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On Stability of Equilibrium Solutions in the Restricted Many-Body Problems

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Introduction

Classical Newtonian many-body problem is a well-known dynamical model

$$\frac{d^2 x_j}{dt^2} = -G \sum_{k=1\,(\neq j)}^{n} m_k \frac{x_j - x_k}{r_{jk}^3}, \quad \frac{d^2 y_j}{dt^2} = -G \sum_{k=1\,(\neq j)}^{n} m_k \frac{y_j - y_k}{r_{jk}^3},$$

$$\frac{d^2 z_j}{dt^2} = -G \sum_{k=1\,(\neq j)}^{n} m_k \frac{z_j - z_k}{r_{jk}^3}, \quad (j, k = 1, 2, \ldots, n)$$

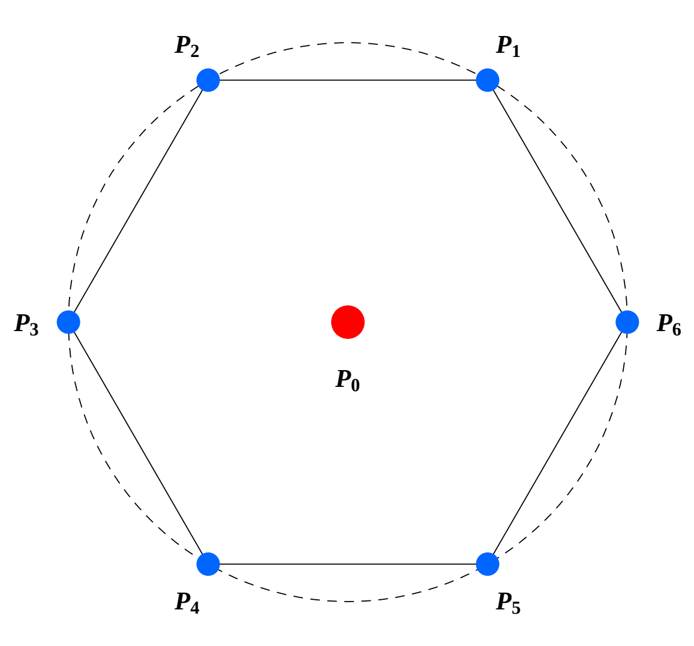
where

$$r_{jk} = \sqrt{(x_j - x_k)^2 + (y_j - y_k)^2 + (z_j - z_k)^2} \; .$$

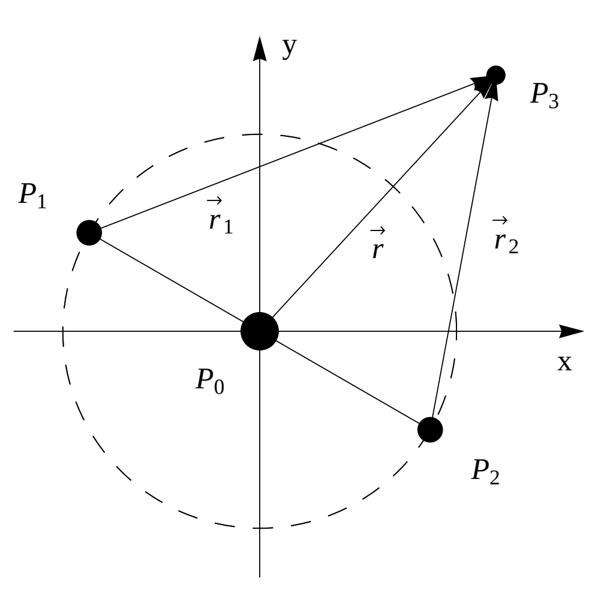
It is well-known also that it is not integrable, in general.
To simplify the three-body problem, Leonard Euler introduced the restricted three-body problem.
The essence: the third body has so small mass that it doesn't influence on the two primaries whose motion is determined by the corresponding solution of the two-body problem. This idea turned out to be very productive, it stimulated development of qualitative theory of differential equations and resulted in the KAM-theory. So it was quite natural that different generalizations of the restricted three-body problem were proposed.

Model

Restricted many-body problem is a generalization of the famous restricted three-body problem.
To formulate a restricted many-body problem it is sufficient to find an exact particular solution of the equations of motion. Then we can add one more body of infinitesimal mass to the system and the problem is to describe its motion in the gravitational field of primary bodies.
Here we consider a system of $(n+1)$ bodies $P_0, P_1, \ldots, P_n$. The body P_0 rests at the origin while the others having equal masses move uniformly about their common center of mass on the same circular orbit and form a regular polygon at any instant of time.



In the simplest case of the restricted four-body problem $(n = 2)$ all formulars and calculations are not too bulky and so we consider this case as an example. The polygon degenerates into a line $P_1 P_2$ and the corresponding solution of the three-body problem is known as Euler collinear solution. We are interested in the motion of the body P_3 having negligible mass in the gravitational field of P_0, P_1, P_2.



A general solution of the equations of motion is not known but some particular solutions can be found. And the problem is to investigate their stability.

Hamiltonian of the system

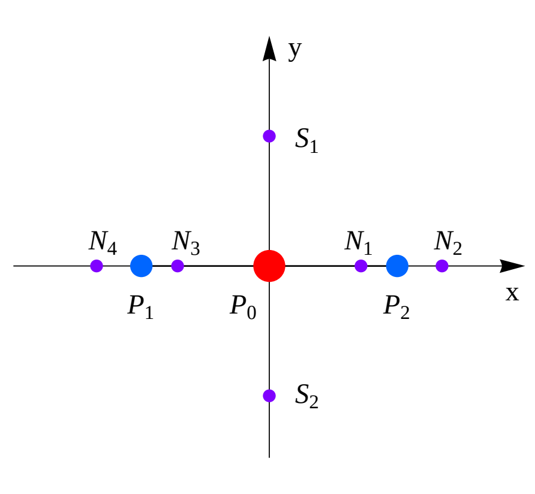
It is convenient to analyze motion of the system in the rotating coordinate frame where the primaries P_0, P_1, P_2 rest. The Hamiltonian of the system is defined in a standard way and is given by (we use cylindrical coordinates ρ, φ, z and polar angle v as independent variable, $\mu = m/m_0$)

```
H = ρ'[v] Pρ[v] + φ'[v] Pφ[v] + z'[v] Pz[v] - Lpol /. rul2 //
   Collect[#, {Pρ[v], Pφ[v], Pz[v]}, Simplify] &
```

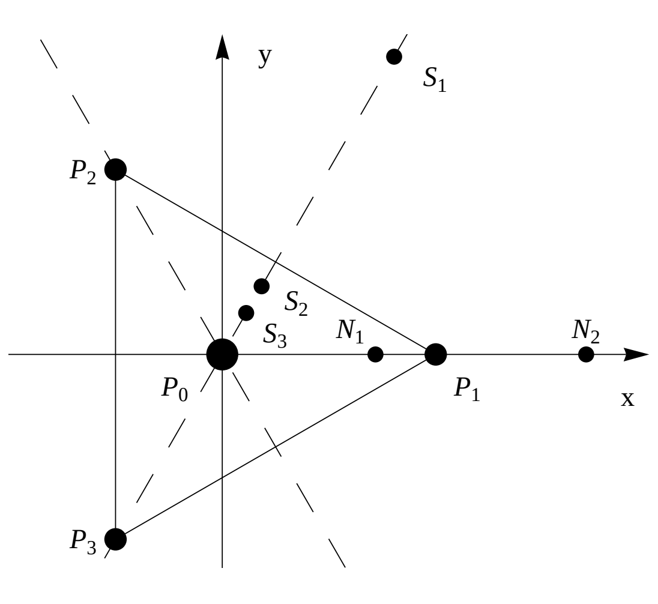
$$\frac{Pz[v]^2}{2} + \frac{P\rho[v]^2}{2} - P\varphi[v] + \frac{P\varphi[v]^2}{2\rho[v]^2} - \frac{1}{4+\mu}\, 4 \left(\frac{1}{\sqrt{z[v]^2 + \rho[v]^2}} + \mu\left(\frac{1}{\sqrt{1 + z[v]^2 - 2\,\mathrm{Cos}[\varphi[v]]\,\rho[v] + \rho[v]^2}} + \frac{1}{\sqrt{1 + z[v]^2 + 2\,\mathrm{Cos}[\varphi[v]]\,\rho[v] + \rho[v]^2}}\right)\right)$$

One can easily obtain equations of motion and show that there are six equilibrium positions of the body P_3.

The corresponding positions of the bodies P_0, P_1, P_2, P_3 are shown below.



The points N_1, N_2, N_3, N_4 correspond to the ***radial equilibrium solutions***, while the points S_1, S_2 represent the ***bisector equilibrium solutions***.
In the case of $n = 3$ some of the corresponding equilibrium positions are shown below.



Hamiltonian in the neighbourhood of the bisector equilibrium solution

In order to obtain equations of the perturbed motion we add small perturbations $q_1, q_2, q_3, p_1, p_2, p_3$ to the bisector equilibrium solution according to the following rule.

$$\text{rul4} = \left\{\rho \to (R + \delta\, q1\, \&),\ P\rho \to (\delta\, p1\, \&),\ \varphi \to \left(\frac{\pi}{2} + \delta\,\frac{q2}{R}\,\&\right),\ P\varphi \to \left(R^2 + \delta\, R\, p2\, \&\right),\ z \to (\delta\, q3\, \&),\ Pz \to (\delta\, p3\, \&)\right\};$$

Here a multiplier δ is introduced to simplify further calculations.

In order to get an expansion of the Hamiltonian in powers of perturbations $q_1, q_2, q_3, p_1, p_2, p_3$ in the neighbourhood of the equilibrium solution it is sufficient to find its expansion in powers of δ about the point $\delta = 0$. The corresponding expansion up to the fourth order is given by

$$H = H_2 + H_3 + H_4 + \ldots,$$

where a quadratic term H_2 is given by

```
H2 = Coefficient[Hexp, δ, 2] // Collect[#, {p1, p2, p3, q1, q2, q3}, Simplify[#, R > 1] &] &
```

$$\frac{p1^2}{2} + \frac{p2^2}{2} + \frac{p3^2}{2} - 2\,p2\,q1 - \frac{12\,q2^2\,\mu}{(1+R^2)^{5/2}\,(4+\mu)} + \frac{2\,q3^2\left(\frac{1}{R^3} + \frac{2\mu}{(1+R^2)^{3/2}}\right)}{4+\mu} + q1^2\left(\frac{3}{2} - \frac{4}{R^3\,(4+\mu)} - \frac{12\,R^2\,\mu}{(1+R^2)^{5/2}\,(4+\mu)} + \frac{4\,\mu}{(1+R^2)^{3/2}\,(4+\mu)}\right)$$

The terms of the third and fourth order are obtained in a similar way.

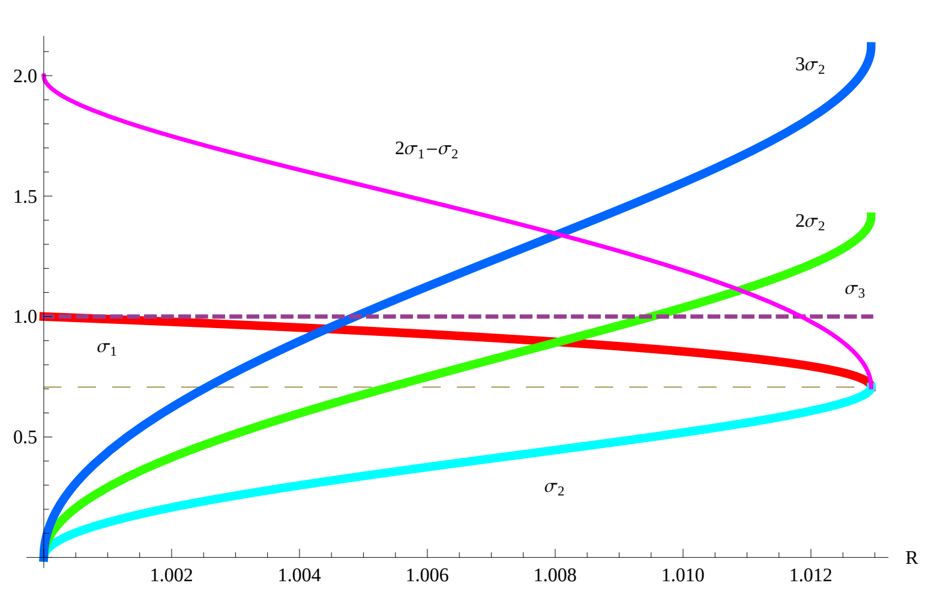
Characteristic exponents

If mass parameter μ belongs to the interval $0 < \mu < \mu_{max} = 0.0853217$ (or equilibrium distance $R \in (1, R_{max})$) characteristic exponents of the system are purely imaginary numbers $(\lambda_{1,2} = \pm i\,\sigma_1,\ \lambda_{3,4} = \pm i\,\sigma_2,\ \lambda_{5,6} = \pm i\,\sigma_3)$, where σ_1, σ_2 and σ_3 are given by

```
eigenval = {σ1 → (σ /. solPol[[6]]), σ2 → (σ /. solPol[[4]]), σ3 → (σ /. solPol[[2]])}
```

$$\left\{\sigma_1 \to \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 - 12\,b + 4\,b^2}},\ \sigma_2 \to \frac{\sqrt{1 - \sqrt{1 - 12\,b + 4\,b^2}}}{\sqrt{2}},\ \sigma_3 \to 1\right\}$$

Their dependences on the parameter R are shown below.



This picture shows that there are five values of the parameter R in the interval $1 < R < R_{max} = 1.01294$ for which the resonance conditions of the third $(\sigma_1 = 2\sigma_2,\ \sigma_3 = 2\sigma_2)$ and the fourth order $(\sigma_1 = 3\sigma_2,\ \sigma_3 = 3\sigma_2,\ 2\sigma_1 - \sigma_2 = \sigma_3)$ are fulfilled.

Note that stability of the equilibrium solutions in the cases of resonance should be analyzed separately.

Normalization of H_2

In order to normalize the quadratic part of the Hamiltonian, we construct the following canonical transformation

$$\begin{aligned}\text{rul5} = \Big\{ & q1 \to 2\,c[1]\,p1 + 2\,c[2]\,p2, \\ & q2 \to -\frac{c[1]\,q1\,\sigma_1\,(1 + b - \sigma_1^2)}{b} + \frac{c[2]\,q2\,\sigma_2\,(1 + b - \sigma_2^2)}{b}, \\ & p1 \to -2\,c[1]\,q1\,\sigma_1 + 2\,c[2]\,q2\,\sigma_2, \\ & p2 \to c[1]\,p1\,(1 + b - \sigma_1^2) + c[2]\,p2\,(1 + b - \sigma_2^2)\Big\};\end{aligned}$$

$$\text{norm} = \left\{c[1] \to \sqrt{\frac{\sigma_1}{(\sigma_1^2 - \sigma_2^2)\,(3 - b + \sigma_1^2)}},\ c[2] \to \sqrt{\frac{\sigma_2}{(\sigma_1^2 - \sigma_2^2)\,(3 - b + \sigma_2^2)}}\right\};$$

```
H2norm = (H2a /. rul5 /. norm /. eigenval //
    Collect[#, {q1, q2, q3, p1, p2, p3}, FullSimplify] &);
```

Then quadratic part of the Hamiltonian ***H2a*** takes a form

$$\text{H2norm} = \frac{1}{2}\,\sigma_1\,(p1^2 + q1^2) - \frac{1}{2}\,\sigma_2\,(p2^2 + q2^2) + \frac{1}{2}\,\sigma_3\,(p3^2 + q3^2);$$

We see that ***H2norm*** is not a positively definite quadratic form and, hence, stability of the bisector equilibrium solutions depends on higher order terms in the expansion of the Hamiltonian. Thus, the stability problem can be solved only in a strict nonlinear formulation.

And we'll invetigate stability of the bisector equilibrium solutions for distances $R \in (1, R_{max})$ considering R as parameter instead of μ.

Arnold's Theorem (for Hamiltonian system with two degrees of freedom)

Let the Hamiltonian of the system

$$H = H_2 + H_3 + H_4 + \ldots,$$

be represented in the form

$$H = \sigma_1\,r_1 - \sigma_2\,r_2 + c_{20}\,r_1^2 + c_{11}\,r_1\,r_2 + c_{02}\,r_2^2 + O\left((r_1 + r_2)^{5/2}\right),$$

where $\sigma_j \geq 0$ and $r_j = \frac{1}{2}(q_j^2 + p_j^2), \quad (j = 1, 2)$,

and the following conditions are satisfied:
a) characteristic exponents of the linearized system $\lambda_{1,2} = \pm i\,\sigma_1, \lambda_{3,4} = \pm i\,\sigma_2$ are purely imaginary numbers;
b) there are no resonances in the system up to the fourth order or

$$n_1\,\sigma_1 + n_2\,\sigma_2 \neq 0,$$

where n_1, n_2 are integers satisfying the condition $0 < |n_1| + |n_2| \leq 4$;
c)

$$c_{20}\,\omega_2^2 + c_{11}\,\omega_1\,\omega_2 + c_{02}\,\omega_1^2 \neq 0,$$

Then equilibrium solution of the Hamiltonian system is **stable in the Liapunov sense**.

Arnold's Theorem (for Hamiltonian system with three degrees of freedom)

Let the Hamiltonian of the system be represented in the form

$$H = H_0 + O\left((r_1 + r_2 + r_3)^{5/2}\right),$$

where

$$H_0 = \sigma_1\,r_1 - \sigma_2\,r_2 + \sigma_3\,r_3 + c_{11}\,r_1^2 + c_{12}\,r_1\,r_2 + c_{13}\,r_1\,r_3 + c_{23}\,r_2\,r_3 + c_{22}\,r_2^2 + c_{33}\,r_3^2,$$

parameters $\sigma_j \geq 0$, $r_j = \frac{1}{2}(q_j^2 + p_j^2), \quad (j = 1, 2, 3)$

and the following conditions are satisfied:
a) characteristic exponents of the linearized system $\lambda_{1,2} = \pm i\,\sigma_1, \lambda_{3,4} = \pm i\,\sigma_2, \lambda_{5,6} = \pm i\,\sigma_3$ are purely imaginary numbers;
b) there are no resonances in the system up to the fourth order or

$$n_1\,\sigma_1 + n_2\,\sigma_2 + n_3\,\sigma_3 \neq 0,$$

where n_1, n_2, n_3 are integers satisfying the condition $0 < |n_1| + |n_2| + |n_3| \leq 4$;
c) for $r_1 = r_2 = r_3 = 0$ at least one of the conditions

$$D_3 = \det\left(\frac{\partial^2 H_0}{\partial r_j\,\partial r_k}\right) \neq \text{ or } D_4 = \det\begin{pmatrix}\frac{\partial^2 H_0}{\partial r_j\,\partial r_k} & \frac{\partial H^*_0}{\partial r_j} \\ \frac{\partial H^*_0}{\partial r_k} & 0\end{pmatrix} \neq 0.$$

Then equilibrium solution of the Hamiltonian system is **stable for the majority of the initial conditions**.

Markeev's Theorems (Resonance cases)

The cases of third- and fourth-order resonances must be analysed separately on the basis of Markeev's theorems.

Third-order resonance

Let the Hamiltonian of the system

$$H = H_2 + H_3 + \ldots,$$

be represented in the form

$$H = \sigma_1\,r_1 - \sigma_2\,r_2 + \sigma_3\,r_3 + B\cos[n_1\,\varphi_1 + n_2\,\varphi_2 + n_3\,\varphi_3]\,r_1^{n_1/2}\,r_2^{n_2/2}\,r_3^{n_3/2} + \tilde{H}(r_j, \varphi_j),$$

where $n_1\,\sigma_1 + n_2\,\sigma_2 + n_3\,\sigma_3 = 0$ $(|n_1| + |n_1| + |n_1| = 3)$, B is some constant and $\tilde{H}(r_j, \varphi_j)$ is 2π-periodic function with respect to variables φ_j and $\tilde{H}(r_j, \varphi_j) = O\left((r_1 + r_2 + r_3)^2\right)$.

Then equilibrium solution of the Hamiltonian system is **unstable** if $B \neq 0$.

Fourth-order resonance

Let the Hamiltonian (16) be represented in the form

$$H = \sigma_1\,r_1 - \sigma_2\,r_2 + \sigma_3\,r_3 + c_{11}\,r_1^2 + c_{22}\,r_2^2 + c_{33}\,r_3^2 + c_{12}\,r_1\,r_2 + c_{13}\,r_1\,r_3 + c_{23}\,r_2\,r_3 + B\,r_1^{n_1/2}\,r_2^{n_2/2}\,r_3^{n_3/2}\cos[n_1\,\varphi_1 + n_2\,\varphi_2 + n_3\,\varphi_3] + O\left((r_1 + r_2)^{5/2}\right),$$

where $n_1\,\sigma_1 + n_2\,\sigma_2 + n_3\,\sigma_3 = 0$ $(|n_1| + |n_1| + |n_1| = 4)$ and B is some constant.

Then equilibrium solution of the Hamiltonian system is **unstable** if

$$\left|c_{11}\,n_1^2 + c_{22}\,n_2^2 + c_{33}\,n_3^2 + c_{12}\,n_1\,n_2 + c_{13}\,n_1\,n_3 + c_{23}\,n_2\,n_3\right| < \left|B\,n_1^{n_1/2}\,n_2^{n_2/2}\,n_3^{n_3/2}\right|$$

Normalization of third-order term in the Hamiltonian

To apply Arnold-Markeev theorems we have to normalize successively the terms H_3, H_4 in the Hamiltonian.

The third order term may be represented in the following general form

$$H_3 = \sum_{i+j+k+l+m+n=3} h^{(3)}_{ijklmn}\,q_1^i\,q_2^j\,q_3^k\,p_1^l\,p_2^m\,p_3^n$$

where coefficients $h^{(3)}_{ijklmn}$ are quite cumbersome functions of parameter μ.
The generating function for canonical transformation normalizing this term is sought in the form

$$S_3 = q_1\,\tilde{p}_1 + q_2\,\tilde{p}_2 + q_3\,\tilde{p}_3 + \sum_{i+j+k+l+m+n=3} s^{(3)}_{ijklmn}\,q_1^i\,q_2^j\,q_3^k\,\tilde{p}_1^l\,\tilde{p}_2^m\,\tilde{p}_3^n$$

where 56 coefficients $s^{(3)}_{ijklmn}$ are to be found from the condition that coefficients $h^{(3)}_{ijklmn}$ become zeros (this can be done if there is no third-order resonances in the system).

On substituting all the coefficients found we observe that the third-order term in the Hamiltonian disappears.

```
H2new + H3new /. δ → 1 /. sol13a /. sol13b /. sol13c /. sol13d /. sol13e /. sol13f /. sol13g //
 Simplify
```

$$\frac{1}{2}\left((p1^2 + q1^2)\,\sigma_1 - (p2^2 + q2^2)\,\sigma_2 + (p3^2 + q3^2)\,\sigma_3\right)$$

Third-Order resonance

There are two third-order resonances in the system considered, namely, $\sigma_1 = 2\sigma_2$ and $2\sigma_2 = \sigma_3 = 1$. In the second case equations determining the corresponding coefficients $s_3(i, j, k, l, m, n)$ do not contain terms of the form $h_3(i, j, k, l, m, n)$. Hence, there exists trivial solution of this system and the corresponding terms $h_3(i, j, k, l, m, n)$ in the transformed hamiltonian will disappear.

In the case of $\sigma_1 = 2\sigma_2$ coefficients $s_3(i, j, k, l, m, n)$ in the expansion of generating function are chosen in such a way that the Hamiltonian takes a form

$$H = 2\,r_1\,\sigma_2 - r_2\,\sigma_2 + r_3\,\sigma_3 + B\sqrt{r_1}\,r_2\cos(\varphi_1 + 2\,\varphi_2) + O\left((r_1 + r_2 + r_3)^2\right),$$

where

$$B = \frac{1}{\sqrt{2}}\left(h^{(3)}_{000\,120} - h^{(3)}_{020\,100} - h^{(3)}_{110\,010}\right).$$

According to Markeev's theorem, if paraameter $B \neq 0$ then equilibrium solution is unstable. Calculation of parameter B gives the following result:

```
B /. sol13h /. rulH3 /. norm /. σ1 → 2 σ2 /. eigenval /. rulb /. solμ /. R → R3res1 // N[#, 10] &
-0.3658222119
```

We see that parameter $B \neq 0$. On the basis of the Markeev theorem we can conclude that **the bisector equilibrium solutions are unstable for $\mu = \mu_1 = 0.0529423$ when the third order resonance takes place.**

Normalization of the fourth-order term

Normalization of the fourth order term can be done similarly but calculations are much more cumbersome.

The fourth-order term is represented in the following general form

$$H_4 = \sum_{i+j+k+l+m+n=4} h^{(4)}_{ijklmn}\,q_1^i\,q_2^j\,q_3^k\,p_1^l\,p_2^m\,p_3^n.$$

with coefficients $h^{(4)}_{ijklmn}$ depending on the parameter μ.
The generating function for canonical transformation normalizing this term is sought in the form

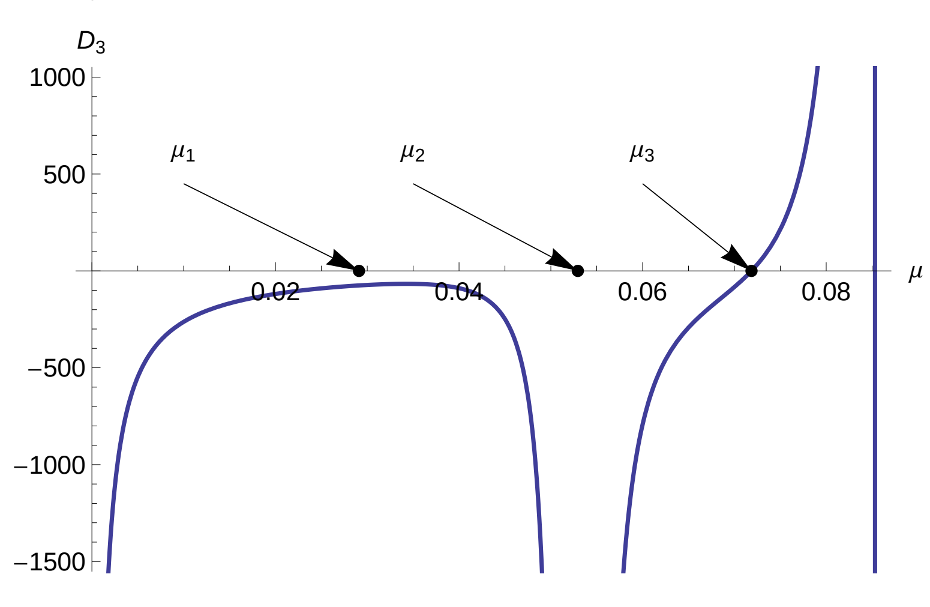
$$S_4 = q_1\,\tilde{p}_1 + q_2\,\tilde{p}_2 + q_3\,\tilde{p}_3 + \sum_{i+j+k+l+m+n=4} s^{(4)}_{ijklmn}\,q_1^i\,q_2^j\,q_3^k\,\tilde{p}_1^l\,\tilde{p}_2^m\,\tilde{p}_3^n$$

where 126 coefficients $s^{(4)}_{ijklmn}$ are to be found from the condition that the fourth-order term H_4 takes the simplest form (note that this term can not be cancelled).
In the case of absence of resonances up to the fourth order the Hamiltonian is reduced to the form

$$H_0 = \sigma_1\,r_1 - \sigma_2\,r_2 + \sigma_3\,r_3 + c_{11}\,r_1^2 + c_{12}\,r_1\,r_2 + c_{13}\,r_1\,r_3 + c_{23}\,r_2\,r_3 + c_{22}\,r_2^2 + c_{33}\,r_3^2,$$

Now we can calculate the determinant

$$D_3 = \det\left(\frac{\partial^2 H_0}{\partial r_j\,\partial r_k}\right)$$

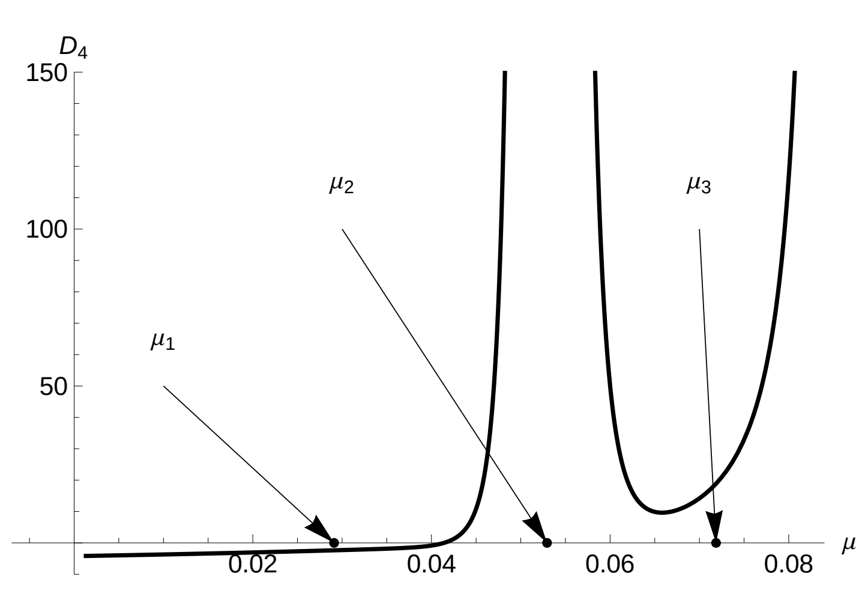


Here μ_1 corresponds to the case of fourth-order resonance $\sigma_1 = 3\sigma_2$, μ_2 corresponds to the case of third-order resonance $\sigma_1 = 2\sigma_2$, and $D_3 = 0$ for $\mu = \mu_3$.

Similar calculations of the determinant

$$D_4 = \det\begin{pmatrix}\frac{\partial^2 H_0}{\partial r_j\,\partial r_k} & \frac{\partial H^*_0}{\partial r_j} \\ \frac{\partial H^*_0}{\partial r_k} & 0\end{pmatrix}.$$

give the following result



We see that in the case of $\mu = \mu_3$ the condition $D_4 \neq 0$ is fulfilled.
Conclusion: **In the absence of the resonances up to the fourth order inclusively the equilibrium positions S_1, S_2 are stable for the majority of the initial conditions.**

Fourth-Order Resonance Case

Note that only in the case of the fourth-order resonance $\sigma_1 = 3\sigma_2$ the corresponding terms in the Hamiltonian expansion can not be eliminated and it takes a form

$$H = \sigma_1\,r_1 - \sigma_2\,r_2 + c_{11}\,r_1^2 + c_{22}\,r_2^2 + c_{33}\,r_3^2 + c_{12}\,r_1\,r_2 + c_{13}\,r_1\,r_3 + c_{23}\,r_2\,r_3 + B\,r_1^{n_1/2}\,r_2^{n_2/2}\,r_3^{n_3/2}\cos[n_1\,\varphi_1 + n_2\,\varphi_2 + n_3\,\varphi_3] + O\left((r_1 + r_2)^{5/2}\right),$$

Now we can easily check the condition

$$\left|c_{11}\,n_1^2 + c_{22}\,n_2^2 + c_{33}\,n_3^2 + c_{12}\,n_1\,n_2 + c_{13}\,n_1\,n_3 + c_{23}\,n_2\,n_3\right| < \left|B\,n_1^{n_1/2}\,n_2^{n_2/2}\,n_3^{n_3/2}\right|$$

The calculations give the following result:

$$|c_{11} + 9\,c_{22} + 3\,c_{12}| = 21.4801 > \left|3\sqrt{3}\;B\right| = 8.99408$$

Therefore, the **fourth-order resonance $\sigma_1 = 3\sigma_2$ doesn't influence on stability of the equilibrium positions** S_1, S_2.

Conclusion

1. Analyzing stability of the equilibrium positions S_1, S_2 in linear approximation, we have shown that they are stable only if parameter μ is sufficiently small, namely, $0 < \mu < \mu_{max} = 0.0853217$.

2. In the planar case $(q_3 = p_3 = 0)$ the equilibrium positions S_1, S_2 are stable in Liapunov sense for all values of μ from the interval of their linear stability, except for the value $\mu_2 = 0.0529423$ for which the third-order resonance takes place.

3. In the spatial case the equilibrium positions S_1, S_2 are stable for the majority of initial conditions for all values of μ from the interval of their linear stability, except for the value $\mu_2 = 0.0529423$.

Thank you very much for your attention!!!