Obstructions to integrability of Hamiltonian systems using high order variational equations

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The problem

To study the integrability of a Hamiltonian $H(q,p)$ real analytic on some domain Ω of $\mathbb{R}^{2n}$. We consider the extension to a complex domain $\hat{\Omega}$ of $\mathbb{C}^{2n}$.

If $x = \{q,p\} \in \mathbb{C}^{2n}$ we consider solutions $x(t)$ with $t \in \hat{D} \subset \mathbb{C}$. The image of $\hat{D}$ by x is a **Riemann surface** Γ. We shall **complete** Γ by adding fixed points, singularities and points at infinity to obtain $\bar{\Gamma}$.

We shall consider integrability in the **Liouville-Arnol'd** sense:

There exist n first integrals $f_1, f_2, \ldots, f_n$ independent almost everywhere and in involution. Usually it is taken $f_1 = H$. In general the functions $f_1, f_2, \ldots, f_n$ will be considered **meromorphic in a neighbourhood of a given solution** $x(t)$.

Typically Hamiltonian systems are **non-integrable**.

The problem is **how to detect/prove** the **non-integrability**.

Theoretical results: using first order variational equations

We present results based of **differential Galois theory** because:

1) They do not require to be in the **perturbative setting** $H = H_0 + \varepsilon H_1$,

2) They can be extended to **variational equations of higher order**.

Consider the **ODE** $\dot{x} = f(x(t)), x(t_0) = x_0$ a **regular** point of f, $x \in \mathbb{C}^m$. Let $x(t)$ be a **solution**.
The **first VE** along $x(t)$ is $\frac{d}{dt}A = Df(x(t))A$, $A(t_0) = Id$.
Consider **closed paths**, ψ, on Γ with base point x_0. One can associate to each ψ the corresponding monodromy matrix M_1^{ψ}. The set of all these matrices form the **monodromy group** M_1.

In general let

$$\frac{d}{dt}A = B(t)A(t),$$

with the entries of B in a suitable **field of functions** K, the **meromorphic functions on** $\bar{\Gamma}$, and let $\xi_{i,j}$ be the **elements of a fundamental matrix**. Consider the **extension** $L = K(\xi_{1,1}, \xi_{2,1}, ..., \xi_{m,m})$.
$G = \mathrm{Gal}(L \mid K)$ denotes the **Galois group** of the extension.

The **closure** of the monodromy group is the Galois group.

Theorem 1 (**Morales–Ramis** *Meth. & Appl. of Analysis* **8**, 2001) Under the assumptions above if a Hamiltonian is integrable in a neighbourhood of Γ then the **identity component** G^0 of the Galois group of the first order VE along Γ is **commutative**.

G^0 commutative $\Longrightarrow$ nothing against integrability.

This can happen, typically, for families of Hamiltonian systems depending on parameters, for some exceptional sets of parameters.

This suggests to try to detect **non-integrability** using **higher order variational equations**, methods introduced recently.

Recalling some concepts

Galois group $G =\mathrm{Gal}(L \mid K)$: automorphisms of L which leave K invariant.

It is an **algebraic group**: The elements satisfy some algebraic conditions (polynomials in an ideal in $\mathbb{C}[X_1, ..., X_m]$) and the group operation and passing to the inverse are algebraic.

Whenever we refer to some topological concept (closure, component,...) one should understand that the **Zariski's topology** is used.
The closed sets are the zeros of an ideal. Note that any two open sets are not disjoint. In particular it is not Hausdorff.

Using higher order variational equations

Let $\varphi(t, x_0)$ be the solution of $\dot{x} = f(x(t))$ with $\varphi(0, x_0) = x_0$.

We consider as **fundamental solutions of the k-th order VE, VE$_k$** based on x_0, the string

$$(\varphi^{(1)}(t), \varphi^{(2)}(t), \ldots, \varphi^{(k)}(t))$$

such that

$$\varphi(t, y_0) = \varphi(t, x_0) + \varphi^{(1)}(t)(y_0 - x_0) + \ldots + \varphi^{(k)}(t)(y_0 - x_0)^k + \ldots,$$

i.e., **the coefficients of the k-jet**.

$\varphi^{(k)}(t)$ satisfy **linear non-homogeneous ODE**. Solved by **quadrature**.

$\frac{d}{dt}\varphi^{(k)}$, $k > 1$ depends on $\varphi^{(j)}$ for $j < k$ in a **nonlinear way**. It can be made linear by **adding additional variables**.

Then one can introduce the **k-th order Galois group G_k**. Loosely we can talk about the **k-th order monodromy M_k^ψ** along a path ψ.

The **composition** of elements in M_k^Γ as a group is equivalent to the **composition of jets**.

Theorem 2 (**Morales–Ramis–S** *Ann. Scient. Éc. Norm. Sup.* 4^e série **40**, 2007)
Under the assumptions above if a Hamiltonian is integrable in a neighbourhood of Γ then for any $k \geq 1$ **the identity component** $(G_k)^0$ **of** G_k **is commutative**.

This result gives rise to **non-integrability criteria to all orders**. Note that these criteria can depend strongly on the reference solution $x(t)$ and on the paths taken on it.

Main references for present work:

R. Martínez, C.S., Non-Integrability of Hamiltonian Systems Through High Order Variational Equations: Summary of Results and Examples, *Regular and Chaotic Dynamics*, 2009, Vol. 14, 323–349. (MS09a)

R. Martínez, C.S., Non-integrability of the degenerate cases of the Swinging Atwood's Machine using higher order variational equations. Submitted, 2009. (MS09b)

R. Martínez, C.S., Efficient numerical implementation of integrability criteria based on high order variational equations. Preprint, 2009. (MS09c)

Difficulties to obtain obstructions to integrability by using the MRS theorem

1) We should **know an orbit** $x(t)$,

2) One should find **suitable paths** ψ_1, ψ_2 **in** $\mathbb{C}$ **giving rise to closed loops under** x,

3) Should exist a k **such that** $M_k^{\psi_1}, M_k^{\psi_2}$ **are in** $(G_k)^0$,

4) One should **check that** $M_k^{\psi_1}$ **and** $M_k^{\psi_2}$ **do not commute**.

Some "simple" cases

There exist **periodic solutions (p.s.) with real period having also an additional complex period**. Take ψ_1, ψ_2 as these periods.

This is the case, e.g., if there exists an **invariant plane containing an homoclinic loop**. Close to it there are p.s. Under some conditions they also have an imaginary period.
As an example: Assume in the invariant plane $H(q,p) = \frac{1}{2}(p,p) + U(q)$ and $\psi(t)$ is homoclinic to $P = (0,0)$. $\psi(t)$ is limit of p.s. ψ_1. The changes $q = v, p = \mathrm{i}\, w, t = -\mathrm{i}\, s$ lead to $K(v,w) = \frac{1}{2}(w,w) - U(v)$ and P becomes elliptic, having nearby p.s. ψ_2 in the variable s.

Some Lemmas

Lemma 1 Assume $M_1^{\psi_1}, M_1^{\psi_2} \in (G_1)^0$ for two closed paths ψ_1, ψ_2 in Γ and they commute. Let us assume that there exists $k > 1$ such that $M_k^{\psi_1}$ and $M_k^{\psi_2}$ do not commute. Then the Hamiltonian is non-integrable in a neighbourhood of Γ.

Ideas of the proof: Take a suitable Riemann "sub"surface $\Gamma' \subset \Gamma$ a tubular neighbourhood of $\psi_1 \cup \psi_2$. If $M_1^{\psi_1}, M_1^{\psi_2} \in (G_1)^0$, then $G_1^{\Gamma'}$ is connected. Also $G_k^{\Gamma'}$ is connected (we only add "transcendental" parts to the group by the quadratures). Hence $(G_k^{\Gamma'})^0$ coincides with $G_k^{\Gamma'}$. By MRS theorem H is non-integrable in a neighbourhood of Γ'. Hence, it is non-integrable in a neighbourhood of Γ.

Lemma 2 Criteria to ensure $M_1^{\psi_1}, M_1^{\psi_2} \in (G_1)^0$:
Assume that in same basis the normal part of VE_k in both matrices is made of 2×2 blocs. And that then, comparing blocs of $M_1^{\psi_1}, M_1^{\psi_2}$ either

a) The blocs are unipotent and commute (e.g., they are the identity), or

b) At least for one matrix the blocs are non-resonant and they commute.

Examples

We consider a few examples of different type:

i) All the discussion reduces to a **local problem around a singularity**. We shall present a **degenerate Hénon-Heiles system**, DHH, which requires $k = 3$ and examples which require **arbitrary** k, GDHH.

ii) A **nonlinear spring-pendulum problem** NLSP with 3 singularities, but only two essential, which, after some analytic work, requires **tools from qualitative theory of ODE**.

iii) The **degenerate cases of the Swinging Atwood Machine** SAM, with 2 singularities, which, after some analytic work, requires **bounds on some complicated expressions involving hypergeometric functions**.

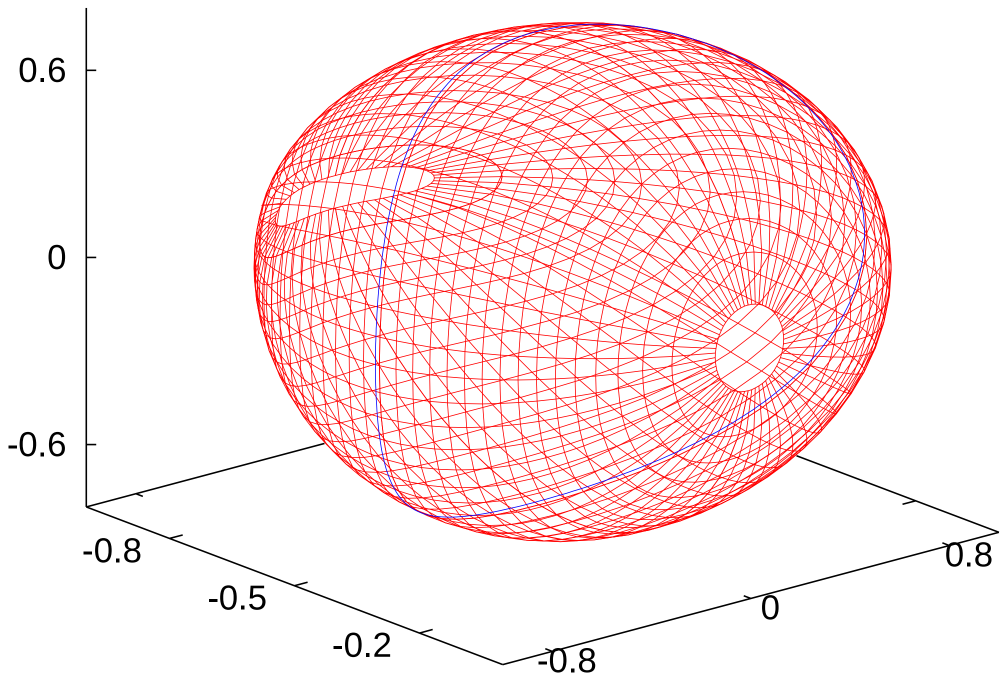
In last two cases the local parts are easy. The difficulties appear in **the paths joining the singularities**. These are **global problems**.

A degenerate Hénon-Heiles problem (see MRS paper)

Hénon-Heiles family (HHF) (classical $b = -1$):

$$H = \frac{1}{2}(x_3^2 + x_4^2 + x_1^2 + x_2^2) + \frac{1}{3}x_1^3 + bx_1x_2^2,$$

non-integrable for all b except **four values**. For **3 of them** integrability is proved. Remaining case: $b = 1/2$, degenerate Hénon-Heiles **DHH**. Fixed points $P_{ee} = (0,0,0,0)$, $P_{hp} = (-1,0,0,0)$.



The global W^c_{hp} manifold. Coincides with a family of periodic orbits.

Separatrix Γ_0 on the invariant plane $\{x_2 = x_4 = 0\}$ and $H = h_0 = 1/6$:

$$x_1(t) = \frac{3/2}{\cosh^2(t/2)} - 1, \quad x_3(t) = \frac{-(3/2)\sinh(t/2)}{\cosh^3(t/2)}, \quad \textbf{singularity } t_* = \pi \mathrm{i}\,.$$

Double-periodic solutions for $h < h_0$: ψ_1, ψ_2 paths along real, imaginary periods. Then, to have integrability

$$[M_k^{\psi_1}, M_k^{\psi_2}] = M_k^{\psi_2^{-1}} \circ M_k^{\psi_1^{-1}} \circ M_k^{\psi_2} \circ M_k^{\psi_1}$$

should be trivial. The path **can be deformed** to loop γ around t_*.

Reduces to **local checks**: M_2^{γ} trivial, but some components of $\varphi^{(3)}$ are **different from zero**. E.g. $x_{2;2,2,2} = \frac{72}{5} 2\pi \mathrm{i}\,.$

In general we can have solutions with **several singularities** and we can have also **new singularities** in the coefficients of the variational equations. If $(G_1)^0$ or $(M_1)^0$ is commutative we have to go to **higher order variational equations**.

Some systems requiring order k variational equations

As an extension of DHH consider, for $n \geq 2$, the **generalised degenerate HH** problem, GDHH, with Hamiltonian

$$H = \frac{1}{2}(x_3^2 + x_4^2) + \frac{1}{2}x_1^2 + \frac{1}{3}x_1^3 + (1 + x_1)\frac{1}{n!}x_2^n.$$

For $n = 2$ gives DHH. Consider now $n \geq 3$.

Fixed points: the origin and P_{hp}, as before. Now the origin is elliptic-parabolic while P_{hp} is still hyperbolic-parabolic. This difference implies that the results for $n = 2$ and $n \geq 3$ are slightly different. The Hamiltonian GDHH has the same invariant plane and the same separatrix Γ_0 as DHH.

Theorem 3 (MS09a) For any $n \geq 3$ system GDHH is non-integrable in a neighbourhood of the separatrix Γ_0 sitting on the $q_2 = p_2 = 0$ plane. To decide the non-integrability one should use the order $n-1$ monodromy M_{n-1}^{γ} along a suitable path, all the lower order monodromies $M_k^{\gamma}, k < n-1$ being trivial.

The proof is local and reduces to analytic computations along loops around $t^* = \pi\mathrm{i}$ and then one has:

all the elements $x_{i;k_1 \ldots, k_{n-1}}$, $i \in \{2, 4\}$, $k_j \in \{2, 4\}$ for all j, are different from zero with the exception of the element $x_{4;2,\ldots,2}$.

A nonlinear spring-pendulum problem

The Hamiltonian

$$H = \frac{1}{2}\left(x_3^2 + \frac{x_4^2}{x_1^2}\right) - x_1\cos(x_2) + \frac{k}{2}(x_1-1)^2 - \frac{a}{3}(x_1-1)^3,\ k>0$$

Known results (Maciejewski-Przybylska-Weil, *J. Phys. A* **37**, 2004):

If $k+a\neq 0,\ k>0$ the system is **non-integrable**.

A simple solution for $a=-k$, which is a separatrix,

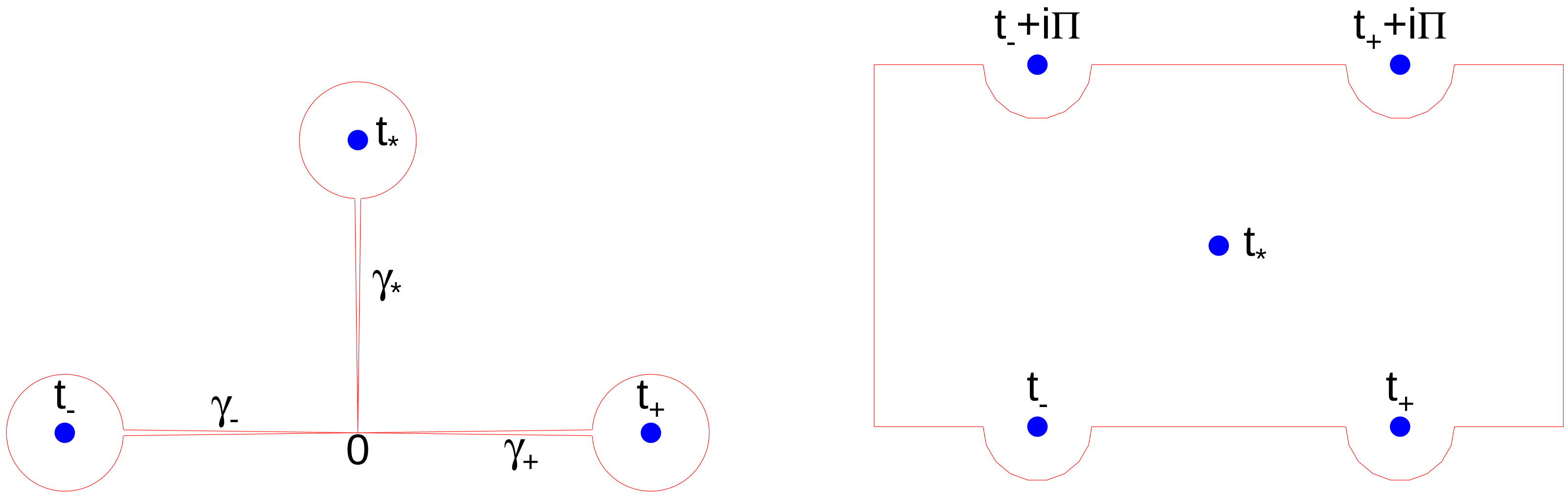
$$x_1(t) = \rho + \frac{\alpha}{\cosh^2(\beta t)}, \quad x_3(t) = \dot{x}_1(t) \quad x_2 = x_4 = 0,$$

with $\rho<0, \alpha>0, \beta>0$ **depending on** k. It has a **singularity** at $t_* = \frac{\pi \mathrm{i}}{2\beta}$. MPW checked that for $a=-k$ **no obstruction** appears up to order 7.

But **other singularities** appear for the VE at $\pm\hat{t}$, where $x_1(\pm\hat{t})=0$.

$$\pm\hat{t} = \pm\frac{1}{\beta}\log\left(\sqrt{-\frac{\alpha}{\rho}} + \sqrt{-\frac{\alpha}{\rho}-1}\right)$$

Theorem 4 (MS09a) System NLSP is non-integrable when $a = -k$.



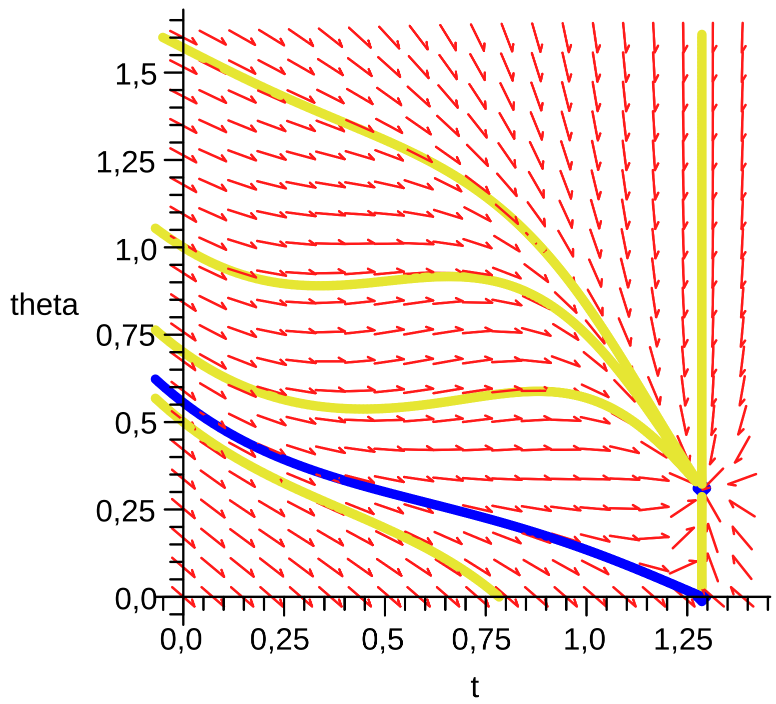
Left: Path for the proof of Theorem 4 around the three singularities of the variational equations. Right: the path deformed to a period parallelogram, avoiding the singularities. Here $i\,\Pi$ denotes the imaginary period on the energy level h.

Steps of the proof:

- Along $\gamma_{+,*,-}$ **the variation of the entries of $\varphi^{(1)}$ and $\varphi^{(2)}$ cancels**. It is worth to mention that for $\varphi^{(1)}$ the singularities **do not give rise to logarithmic terms**.
- The path **can be deformed** as in the right plot and be seen as a **commutator** $\psi_2^{-1} \circ \psi_1^{-1} \circ \psi_2 \circ \psi_1$ where ψ_1 and ψ_2 are the paths giving rise to real and imaginary periods.
- The **NVE$_1$ are non-resonant and commute**. By Lemma 2, $M_1^{\psi_j} \in (G_1)^0$, $j = 1, 2$.
- Along γ_* the variation of the entries of $\varphi^{(3)}$ cancels.
- Use the **symmetries to reduce to the study around γ_+**. Finally prove $x_{2;2,2,2} = -x_{4;2,2,4} \neq 0$ and $x_{2;2,4,4} = -x_{4;4,4,4} \neq 0$. These are the only non-zero components of $\varphi^{(3)}$ along the full path. By Lemma 1 the proof is finished. (The identities of components follow from **symplectiness**).

The **most delicate part** is to prove that some elements in $\varphi^{(3)}$ are non-zero. They depend on products of two numbers b, d which must be, both, non-zero. They are related to the solutions obtained by **Frobenius method** at t_+ when transported to $t = 0$.

After some **change of variable**, using a new time, t being considered as a dependent variable and a **suitable blow-up** the problem is equivalent to show that **the stable manifold** of a saddle located at $t = t_+$, $\theta = 0$, reaches $t = 0$ for some value $0 < \theta < \pi/2$.



Comparison techniques, different for different ranges of the parameter k, are used to conclude the proof.

The Swinging Atwood's Machine

A classical mechanical device. If we **include pulleys** the Hamiltonian is

$$H(r,\theta,p_r,p_\theta)=\frac{1}{2}\left[\frac{p_r^2}{M_t}+\frac{(p_\theta+Rp_r)^2}{mr^2}\right]+gr(M-m\cos\theta)+gR(m\sin\theta-M\theta),$$

where $M_t = M + m + 2\frac{I_p}{R^2}$.

Theorem 5 (PPSRMWS, Swinging Atwood's machine: experimental and theoretical studies, preprint 2009):
For every physically consistent value of the parameters, the SAM with pulleys is meromorphically non-integrable.

This can be already detected using Theorem 1.

Without pulleys and normalising constants

$$H = (x_3^2/(1+\mu) + x_4^2 x_1^{-2})/2 + x_1(\mu - \cos(x_2)),$$

where μ is a mass ratio, $\mu > 1$ in the domain of interest.

Known to be **non-integrable if** $\mu \neq \mu_p$ where $\mu_p = 1 + \frac{4}{p^2+p-4}$, $p \in \mathbb{N}$, $p > 2$ and integrable if $\mu = \mu_2 = 3$. (Casasayas- Nunes-Tufillaro, *J. Physique* **51**, 1990). Follows also from Theorem 1, but was proved using Ziglin's Thm.

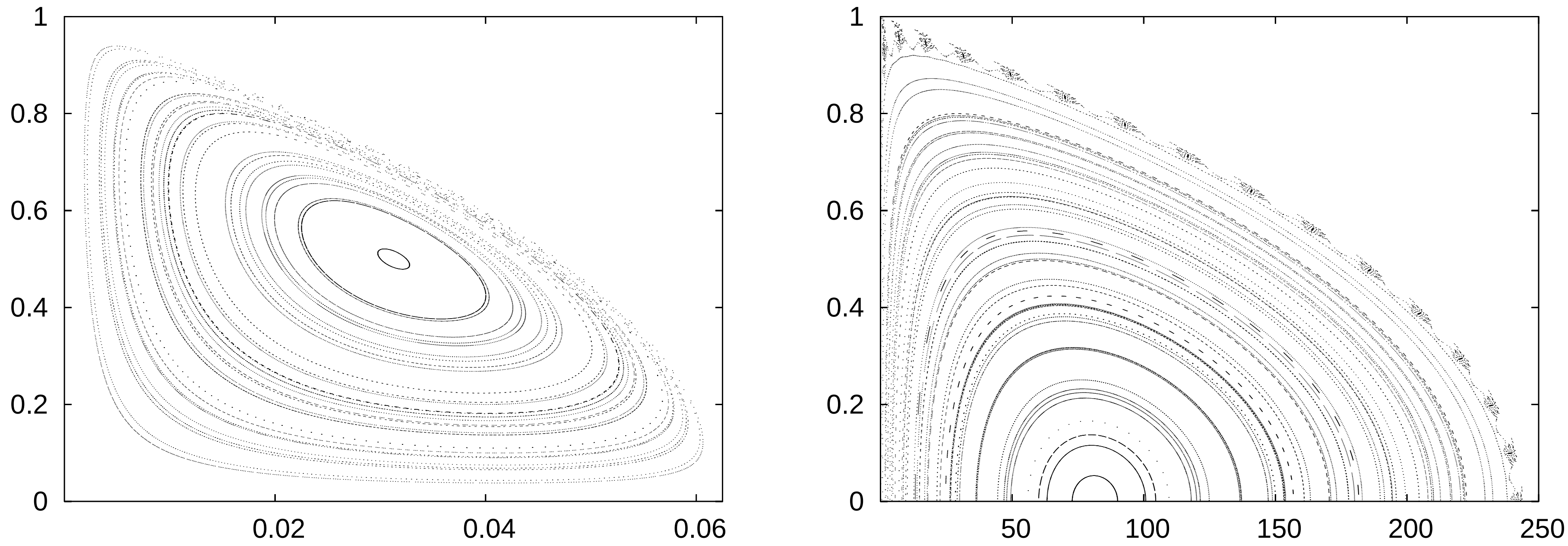
It remains to study the **exceptional cases**, which can not be decided using Theorem 1. The figure shows $\mu_2=3, \mu_{62}=1953/1951$ on $H^{-1}(\frac{1}{2(1+\mu)})$.

A simple solution

$$x_1(t) = \frac{1}{a}\left(1 - t^2\right), \quad x_2(t) = 0, \quad x_3(t) = (1-\mu_p)t, \quad x_4(t) = 0,$$

where $a = p^2 + p - 2$. Note that $r(t_\pm) = r(\pm 1) = 0$.

Theorem 6 (MS09b) The degenerate cases $\mu = \mu_p = 1 + \dfrac{4}{p^2+p-4}, p > 2$ of the SAM are non-integrable.



Steps of the proof

- The **solution above** has been used and a path $\gamma = \gamma_-^{-1} \circ \gamma_+^{-1} \circ \gamma_- \circ \gamma_+$ is selected, where $\gamma_\pm$ surround $t_\pm$. Then $(M_k^{\gamma_-})^{-1} \circ (M_k^{\gamma_+})^{-1} \circ M_k^{\gamma_-} \circ M_k^{\gamma_+}$, is simply represented as M_k^γ.

- One checks $M_1^{\gamma_+}, M_1^{\gamma_-} \in (G_1)^0$. We can apply Lemma 1.

- The solutions $\varphi^{(k)}$ satisfy **symmetries** as functions of t and some elements are $\equiv 0$. But $\varphi^{(1)}$ contains **logarithmic terms**. The final third order jet along γ, M_3^γ, follows from $M_3^{\gamma_+}$ and several **additional integrals**. They come from **changes in the determinations** of the involved integrands.

- Some components of $M_3^{\gamma_+}$ are zero and some are $\neq 0$. Then a **part of** M_3^γ can be computed. This is enough to prove Theorem 6.

The cases p **odd or even** are similar, exchanging the role of some variables. The proof is given for p odd.

Remark. One proves $M_1^\gamma = Id$. For M_2^γ all elements are zero except, perhaps $x_{1;4,4}$ and $x_{2;3,4}$ (numerically seem $= 0$). If not, **we are done**. Otherwise we pass to M_3^γ and one proves $x_{2;2,2,4}$ **and** $x_{4;2,4,4}$ **are real and non-zero at the end of** γ, independently of the values of $x_{1;4,4}, x_{2;3,4}$.

Some technical details

The **NVE**$_1$ have fundamental solutions ξ_1, ξ_2 which can be chosen as follows: ξ_1 is a multiple of $P_{p-1}^{(1,1)}$ (Jacobi) normalised as $\xi_1(1) = 1$.

$$\xi_2(t) = \left[-\frac{1}{2}(a+2)\left(\log(1+t) - \log(1-t)\right)\xi_1(t) + \psi(t) \right] + g(t),$$

where $\psi(t) = -\dfrac{2t}{1-t^2}$ and $g(t)$ is the unique polynomial solution of degree $p-2$ of the equation

$$(1-t^2)\frac{d^2 g}{dt^2}(t) - 4t\frac{dg}{dt}(t) + ag(t) = (2a+4)\left(\frac{d\xi_1(t)}{dt} - \frac{t\xi_1(t) - s(t)}{1-t^2} \right).$$

After some analytic work one introduces

$$k(t) = \frac{1}{2}\xi_1^2(t) - \frac{1}{a}(1-t^2)\dot{\xi}_1^2(t)$$

and $K(t) = \int_0^t k(s)ds$.

Then $x_{2;2,2,4}, x_{4;2,4,4} > 0$ at the end of γ if and only if

$$Z := \int_0^1 \left[\frac{1}{1+\mu}K^2(t) - \frac{1}{12a}(1-t^2)\xi_1^4(t) \right] dt > 0.$$

Last inequality is checked **directly** (using **exact rational arithmetic**) until $p = 3162$, the first value for which $a(p) > 10^7$.

For larger values it is proved using **rigorous analytic bounds** which require computations with Bessel functions and asymptotic properties of Gegenbauer polynomials.

This concludes the proof of theorem 6.

Comments on jet transport

It has been widely used (along a large number of different complex paths, reaching order 30 and based on Taylor integrators) for many tests that gave the hints for the analytic proofs.

THANK YOU FOR YOUR ATTENTION