Variational and topological properties of *n*-body minimizers

D.L. Ferrario

University of Milano-Bicocca

Dynamics, Topology and Computations
Będlewo, 2009-06-02

1. Periodic solutions of the n-body problem
2. Symmetries
3. Computer algebra and symmetries
4. Numerical minimization
5. Examples
6. Concluding remarks

1. Periodic solutions of the n-body problem
2. Symmetries
3. Computer algebra and symmetries
4. Numerical minimization
5. Examples
6. Concluding remarks

1. The n-body problem

Consider $n > 1$ material points in a euclidean space $\mathbb{R}^d$, with masses $m_i \geq 0$, $i = 1 \ldots n$, and mutual interaction potential $U(x)$, where $x = (x_1, x_2, \ldots, x_n)$ and $x_i \in \mathbb{R}^d$.
Hypotheses on $U(x)$ may be quite general [1], but the basic example is given by homogeneous isotropic potentials of the type

$$U_\alpha(x) = \sum_{\substack{i<j \\ i,j=1}}^{n} \frac{m_i m_j}{|x_i - x_j|^\alpha}.$$

[1] V. Barutello, DLF, and S. Terracini. On the singularities of generalized solutions to n-body-type problems. *Int. Math. Res. Not. IMRN*, pages Art. ID rnn 069, 78, 2008.

1. (The n-body problem)

Newton equations of the motion are (m = mass matrix and corresponding gradient ∇):

$$m\ddot{x} = \frac{\partial U}{\partial x} \iff \ddot{x} = \nabla U.$$

Regular solutions can be defined for all $t \in \mathbb{R}$, or in bounded intervals due to the occurrence of singularities: collisions ($x(t) \to \Delta$, where $\Delta = \{x : \exists(i,j) : x_i = x_j, i \neq j\}$, $U(\Delta) = +\infty$) or Xia escapes to infinity in finite time.

[2] Z. Xia. The existence of non collision singularities in newtonian systems. *Ann. of Math.*, 135:411–468, 1992.

[3] F. Diacu. Singularities of the N-body problem. In *Classical and celestial mechanics (Recife, 1993/1999)*, pages 35–62. Princeton Univ. Press, Princeton, NJ, 2002.

2. Lagrangian action

Solutions are critical points for the Lagrangean action functional, defined by

$$\mathcal{A}(x) = \int_{t_0}^{t_1} L(x(t), \dot{x}(t))\, dt$$

where (in rotating coordinate frames)

$$L(x, \dot{x}) = K + U = \sum_i \frac{1}{2} m_i |\dot{x}_i + \Omega x_i|^2 + \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^\alpha}.$$

Here $x = x(t) \in H^1([t_0, t_1], \mathcal{X})$, $t_0 < t_1 \in \mathbb{R}$.
Periodic solutions are critical points for

$$\mathcal{A}_\omega = \mathcal{A} \colon H^1(\mathbb{R}/T\mathbb{Z}, \mathcal{X}) \to \mathbb{R} \cup \infty.$$

[4] A. Ambrosetti and V. Coti Zelati. *Periodic solutions of singular Lagrangian systems*. Progress in Nonlinear Differential Equations and their Applications, 10. Birkhäuser Boston Inc., Boston, MA, 1993.

3. Action minimizers: coercivity and collisions

The action functional

$$\mathcal{A}_\omega \colon H^1(\mathbb{R}/T\mathbb{Z}, \mathcal{X}) \to \mathbb{R} \cup \infty.$$

is not *coercive*, for all ω such that $\omega/2\pi \in \mathbb{Z}$, otherwise it is coercive. To obtain coercivity: take $\omega \notin 2\pi\mathbb{Z}$, winding-number constraints, "topological" (homotopy class) constraints braid group, add anti-symmetry $x(t + T/2) = -x(t), \ldots$

$$\textit{Coercive}: \lim_{|x|\to\infty} \mathcal{A}(x) = +\infty$$

$|x| = H^1$-norm

[5] William B. Gordon. A minimizing property of Keplerian orbits. *Amer. J. Math.*, 99(5):961–971, 1977.

4. Symmetries and Palais symmetric criticality

Let G be a finite (compact Lie) group, acting linearly on X (Banach space), and $f : X \to \mathbb{R}$ a G-equivariant functional. If $x \in X$ is a critical point for

$$f^G : X^G \subset X \to \mathbb{R}$$

then x is a critical point for f.
Subspace of points fixed by G: $X^G = \{x \in X : Gx = x\}$, that is, symmetric (i.e. G-equivariant) periodic trajectories in $X = H^1(\mathbb{R}/T\mathbb{Z}, \mathcal{X})$, where

$$(g \cdot x)(t) = gx(g^{-1}t).$$

[6] R.S. Palais. The principle of symmetric criticality. *Comm. Math. Phys.*, 69(1):19–30, 1979.

4. (Symmetries and Palais symmetric criticality)

For many groups G on $X = H^1(\mathbb{R}/T\mathbb{Z}, \mathcal{X})$:

(1) $\mathcal{A}^G \colon X^G \to \mathbb{R}$ is coercive (e.g. $x(t + T/2) = -x(t)$).

(2) X^G has better topology (e.g. finite rotation groups in the euclidean space).

(3) Local minimizers of $\mathcal{A}^G$ are collisionless.

(4) Properties of minimizers can be deduced by algebraic properties of the G-action (angular momentum, being non-homographic, colliding solutions,…).

(5) More equal masses, more possible symmetries.

[7] DLF and S. Terracini. On the existence of collisionless equivariant minimizers for the classical n-body problem. *Invent. Math.*, 155(2):305–362, 2004.

4. (Symmetries and Palais symmetric criticality)

For many groups G on $X = H^1(\mathbb{R}/T\mathbb{Z}, \mathcal{X})$:

(1) $\mathcal{A}^G \colon X^G \to \mathbb{R}$ is coercive (e.g. $x(t + T/2) = -x(t)$).

(2) X^G has better topology (e.g. finite rotation groups in the euclidean space).

(3) Local minimizers of $\mathcal{A}^G$ are collisionless.

(4) Properties of minimizers can be deduced by algebraic properties of the G-action (angular momentum, being non-homographic, colliding solutions,…).

(5) More equal masses, more possible symmetries.

[7] DLF and S. Terracini. On the existence of collisionless equivariant minimizers for the classical n-body problem. *Invent. Math.*, 155(2):305–362, 2004.

4. (Symmetries and Palais symmetric criticality)

For many groups G on $X = H^1(\mathbb{R}/T\mathbb{Z}, \mathcal{X})$:

(1) $\mathcal{A}^G \colon X^G \to \mathbb{R}$ is coercive (e.g. $x(t + T/2) = -x(t)$).

(2) X^G has better topology (e.g. finite rotation groups in the euclidean space).

(3) Local minimizers of $\mathcal{A}^G$ are collisionless.

(4) Properties of minimizers can be deduced by algebraic properties of the G-action (angular momentum, being non-homographic, colliding solutions,...).

(5) More equal masses, more possible symmetries.

[7] DLF and S. Terracini. On the existence of collisionless equivariant minimizers for the classical n-body problem. *Invent. Math.*, 155(2):305–362, 2004.

4. (Symmetries and Palais symmetric criticality)

For many groups G on $X = H^1(\mathbb{R}/T\mathbb{Z}, \mathcal{X})$:

(1) $\mathcal{A}^G \colon X^G \to \mathbb{R}$ is coercive (e.g. $x(t + T/2) = -x(t)$).

(2) X^G has better topology (e.g. finite rotation groups in the euclidean space).

(3) Local minimizers of $\mathcal{A}^G$ are collisionless.

(4) Properties of minimizers can be deduced by algebraic properties of the G-action (angular momentum, being non-homographic, colliding solutions,…).

(5) More equal masses, more possible symmetries.

[7] DLF and S. Terracini. On the existence of collisionless equivariant minimizers for the classical n-body problem. *Invent. Math.*, 155(2):305–362, 2004.

4. (Symmetries and Palais symmetric criticality)

For many groups G on $X = H^1(\mathbb{R}/T\mathbb{Z}, \mathcal{X})$:

(1) $\mathcal{A}^G \colon X^G \to \mathbb{R}$ is coercive (e.g. $x(t+T/2) = -x(t)$).

(2) X^G has better topology (e.g. finite rotation groups in the euclidean space).

(3) Local minimizers of $\mathcal{A}^G$ are collisionless.

(4) Properties of minimizers can be deduced by algebraic properties of the G-action (angular momentum, being non-homographic, colliding solutions,…).

(5) More equal masses, more possible symmetries.

[7] DLF and S. Terracini. On the existence of collisionless equivariant minimizers for the classical n-body problem. *Invent. Math.*, 155(2):305–362, 2004.

4. (Symmetries and Palais symmetric criticality)

For many groups G on $X = H^1(\mathbb{R}/T\mathbb{Z}, \mathcal{X})$:

(1) $\mathcal{A}^G \colon X^G \to \mathbb{R}$ is coercive (e.g. $x(t + T/2) = -x(t)$).

(2) X^G has better topology (e.g. finite rotation groups in the euclidean space).

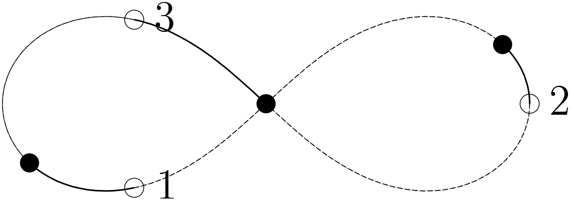
(3) Local minimizers of $\mathcal{A}^G$ are collisionless.

(4) Properties of minimizers can be deduced by algebraic properties of the G-action (angular momentum, being non-homographic, colliding solutions,...).

(5) More equal masses, more possible symmetries.

[7] DLF and S. Terracini. On the existence of collisionless equivariant minimizers for the classical n-body problem. *Invent. Math.*, 155(2):305–362, 2004.

5. Well-known example



A. Chenciner and R. Montgomery's Figure-eight. [▷].

[8] Alain Chenciner and Richard Montgomery. A remarkable periodic solution of the three-body problem in the case of equal masses. *Ann. of Math. (2)*, 152(3):881–901, 2000.

[9] Carles Simó. New families of solutions in N-body problems. In *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, volume 201 of *Progr. Math.*, pages 101–115. Birkhäuser, Basel, 2001.

[10] Cristopher Moore. Braids in classical dynamics. *Phys. Rev. Lett.*, 70(24):3675–3679, 1993. *(handwritten picture)*

6. Action types

The left action of G on $X = H^1(\mathbb{T}^1, \mathcal{X})$ induces an ascending chain of subgroups

$$K \lhd G_0 \lhd G$$

where:

(1) The *core* $K = \{g \in G : \forall t \in \mathbb{T}^1, gt = t\}$. $\implies$ G/K acts faithfully on $\mathbb{T}^1$, and hence it is a finite subgroup of $O(2)$.

(2) The *cyclic* part $G_0 = \{g \in G : \forall t \in \mathbb{T}^1, gt \neq t\} \cup K$.

[11] V. Barutello, DLF, and S. Terracini. Symmetry groups of the planar three-body problem and action-minimizing trajectories. *Arch. Ration. Mech. Anal.*, 190(2):189–226, 2008.

[12] DLF. Symmetry groups and non-planar collisionless action-minimizing solutions of the three-body problem in three-dimensional space. *Arch. Ration. Mech. Anal.*, 179(3):389–412, 2006.

6. (Action types)

For all times t and all $g \in K$, the configuration $x(t)$ is fixed by g, since $x(gt) = x(t) = gx(t)$, hence the real configuration space is $\mathcal{X}^K \subset \mathcal{X}$, and hence

$$X^K \cong H^1(\mathbb{T}^1, \mathcal{X}^K), \qquad X^G = \left(X^K\right)^{G/K};$$

the quotient group G/K acts on X^K, and so that by replacing the configuration space X with X^K one is left with the action (cyclic or dihedral) of $G/K \subset O(2)$ on X^K.
Action of cyclic/brake/dihedral type according to G/K. In any case, at most 2 generators.
Examples:

(1) Collinear/planar n-body problem.

(2) Isosceles, rhomboidal, tetrahedral, multiple binaries, platonic, …

6. (Action types)

For all times t and all $g \in K$, the configuration $x(t)$ is fixed by g, since $x(gt) = x(t) = gx(t)$, hence the real configuration space is $\mathcal{X}^K \subset \mathcal{X}$, and hence

$$X^K \cong H^1(\mathbb{T}^1, \mathcal{X}^K), \qquad X^G = \left(X^K\right)^{G/K};$$

the quotient group G/K acts on X^K, and so that by replacing the configuration space X with X^K one is left with the action (cyclic or dihedral) of $G/K \subset O(2)$ on X^K.
Action of cyclic/brake/dihedral type according to G/K. In any case, at most 2 generators.
Examples:

(1) Collinear/planar n-body problem.

(2) Isosceles, rhomboidal, tetrahedral, multiple binaries, platonic, ...

6. (Action types)

For all times t and all $g \in K$, the configuration $x(t)$ is fixed by g, since $x(gt) = x(t) = gx(t)$, hence the real configuration space is $\mathcal{X}^K \subset \mathcal{X}$, and hence

$$X^K \cong H^1(\mathbb{T}^1, \mathcal{X}^K), \qquad X^G = \left(X^K\right)^{G/K};$$

the quotient group G/K acts on X^K, and so that by replacing the configuration space X with X^K one is left with the action (cyclic or dihedral) of $G/K \subset O(2)$ on X^K.
Action of cyclic/brake/dihedral type according to G/K. In any case, at most 2 generators.
Examples:

(1) Collinear/planar n-body problem.

(2) Isosceles, rhomboidal, tetrahedral, multiple binaries, platonic, …

7. Non-trivial cores (finite subgroups of $SO(3)$)

(1) Tetrahedral group of order 12 [▷].

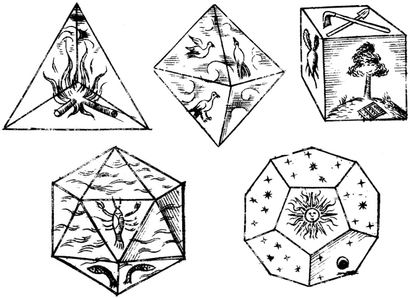
(2) Octahedral group of order 24 [▷].

(3) Icosahedral group of order 60 [▷].

(4) Dihedral group of order $2k$: 4*B* [▷], 6*B* [▷].

(5) Cyclic group (rotation).

(6) Not $SO(3)$: Prism/antiprism 4*B* [▷], 6*B* [▷]. Antipodal (group of order 2).



7. Non-trivial cores (finite subgroups of $SO(3)$)

(1) Tetrahedral group of order 12 [▷].

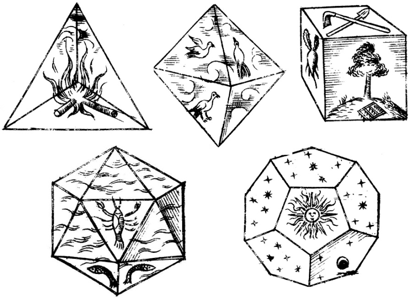
(2) Octahedral group of order 24 [▷].

(3) Icosahedral group of order 60 [▷].

(4) Dihedral group of order $2k$: $4B$ [▷], $6B$ [▷].

(5) Cyclic group (rotation).

(6) Not $SO(3)$: Prism/antiprism $4B$ [▷], $6B$ [▷]. Antipodal (group of order 2).



7. Non-trivial cores (finite subgroups of $SO(3)$)

(1) Tetrahedral group of order 12 [▷].

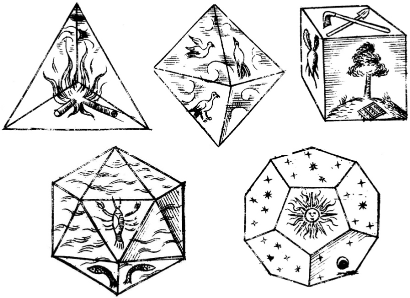
(2) Octahedral group of order 24 [▷].

(3) Icosahedral group of order 60 [▷].

(4) Dihedral group of order $2k$: $4B$ [▷], $6B$ [▷].

(5) Cyclic group (rotation).

(6) Not $SO(3)$: Prism/antiprism $4B$ [▷], $6B$ [▷]. Antipodal (group of order 2).



7. Non-trivial cores (finite subgroups of $SO(3)$)

(1) Tetrahedral group of order 12 [▷].

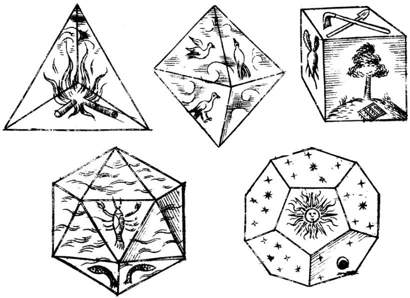
(2) Octahedral group of order 24 [▷].

(3) Icosahedral group of order 60 [▷].

(4) Dihedral group of order $2k$: $4B$ [▷], $6B$ [▷].

(5) Cyclic group (rotation).

(6) Not $SO(3)$: Prism/antiprism $4B$ [▷], $6B$ [▷]. Antipodal (group of order 2).



7. Non-trivial cores (finite subgroups of $SO(3)$)

(1) Tetrahedral group of order 12 [▷].

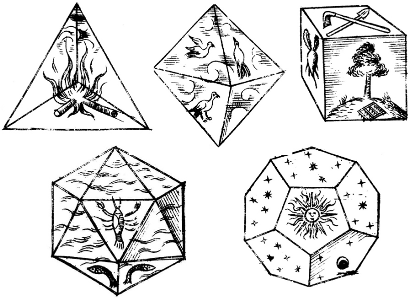
(2) Octahedral group of order 24 [▷].

(3) Icosahedral group of order 60 [▷].

(4) Dihedral group of order $2k$: $4B$ [▷], $6B$ [▷].

(5) Cyclic group (rotation).

(6) Not $SO(3)$: Prism/antiprism $4B$ [▷], $6B$ [▷]. Antipodal (group of order 2).



7. Non-trivial cores (finite subgroups of $SO(3)$)

(1) Tetrahedral group of order 12 [▷].

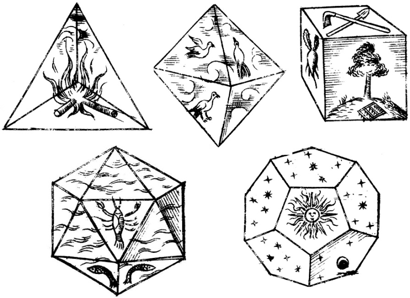
(2) Octahedral group of order 24 [▷].

(3) Icosahedral group of order 60 [▷].

(4) Dihedral group of order $2k$: $4B$ [▷], $6B$ [▷].

(5) Cyclic group (rotation).

(6) Not $SO(3)$: Prism/antiprism $4B$ [▷], $6B$ [▷]. Antipodal (group of order 2).



8. Action on paths

If G is of cyclic type:

$$X^G \cong \left\{ x \colon [0,1] \to \mathcal{X}^K : x(1) = gx(0) \right\}$$

where gK is the generator of the cyclic part $G_0/K \subset G/K$ and (up to rescaling) $[0,1]$ is the fundamental domain in $\mathbb{T}^1$ of the G-action.

If G is of dihedral type:

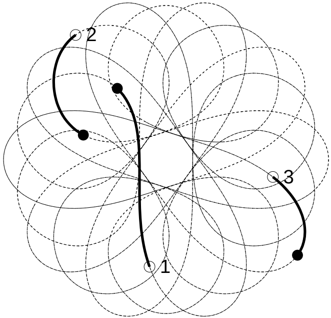
$$X^G \cong \left\{ x \colon [0,1] \to \mathcal{X}^K : h_0 x(0) = x(0),\ h_1 x(1) = x(1) \right\}$$

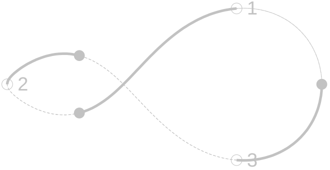
where h_0K and h_1K are two consecutive generators of order 2 of the dihedral group G/K. Hence if $Y_0 \subset \mathcal{X}^K$ and $Y_1 \subset \mathcal{X}^K$ are the linear subspaces fixed by h_0K and h_1K,

$$X^G \cong \left\{ x \colon [0,1] \to \mathcal{X}^K : x(0) \in Y_0,\ x(1) \in Y_1. \right\}$$

9. Reductions: rotating frames

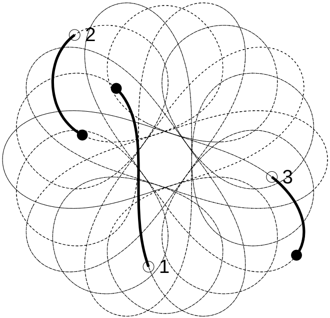
A simplification in the order of G/K can be performed by a simple change of variables (uniformly rotating frame) [8] and [9]. Whenever this is possible (a simple condition on the action, of “rotating type” or not), the order of the cyclic part G_0/K can be reduced to be at most the number of bodies.

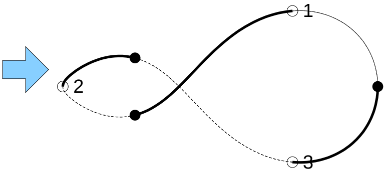




9. Reductions: rotating frames

A simplification in the order of G/K can be performed by a simple change of variables (uniformly rotating frame) [8] and [9]. Whenever this is possible (a simple condition on the action, of "rotating type" or not), the order of the cyclic part G_0/K can be reduced to be at most the number of bodies.





10. Reductions: transitive decomposition

The group G acts on the set of indices $\mathbf{n} = \{1, 2, \ldots, n\}$ by permutations, and hence $\mathbf{n}$ can be decomposed as disjoint sum of G-orbits

$$\mathbf{n} \approx \frac{G}{H_1} + \frac{G}{H_2} + \ldots + \frac{G}{H_l}$$

on which the action is transitive (and with equal masses). The isotropy groups H_i hence determine the permutations on $\mathbf{n}$, and groups with transitive action can be considered as *building blocks* for general symmetry groups.

(1) Union of different choreographies (planar, satellite constellations,...).

(2) Multiple binaries, multiple copies of the same K, ...

[13] DLF. Transitive decomposition of symmetry groups for the n-body problem. *Adv. Math.*, 213(2):763–784, 2007.

10. Reductions: transitive decomposition

The group G acts on the set of indices $\mathbf{n} = \{1, 2, \ldots, n\}$ by permutations, and hence $\mathbf{n}$ can be decomposed as disjoint sum of G-orbits

$$\mathbf{n} \approx \frac{G}{H_1} + \frac{G}{H_2} + \ldots + \frac{G}{H_l}$$

on which the action is transitive (and with equal masses). The isotropy groups H_i hence determine the permutations on $\mathbf{n}$, and groups with transitive action can be considered as *building blocks* for general symmetry groups.

(1) Union of different choreographies (planar, satellite constellations,...).

(2) Multiple binaries, multiple copies of the same K, ...

[13] DLF. Transitive decomposition of symmetry groups for the n-body problem. *Adv. Math.*, 213(2):763–784, 2007.

10. Reductions: transitive decomposition

The group G acts on the set of indices $\mathbf{n} = \{1, 2, \ldots, n\}$ by permutations, and hence $\mathbf{n}$ can be decomposed as disjoint sum of G-orbits

$$\mathbf{n} \approx \frac{G}{H_1} + \frac{G}{H_2} + \ldots + \frac{G}{H_l}$$

on which the action is transitive (and with equal masses). The isotropy groups H_i hence determine the permutations on $\mathbf{n}$, and groups with transitive action can be considered as *building blocks* for general symmetry groups.

(1) Union of different choreographies (planar, satellite constellations,...).

(2) Multiple binaries, multiple copies of the same K, ...

[13] DLF. Transitive decomposition of symmetry groups for the n-body problem. *Adv. Math.*, 213(2):763–784, 2007.

11. CAS and symmetry groups

Elements of the core $K \subset G$: represented as $(A, \sigma) \in O(d) \times \Sigma_n$, where $A \in O(d)$ is an orthogonal matrix and $\sigma \in S_n$ is a permutation of $\mathbf{n} = \{1, 2, \ldots, n\}$. Hence K can be generated by a list of generators

$$[(A_1, \sigma_1), (A_2, \sigma_2), \ldots, (A_k, \sigma_k)].$$

The action on $\mathcal{X} \subset \left(\mathbb{R}^d\right)^{\mathbf{n}}$ is given by

$$((A, \sigma) \cdot x)_i = Ax_{\sigma^{-1}(i)}$$

$$(A, \sigma) \cdot (x_1, \ldots, x_n) = \left(Ax_{\sigma^{-1}(1)}, Ax_{\sigma^{-1}(2)}, \ldots, Ax_{\sigma^{-1}(n)}\right)$$

[14] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.12*; 2008, `(http://www.gap-system.org)`.

Hence G (if G is of cyclic type) can be generated by adding to the generators of K a minimal time translation, encoded as

$$g = (A, \sigma) \in O(d) \times S_n$$

where (A, σ) is in the normalizer of K in $O(d) \times S_n$.
If G is of dihedral type, then it is necessary to add both the minimal time translation and the time reflection h_0, i.e. a pair $(H, \sigma') \in O(d) \times S_n$ such that $h_0^2 \in K$ and h_0 is in the normalizer of G_0. The two generators of G/K h_0K and h_1K will be given by h_0K and gh_0K.
In other words, the action of G can be encoded by the list of generators of K, a representative of the cyclic generator of G_0/K and a representative for a time reflection.

12. Transitive actions

The permutations σ_i can be generated by the permutation representation of K (by multiplication). If the isotropy subgroups are trivial, it turns out to be very effective: consider the matrix group generated by $A_1, \ldots, A_k$, then $\sigma_1, \ldots, \sigma_k$ are nothing but the images of A_i in S_n by the permutation representation of K. For example, if `p` and `q` denote two 3×3 matrices generating the icosahedral group $Y \subset SO(3)$, the corresponding elements `s1` and `s2` in S_{60} can be computed as follows.

```
Y:=GroupWithGenerators([p,q]);
hom:=ActionHomomorphism(Y,Y,OnRight);
s1:=Image(hom,p); s2:=Image(hom,q);
kert:=GroupWithGenerators([ Tuple([p,s1]), Tuple([q,s2]) ] );
```

$$\begin{aligned}\Longrightarrow\ s_1 = {} & (1,11,13)(2,12,14)(3,29,38)(4,30,37)(5,54,42)(6,53,41)(7,31,45)\\ & (8,32,46)(9,56,49)(10,55,50)(15,27,17)(16,28,18)(19,34,39)(20,33,40)\\ & (21,57,43)(22,58,44)(23,36,48)(24,35,47)(25,59,52)(26,60,51)\\ s_2 = {} & (1,49,46)(2,50,45)(3,32,48)(4,31,47)(5,14,9)(6,13,10)(7,52,36)\ldots\end{aligned}$$

14. Methods and properties

(1) **Coercivity:** $\dim \mathcal{X}^G = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(\rho(g)) \#\mathrm{Fix}(\sigma(g)) - d$, where $\rho \colon G \to O(d)$ and $\sigma \colon G \to S_n$. Since coercivity is equivalent to $\dim \mathcal{X}^G = 0$, it can be computed from matrices and permutations.

(2) **Collisions:** Beside general theorems on local/global existence of collisionless minimizers: *a-priori* does the symmetry group *force* collisions (that is, all trajectories are bound to collide)? Can collisions be avoided? Methods include: HasAlwaysCollisions, Rotating circle property, being finite rotation group, …which might be easy to implement or very difficult (homology of complements).

(3) **Rotating frame:** It is possibile to find *rotation axes* in the euclidean space, that can be used to change coordinates and reduce the group order.

(4) **Transitive decomposition:** A very easy method on transitivity for permutation groups, and generator of permutation representations of matrix groups.

14. Methods and properties

(1) **Coercivity:** $\dim \mathcal{X}^G = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(\rho(g)) \#\mathrm{Fix}(\sigma(g)) - d$, where $\rho \colon G \to O(d)$ and $\sigma \colon G \to S_n$. Since coercivity is equivalent to $\dim \mathcal{X}^G = 0$, it can be computed from matrices and permutations.

(2) **Collisions:** Beside general theorems on local/global existence of collisionless minimizers: *a-priori* does the symmetry group *force* collisions (that is, all trajectories are bound to collide)? Can collisions be avoided? Methods include: HasAlwaysCollisions, Rotating circle property, being finite rotation group, ...which might be easy to implement or very difficult (homology of complements).

(3) **Rotating frame:** It is possibile to find *rotation axes* in the euclidean space, that can be used to change coordinates and reduce the group order.

(4) **Transitive decomposition:** A very easy method on transitivity for permutation groups, and generator of permutation representations of matrix groups.

14. Methods and properties

(1) **Coercivity:** $\dim \mathcal{X}^G = \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr}(\rho(g)) \#\operatorname{Fix}(\sigma(g)) - d$, where $\rho \colon G \to O(d)$ and $\sigma \colon G \to S_n$. Since coercivity is equivalent to $\dim \mathcal{X}^G = 0$, it can be computed from matrices and permutations.

(2) **Collisions:** Beside general theorems on local/global existence of collisionless minimizers: *a-priori* does the symmetry group *force* collisions (that is, all trajectories are bound to collide)? Can collisions be avoided? Methods include: HasAlwaysCollisions, Rotating circle property, being finite rotation group, . . . which might be easy to implement or very difficult (homology of complements).

(3) **Rotating frame:** It is possibile to find *rotation axes* in the euclidean space, that can be used to change coordinates and reduce the group order.

(4) **Transitive decomposition:** A very easy method on transitivity for permutation groups, and generator of permutation representations of matrix groups.

14. Methods and properties

(1) **Coercivity:** $\dim \mathcal{X}^G = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(\rho(g)) \# \mathrm{Fix}(\sigma(g)) - d$, where $\rho \colon G \to O(d)$ and $\sigma \colon G \to S_n$. Since coercivity is equivalent to $\dim \mathcal{X}^G = 0$, it can be computed from matrices and permutations.

(2) **Collisions:** Beside general theorems on local/global existence of collisionless minimizers: *a-priori* does the symmetry group *force* collisions (that is, all trajectories are bound to collide)? Can collisions be avoided? Methods include: HasAlwaysCollisions, Rotating circle property, being finite rotation group, ... which might be easy to implement or very difficult (homology of complements).

(3) **Rotating frame:** It is possibile to find *rotation axes* in the euclidean space, that can be used to change coordinates and reduce the group order.

(4) **Transitive decomposition:** A very easy method on transitivity for permutation groups, and generator of permutation representations of matrix groups.

14. Methods and properties

(1) **Coercivity:** $\dim \mathcal{X}^G = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(\rho(g)) \#\mathrm{Fix}(\sigma(g)) - d$, where $\rho \colon G \to O(d)$ and $\sigma \colon G \to S_n$. Since coercivity is equivalent to $\dim \mathcal{X}^G = 0$, it can be computed from matrices and permutations.

(2) **Collisions:** Beside general theorems on local/global existence of collisionless minimizers: *a-priori* does the symmetry group *force* collisions (that is, all trajectories are bound to collide)? Can collisions be avoided? Methods include: HasAlwaysCollisions, Rotating circle property, being finite rotation group, ... which might be easy to implement or very difficult (homology of complements).

(3) **Rotating frame:** It is possibile to find *rotation axes* in the euclidean space, that can be used to change coordinates and reduce the group order.

(4) **Transitive decomposition:** A very easy method on transitivity for permutation groups, and generator of permutation representations of matrix groups.

15. Numerical minimization

Paths are elements of

$$X^G \cong \left\{x\colon [0,1] \to \mathcal{X}^K : x(1) = gx(0)\right\}$$
$$X^G \cong \left\{x\colon [0,1] \to \mathcal{X}^K : h_0x(0) = x(0),\ h_1x(1) = x(1)\right\}$$

Hence, assuming $K = 1$, after the projection

$$x \in \mathcal{X} \mapsto \mathcal{X}^K \ni \frac{1}{|K|}\sum_{g\in K} gx$$

the space X^G can be seen as a linear subspace of

$$H^1([0,1], \mathcal{X}) \cong \mathcal{X} \times H^1(\mathbb{T}^1_0, \mathcal{X}) \times \mathcal{X}$$

by

$$x(t) \mapsto (x(0), x(t) - x(0) - t(x(1) - x(0)), x(1))$$

15. Numerical minimization (cont.)

where $\gamma \in H^1(S^1_0, \mathcal{X}) \iff \gamma(0) = 0$.
Now, the constraints can be read as

$$X^G \cong \left\{ (x, \gamma, y) \in \mathcal{X} \times H^1(\mathbb{T}^1_0, \mathcal{X}) \times \mathcal{X} : y = gx \right\}$$
$$X^G \cong \left\{ (x, \gamma, y) \in \mathcal{X} \times H^1(\mathbb{T}^1_0, \mathcal{X}) \times \mathcal{X} : h_0 x = x,\ h_1 y = y \right\},$$

hence the action part is only in $x, y \in \mathcal{X}$. In both cases (cyclic and dihedral), the contraints can be written as fixed points of an involution of $\mathcal{X}^2$:

$$\underbrace{(x, y) \mapsto (g^{-1}y, gx),}_{\text{cyclic type}} \qquad \underbrace{(x, y) \mapsto (h_0 x, h_1 y)}_{\text{dihedral type}}.$$

The part in γ can be now expanded in trigonometric sine-only series.

16. Approximating trajectories

We therefore approximate trajectories with pairs of configurations $(x, y) \in \mathcal{X}^2$ and k-uples of coefficients (in $\mathcal{X}$) for the Fourier expansion of $\gamma(t)$:

$$u(:,:,:) = (x, a_1, a_2, \ldots, a_k, y), \quad x, y, a_i \in \mathcal{X} \subset \mathbb{R}^{nd},$$

that is $n \times d \times (k+2)$-dimensional arrays, and the group action is just given on $u(:,:,0)$ and $u(:,:,k+1)$.
The action is therefore approximated as

$$\mathcal{A} \colon \mathcal{X}^{k+2} \to \mathbb{R}, \quad \mathcal{A} = \underbrace{\mathcal{K}}_{\text{kinetic}} + \underbrace{\mathcal{U}}_{\text{potential}} .$$

The kinetic quadratic form $\mathcal{K}$ (with or without rotating frame) can be easily computed in terms of the coefficients x, a_i, y. The potential part $\mathcal{U}$ has to be approximated by numerical integration, with suitable mesh of points.

18. Methods

Given the approximated path encoding:

(1) Create the starting path (random, linear, from an expression, …)

(2) Numerical minimization, with known suitable algorithms. Constrained to the linear subspace fixed by an orthogonal involution M, with projection $\frac{I+M}{2}$. Well known tested algorithms: conjugate gradient, Newton-Powell (with Finite-difference or analytic Jacobian), Secant Broyden's Update and Finite-difference Jacobian, IMSL routines, …

(3) Check and improve the result (shooting, multi-shooting, …). Algebraic operations on paths, compute the action, the norm of the gradient, reshaping the array, …

(4) Communication/interface.

18. Methods

Given the approximated path encoding:

(1) Create the starting path (random, linear, from an expression, …)

(2) Numerical minimization, with known suitable algorithms. Constrained to the linear subspace fixed by an orthogonal involution M, with projection $\frac{I+M}{2}$. Well known tested algorithms: conjugate gradient, Newton-Powell (with Finite-difference or analytic Jacobian), Secant Broyden's Update and Finite-difference Jacobian, IMSL routines, …

(3) Check and improve the result (shooting, multi-shooting, …). Algebraic operations on paths, compute the action, the norm of the gradient, reshaping the array, …

(4) Communication/interface.

18. Methods

Given the approximated path encoding:

(1) Create the starting path (random, linear, from an expression, . . .)

(2) Numerical minimization, with known suitable algorithms. Constrained to the linear subspace fixed by an orthogonal involution M, with projection $\frac{I+M}{2}$. Well known tested algorithms: conjugate gradient, Newton-Powell (with Finite-difference or analytic Jacobian), Secant Broyden's Update and Finite-difference Jacobian, IMSL routines, . . .

(3) Check and improve the result (shooting, multi-shooting, . . .). Algebraic operations on paths, compute the action, the norm of the gradient, reshaping the array, . . .

(4) Communication/interface.

18. Methods

Given the approximated path encoding:

(1) Create the starting path (random, linear, from an expression, ...)

(2) Numerical minimization, with known suitable algorithms. Constrained to the linear subspace fixed by an orthogonal involution M, with projection $\frac{I+M}{2}$. Well known tested algorithms: conjugate gradient, Newton-Powell (with Finite-difference or analytic Jacobian), Secant Broyden's Update and Finite-difference Jacobian, IMSL routines, ...

(3) Check and improve the result (shooting, multi-shooting, ...). Algebraic operations on paths, compute the action, the norm of the gradient, reshaping the array, ...

(4) Communication/interface.

18. Methods

Given the approximated path encoding:

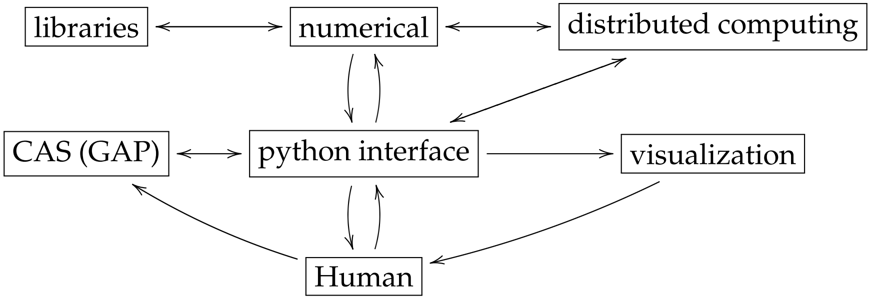
(1) Create the starting path (random, linear, from an expression, ...)

(2) Numerical minimization, with known suitable algorithms. Constrained to the linear subspace fixed by an orthogonal involution M, with projection $\frac{I+M}{2}$. Well known tested algorithms: conjugate gradient, Newton-Powell (with Finite-difference or analytic Jacobian), Secant Broyden's Update and Finite-difference Jacobian, IMSL routines, ...

(3) Check and improve the result (shooting, multi-shooting, ...). Algebraic operations on paths, compute the action, the norm of the gradient, reshaping the array, ...

(4) Communication/interface.

19. Putting things together



20. Interfaces

To integrate symbolic, numeric and interactive visualization tools it is necessary to use different computing paradigms and different specific languages, glued together by some high-level general purpose language. As for `SAGE` (William Stein), the main focus is on *interfaces*, implemented in `python` /`ipython`.

(1) Interactive interface with humans: `Interactive CLI + GAP + geomview/povray`.

(2) Interface with numerical minimization engines: `f95/gf95/gfortran + small parts in C/C++ (interface generated by SWIG)`.

(3) Interface with clusters: `lam-mpi/openmpi`

20. Interfaces

To integrate symbolic, numeric and interactive visualization tools it is necessary to use different computing paradigms and different specific languages, glued together by some high-level general purpose language. As for `SAGE` (William Stein), the main focus is on *interfaces*, implemented in `python` /`ipython`.

(1) Interactive interface with humans: `Interactive CLI + GAP + geomview/povray`.

(2) Interface with numerical minimization engines: `f95/gf95/gfortran + small parts in C/C++ (interface generated by SWIG)`.

(3) Interface with clusters: `lam-mpi/openmpi`

20. Interfaces

To integrate symbolic, numeric and interactive visualization tools it is necessary to use different computing paradigms and different specific languages, glued together by some high-level general purpose language. As for `SAGE` (William Stein), the main focus is on *interfaces*, implemented in `python` /`ipython`.

(1) Interactive interface with humans: `Interactive CLI + GAP + geomview/povray`.

(2) Interface with numerical minimization engines: `f95/gf95/gfortran + small parts in C/C++ (interface generated by SWIG)`.

(3) Interface with clusters: `lam-mpi/openmpi`

20. Interfaces

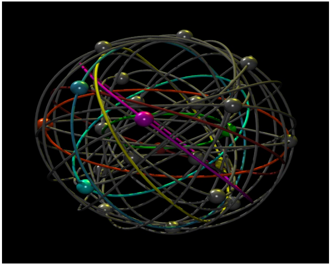
To integrate symbolic, numeric and interactive visualization tools it is necessary to use different computing paradigms and different specific languages, glued together by some high-level general purpose language. As for `SAGE` (William Stein), the main focus is on *interfaces*, implemented in `python` /`ipython`.

(1) Interactive interface with humans: `Interactive CLI + GAP + geomview/povray`.

(2) Interface with numerical minimization engines: `f95/gf95/gfortran + small parts in C/C++ (interface generated by SWIG)`.

(3) Interface with clusters: `lam-mpi/openmpi`

21. Example: rotation symmetries

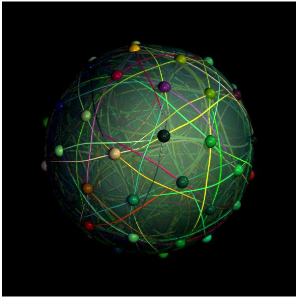


Animation: [▷]
Rendered: [▷]

$$-A_{\mathrm{rot}} = A_1 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

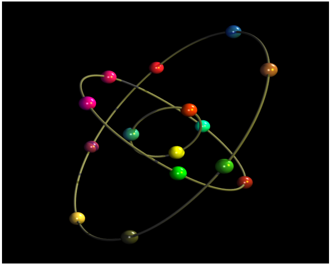
$$\sigma_1 = (1,5,9,13,17,21)(2,6,\ldots$$

$$A_1 = \text{6-gonal rot.}; \sigma_{\mathrm{rot}} = ()$$

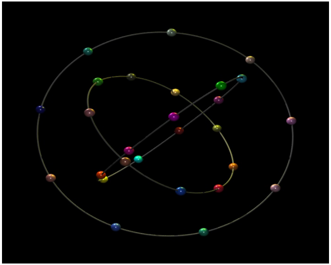


Animation: [▷]
Rendered: [▷]
K=Icosahedral group
$Y \subset SO(3)$;
$x(t+1/2) = -x$.

22. Example: sets of choreographies

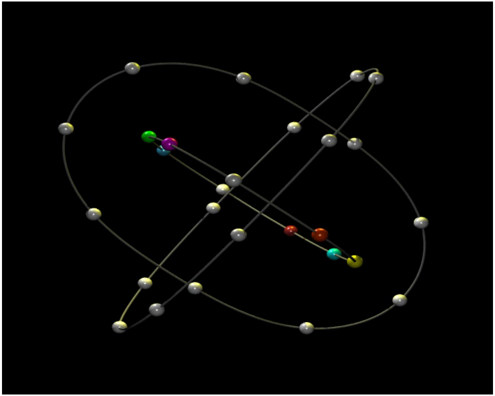


Animation: [▷]
Rendered: [▷]
Constraints: $\sigma = (1, 2, 3)$
$(4, 5, 6, 7, 8)$
$(9, 10, 11, 12, 13, 14, 15)$,
$A = -I_3$.



Animation: [▷]
Rendered: [▷]
Constraints: $\sigma =$
$(1, 2, \ldots, 9)$
$(10, 11, \ldots, 18)$
$(19, 20, \ldots, 27)$; $A = -I_3$.

23. Galileo satellite constellation and Borromean rings



Animation: [▷], Rendered: [▷]

Constraints: cyclic part $\sigma = (1, 2, \ldots, 9)\ (10, 11, \ldots, 18)\ (19, 20, \ldots, 27)$; $A = -I_3$ (as before); $K = \ker \tau$: $A_1 =$ rotation of order 3, $\sigma_1 = (1, 10, 19)(2, 11, 20)(3, 12, 21) \ldots (9, 18, 27)$.

24. Concluding remarks

(1) Periodic solutions of the n-body problem can be studied via computational and theoretical approaches: multi-paradigm approach (different languages, methods, academic practices).

(2) Symmetries yield a rich set of examples (of loop spaces, periodic orbits, mounain-pass theorems, ...), on which apply given techniques. It is an example where computer algebra and numerical methods are both needed (at the same time).

(3) The original goal was more general: study the infinite-dimensional Morse theory for the loop space on the configuration space for the n-body problem (Fadell-Husseini), and the algebraic topology of the corresponding Morse complex.

(4) The Morse complex is actually stratified by the equivariant slices (after being regularized around collisions).

24. Concluding remarks

(1) Periodic solutions of the n-body problem can be studied via computational and theoretical approaches: multi-paradigm approach (different languages, methods, academic practices).

(2) Symmetries yield a rich set of examples (of loop spaces, periodic orbits, mounain-pass theorems, ...), on which apply given techniques. It is an example where computer algebra and numerical methods are both needed (at the same time).

(3) The original goal was more general: study the infinite-dimensional Morse theory for the loop space on the configuration space for the n-body problem (Fadell-Husseini), and the algebraic topology of the corresponding Morse complex.

(4) The Morse complex is actually stratified by the equivariant slices (after being regularized around collisions).

24. Concluding remarks

(1) Periodic solutions of the n-body problem can be studied via computational and theoretical approaches: multi-paradigm approach (different languages, methods, academic practices).

(2) Symmetries yield a rich set of examples (of loop spaces, periodic orbits, mounain-pass theorems, ...), on which apply given techniques. It is an example where computer algebra and numerical methods are both needed (at the same time).

(3) The original goal was more general: study the infinite-dimensional Morse theory for the loop space on the configuration space for the n-body problem (Fadell-Husseini), and the algebraic topology of the corresponding Morse complex.

(4) The Morse complex is actually stratified by the equivariant slices (after being regularized around collisions).

24. Concluding remarks

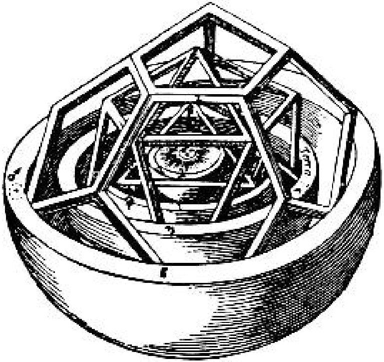
(1) Periodic solutions of the n-body problem can be studied via computational and theoretical approaches: multi-paradigm approach (different languages, methods, academic practices).

(2) Symmetries yield a rich set of examples (of loop spaces, periodic orbits, mounain-pass theorems, ...), on which apply given techniques. It is an example where computer algebra and numerical methods are both needed (at the same time).

(3) The original goal was more general: study the infinite-dimensional Morse theory for the loop space on the configuration space for the n-body problem (Fadell-Husseini), and the algebraic topology of the corresponding Morse complex.

(4) The Morse complex is actually stratified by the equivariant slices (after being regularized around collisions).

25. Kepler model of the solar system (1596)



Thank you