 \to \Sigma^0$ and $\Sigma^2 \to \Sigma^1$ one can also handle the cases $\Sigma^n \to \Sigma^{n-1}$ and $\Sigma^{n-1} \to \Sigma^{n-2}$ through time reversal. Therefore, in $\mathbb{R}^n$ with $n \leq 4$, all transverse heteroclinic connections may be verified numerically.

Other case are more subtle, because manifolds of codimension two or more do not separate the space $\mathbb{R}^n$ into two parts.

Nevertheless, this is current research at our group (in particular Zofia Maczynska).