Chaotic dynamics in isolating segments

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- Φ — a local process on M generated by the vector field f

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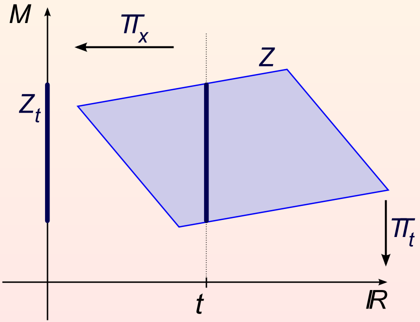
$$\pi_t : \mathbb{R} \times M \to \mathbb{R}$$

$$\pi_x : \mathbb{R} \times M \to M$$



The Poincaré map

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We will study the Poincaré map, using the special set W in the phase space, called isolating segment.

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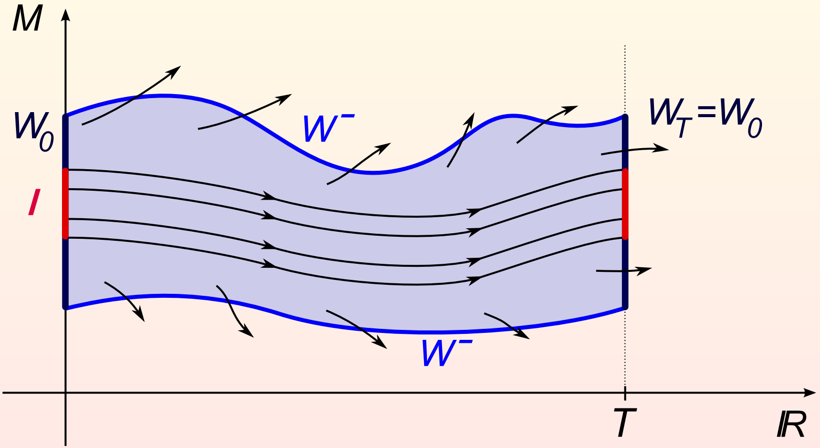
$$\pi_t \circ h = \pi_t \quad \text{and} \quad h([0, T] \times W_0^{\pm}) = W^{\pm}.$$

Illustration



Monodromy map

A homeomorphism h induces a homeomorphism m of the 'end' of the segment W onto itself:

$$m : (W_0, W_0^-) \to (W_0, W_0^-),$$

$$m(x) = \pi_x h(T, \pi_x h^{-1}(0, x)), \quad x \in W_0.$$

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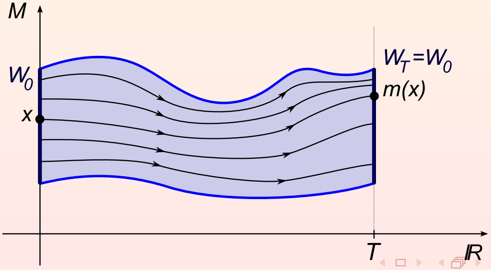

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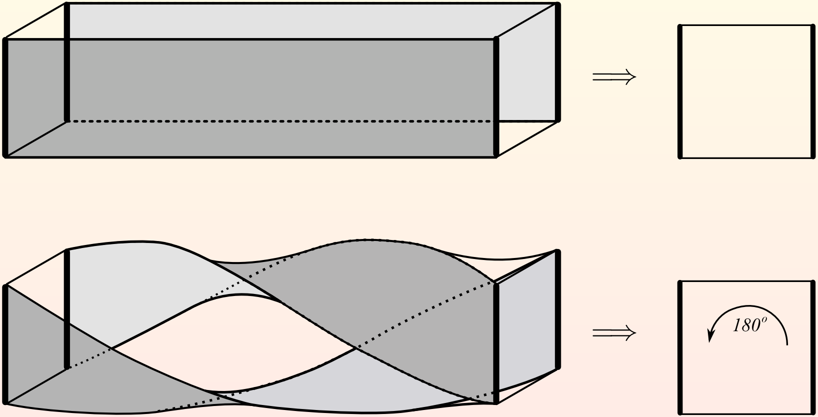
So the isomorphism induced in (singular) homology

$$\mu = H(m) : H(W_0, W_0^-) \to H(W_0, W_0^-)$$

is an invariant of W (it does not depend on h!)


$\Longrightarrow$
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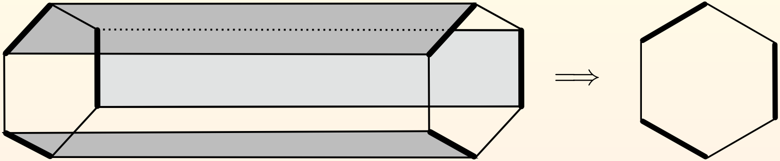
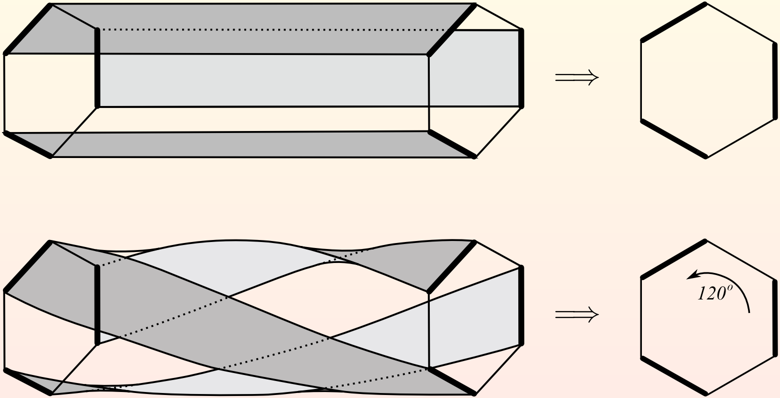
180°




120°

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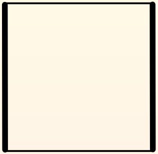
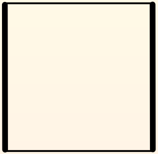
$L(m)$ is well defined for W_0, W_0^- are compact ENRs, hence $H(W_0, W_0^-)$ is of finite type.



$$\Longrightarrow \quad L(\mu) = L\left(\mathrm{id}_{(W_0, W_0^-)}\right) = \mathrm{tr}\,[1] - \mathrm{tr}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -1$$



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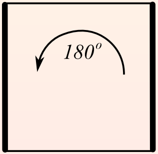

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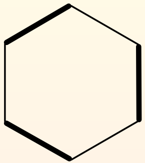

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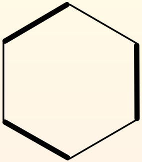

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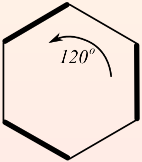

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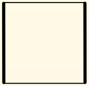
Lefschetz sequence (L_k)

For a periodic isolating segment W we define a sequence of integers by

$$L_k = L(\mu^k), \quad k \geqslant 0$$

where $\mu^k = \underbrace{\mu \circ \ldots \circ \mu}_{k}, \qquad \mu^0 = \mathrm{id}_{H(W_0, W_0^-)}.$



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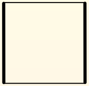

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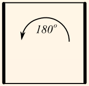
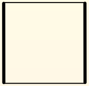


$$\Longrightarrow \qquad L = (-1, 1, -1, 1, \ldots)$$



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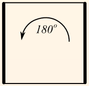

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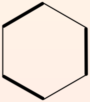

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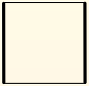

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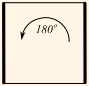

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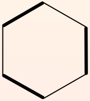

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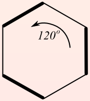

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Dual sequence

We also define the **dual sequence** (L^*_k) for (L_k):

$$L^*_k = (-1)^k \left(\sum_{i=0}^{k} (-1)^i \binom{k}{i} L_i \right), \quad k \geqslant 0.$$

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The definition of L_k^* is motivated by the criterion for detecting chaotic dynamics ([S2, SW, W, PW]).

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The previous formula for some dual Lefschetz numbers

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$$
\begin{aligned}
L_0^* &= L_0 \\
L_1^* &= -L_0 + L_1 \\
L_2^* &= L_0 - 2L_1 + L_2 \\
L_3^* &= -L_0 + 3L_1 - 3L_2 + L_3 \\
L_4^* &= L_0 - 4L_1 + 6L_2 - 4L_3 + L_4 \\
&\dots
\end{aligned}
$$

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3) If (L_n) is a bounded but not constant sequence (so it is periodic), then L_n^* is unbounded.
4) If the periodic sequence (L_n) is of period 2, then L_n^* has the form

$$L_k^* = \begin{cases} L_0 & \text{for} \quad k = 0 \\ (-2)^{k-1}(L_1 - L_0) & \text{for} \quad k \geqslant 1 \end{cases}.$$

Dold relations

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This fact implies a lot of interesting arithmetic properties of (L_n) and (L_n^*).

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In general, for a prime p, the pth Dold relation is of the form

$$L_1 \cong L_p \pmod p.$$

Theorem [PW] — when L_k^* is nonzero

If the sequence of Lefschetz numbers (L_n) is m–periodic and

$$L_1 = \dots = L_{m-1}, \qquad L_1 \neq L_0 = L_m \neq 0$$

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then:

- if m is even, then $L_k^* \neq 0$ for every $k \geqslant 0$,
- if m is odd, then $L_k^* = 0$ iff k is an odd multiplicity of m.

?

Are there any sequences of the form

$$L_1 = \cdots = L_{m-1}, \qquad L_1 \neq L_0 = L_m \neq 0\ ?$$

Lemma

Assume that a_n is an m-periodic sequence of integers satisfying the condition: for each prime p

$$a_p \cong a_1 \pmod{p}.$$

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The simple proof uses the Dirichlet theorem.

Corollary

Any periodic Lefschetz sequence with prime period is of the form considered above.

(Σ_2, σ)

Recall the set

$$\Sigma_2 := \{0,1\}^{\mathbb{Z}}$$

and a bijection $\sigma : \Sigma_2 \to \Sigma_2$ **(left shift)** :

$$\sigma : \Sigma_2 \ni (\ldots, s_{-1}.s_0, s_1, \ldots) \longmapsto (\ldots, s_0.s_1, s_2, \ldots) \in \Sigma_2$$

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  - $\sigma \circ g = g \circ P$ (semi–conjugacy),
  - for every k–periodic sequence $s \in \Sigma_2$ the set $g^{-1}(s)$ contains at least one k–periodic point of P.

Let $U \subset W$ be two T–periodic isolating segments for the equation (C), which 'ends' are the same:

$$U_0 = W_0, \quad U_0^{\pm} = W_0^{\pm}, \quad \mu_U = \mathrm{id}_{H(W_0, W_0^-)}.$$

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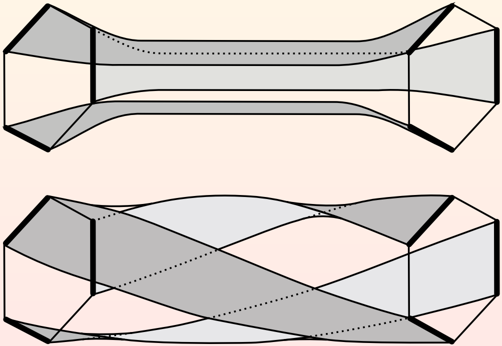

Invariant set

We define the set of all points in W_0 which full trajectories are contained in the bigger segment W:

$$I := \bigcap_{n=-\infty}^{\infty} \{x \in W_0 : \varphi_{(0,t+nT)}(x) \in W_t \ \forall t \in [0, T]\}$$

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I is a compact subset of int W_0, invariant for P.

Semi–conjugacy

Consider a map $g : I \longrightarrow \Sigma_2$ defined as follows

$$g(x)_n = \begin{cases} 0, & \text{if} \quad \forall t \in [0, T] : \varphi_{(0,nT+t)}(x) \in U_t \\ 1, & \text{if} \quad \exists t \in [0, T] : \varphi_{(0,nT+t)}(x) \notin U_t \end{cases}.$$

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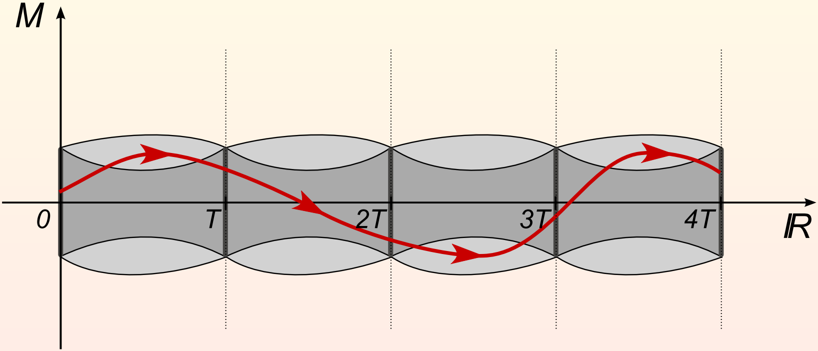
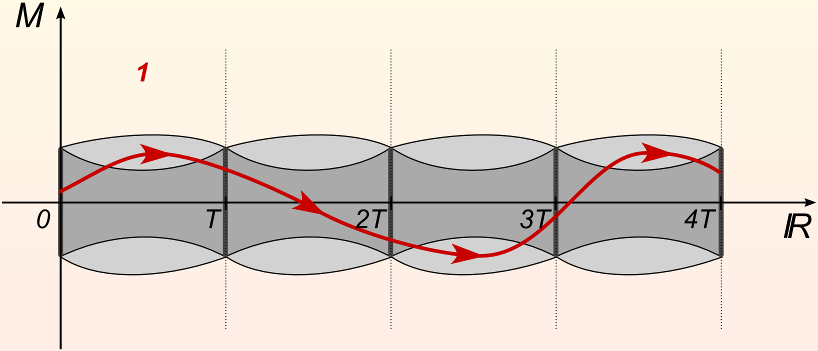
One can prove that $g : I \to \Sigma_2$ is continuous and for the Poincaré map $P = \varphi_{(0,T)}$

$$\sigma \circ g = g \circ P.$$


M
0
T
2T
3T
4T
ℝ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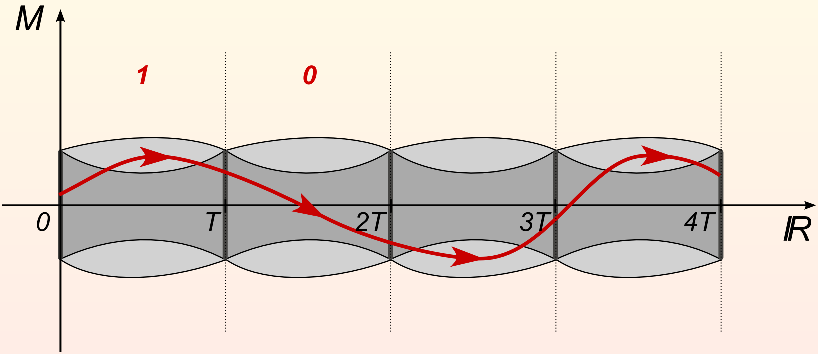
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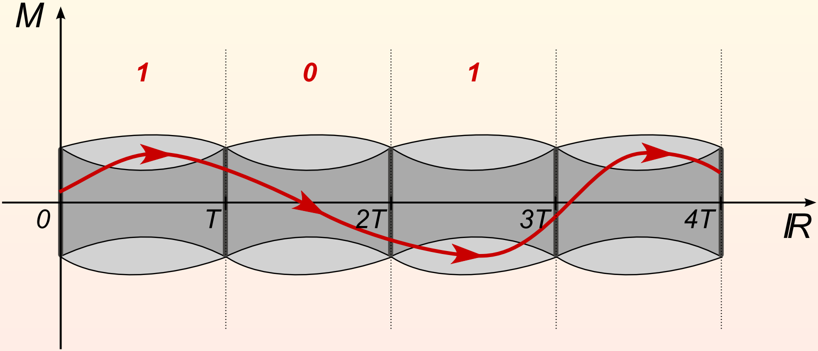
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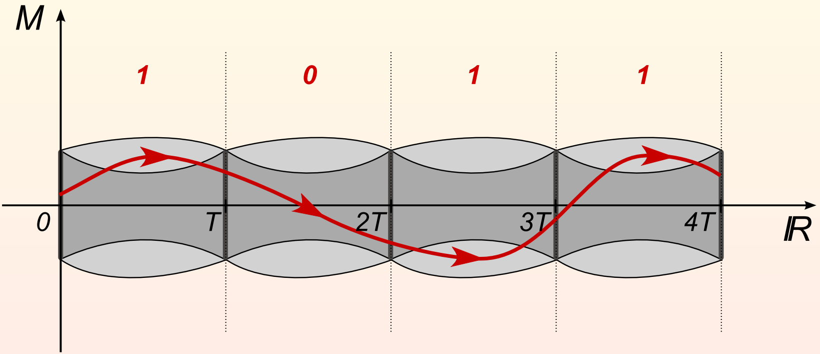
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Notation

By $\Sigma_2(n, k) \subset \Sigma_2$ we denote the set of n–periodic sequences, such that '1' appears exactly k times in the period.

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For $c_k^n \in \Sigma_2(n,k)$ we define the set $F(c_k^n) \subset I$:

$$F(c_k^n) = g^{-1}(c_k^n) \cap \mathrm{Fix}(P^n).$$

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If f is Σ_2–chaotic, then $F(c_k^n)$ is nonempty for every c_k^n!

Theorem (Srzednicki, Wójcik)

The set $F(c_k^n)$ is isolated in $\mathrm{Fix}(P^n)$ for $c_k^n \in \Sigma_2(n,k)$, so the fixed point index $\mathrm{ind}(P^n, F(c_k^n))$ is well-defined and

$$\mathrm{ind}\left(P^n, F(c_k^n)\right) = L_k^*.$$

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$$\mathrm{ind}\left(P^n, F(c_k^n)\right) = L_k^*.$$

In particular, $L_k^* \neq 0$ implies that $F(c_k^n)$ is non-empty for each sequence c_k^n.

The theorem and the compactness of $g(I)$ imply that if $L_k^* \neq 0$, then

$$\overline{\{c_k^n : L_k^* \neq 0,\ n \geqslant k\}} \subset g(I) \subset \Sigma_2,$$

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Recall that the set of all periodic sequences is dense in Σ_2, as well as the set of all periodic sequences with prime period!

Theorem

Assume that $(L_k)_{k\geqslant 0}$ is p–periodic, with p prime and

$$L_1 \neq L_0.$$

Then $g : I \to \Sigma_2$ is surjective. Moreover,

- if $p = 2$, then $F(c_k^n)$ is non-empty for each n–periodic sequence c_k^n,
- if p is an odd prime, then the set $F(c_k^n)$ is nonempty for each n–periodic sequence c_k^n such that k is not an odd multiplicity of p.

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Thank you for your attention.