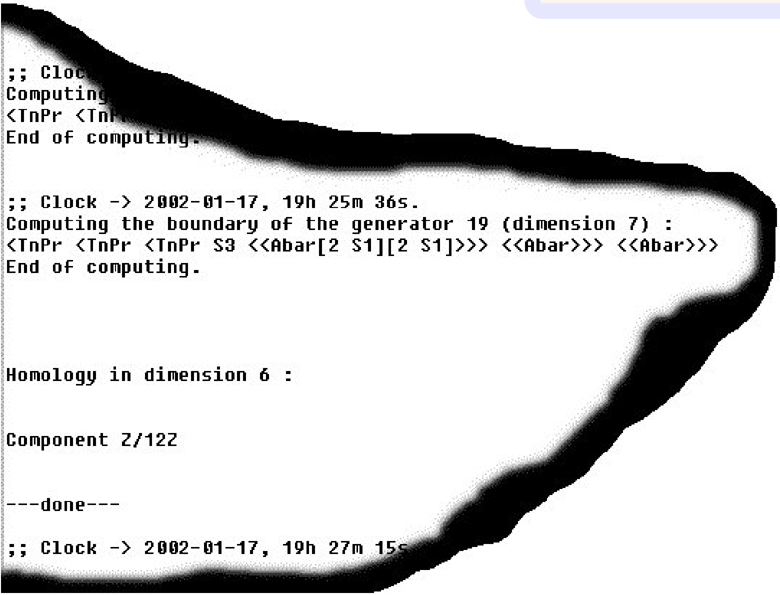
Algorithms for Topological Invariants



Julio Rubio, Universidad de La Rioja
Francis Sergeraert, Institut Fourier, Grenoble
DyToComp Conference
Będlewo, May 31 – June 6, 2009

Semantics of colours:

Blue	=	**“Standard” Mathematics**
Red	=	**Constructive, effective,**
		algorithm, machine object, . . .
Violet	=	**Problem, difficulty, obstacle, disadvantage, . . .**
Green	=	**Solution, essential point, mathematicians, . . .**

Numerical Algorithms for Topological Invariants ???

Topological Invariants

= Invariants of Algebraic Topology ???

= Invariants of Homotopy Type ???

Homology Groups, Homotopy Groups,

Postnikov Invariants, ...

Precise definition of the notion of Algorithm ???

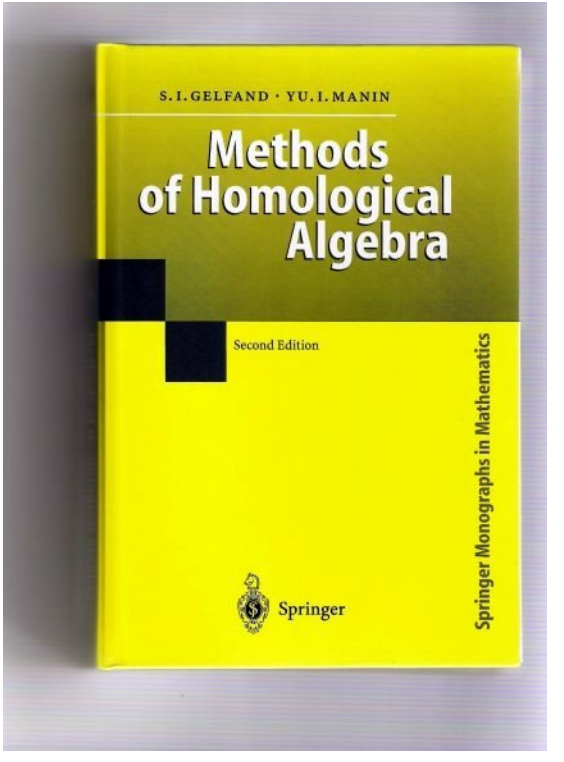
Precise definition of the notion of Invariant ???

Typical example

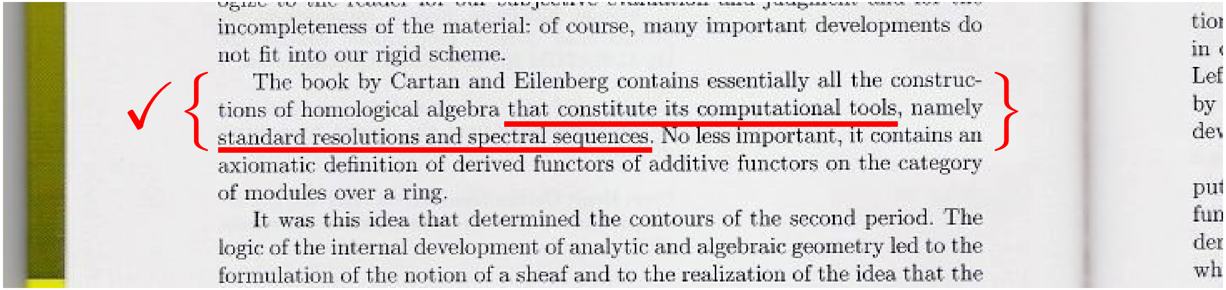
of questionable statement

in a (very good)

classical book.



In the foreword:



incompleteness of the material: of course, many important developments do not fit into our rigid scheme.

The book by Cartan and Eilenberg contains essentially all the constructions of homological algebra that constitute its computational tools, namely standard resolutions and spectral sequences. No less important, it contains an axiomatic definition of derived functors of additive functors on the category of modules over a ring.

It was this idea that determined the contours of the second period. The logic of the internal development of analytic and algebraic geometry led to the formulation of the notion of a sheaf and to the realization of the idea that the

"The book by Cartan and Eilenberg contains essentially all the constructions of homological algebra that constitute its computational tools, namely standard resolutions and spectral sequences."

Essentially false !!

Typical problem not computationnally solved by exact sequences.

J.-P. Serre (1950) computing (?) sphere homotopy groups.

Serre's result: Exact sequence:

$$0 \longleftarrow \mathbb{Z}_6 \longleftarrow \pi_6(S^3) \longleftarrow \mathbb{Z}_2 \longleftarrow 0$$

$$\Rightarrow \quad \pi_6(S^3) = \mathbb{Z}_{12} \ \boxed{\text{or}} \ \mathbb{Z}_2 \oplus \mathbb{Z}_6 \ ???$$

"Solution": "compute" the cohomology class

$$\varepsilon \in H^2(\mathbb{Z}_6, \mathbb{Z}_2) = \mathbb{Z}_2 \text{ classifying the extension.}$$

Needs a representant of the generator of $\mathbb{Z}_6$

in an esoteric chain group $C_6(X_6)$

with X_6 the total space of a terrible fibration

+ a final terrible computation.

Solved one year later by Barrat and Paechter,

thanks to a very specific study:

A NOTE ON $\pi_r(V_{n,m})$

BY M. G. BARRATT AND G. F. PAECHTER

MAGDALEN COLLEGE, OXFORD

Communicated by S. Lefschetz, November 28, 1951

Introduction.—Let $k \geqslant 3$. We shall prove

THEOREM 1.1. *$\pi_{k+3}(S^k)$ has an element of order four.*

Let $V_{k+m,m}$ be the Stiefel Manifold of all orthogonal m-frames in real Euclidean $(k+m)$-space.

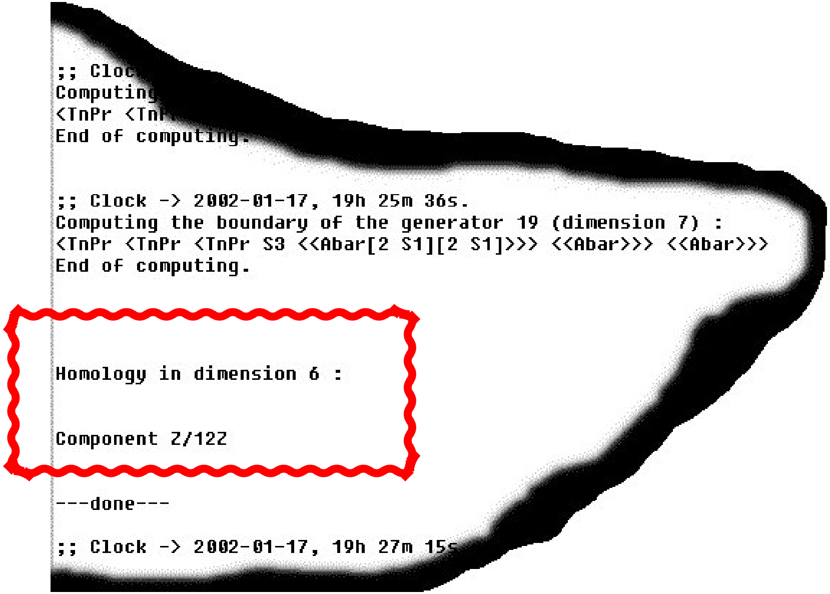
THEOREM 1.2. *The groups $\pi_{k+2}(V_{k+m,m})$ are given by the following table, in which Z_p, Z_∞, are cyclic groups of order p, ∞, respectively.*

$\pi^2 k, m$	$m = 1$	$m = 2$	$m = 3$	$m \geq 4$
$k = 1$	0	Z_∞	$Z_\infty + Z_\infty$	Z_∞
$k = 4s - 2$	Z_2	$Z_2 + Z_2$	Z_2	0
$k = 4s$	Z_2	$Z_2 + Z_2$	$Z_2 + Z_2$	$Z_2 + Z_2$
$k = 4s - 1$	Z_2	Z_4	$Z_2 + Z_\infty$	Z_2
$k = 4s + 1$	Z_2	Z_4	$Z_4 + Z_\infty$	Z_8

Let Y^{n+1} be the $(n-1)$-fold suspension of the real projective plane, so that Y^{n+1} consists of an n-sphere S^n and an $(n+1)$-cell e^{n+1} attached to S^n by a map of degree 2. We prove

THEOREM 1.3 *$\pi_{n+2}(Y^{n+1}) = Z_4$ if $n \geqslant 3$.*

Now “stupidly” obtained by the Kenzo program:



Analysis of the problem:

"Standard" homological algebra is not constructive.

Typical statement:

The sequence $A \xleftarrow{\alpha} B \xleftarrow{\beta} C$ is exact.

Common translation:

$$(\forall b \in B) \quad [(\alpha(b) = 0) \Rightarrow (\boxed{\exists c \in C} \ \underline{\text{st}} \ b = \beta(c))]$$

with $\boxed{\exists c \in C}$ most often non-constructive.

Constructive exactness:

$$A \xleftarrow{\alpha} B \xleftarrow{\beta} C \text{ constructively exact}$$

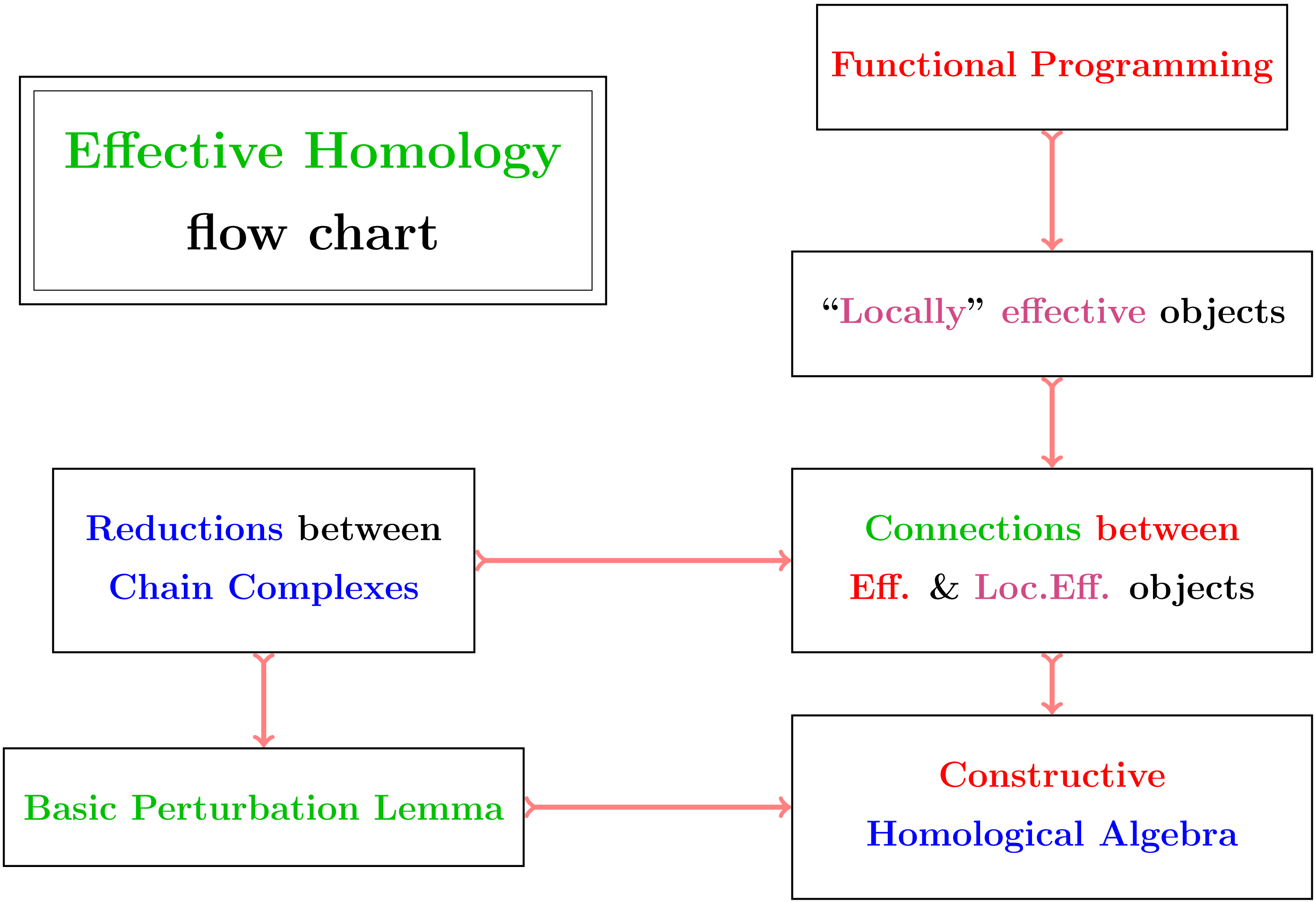
if an algorithm $\rho : \ker \alpha \to C$ is given satisfying:

$$\begin{array}{ccccc} A & \xleftarrow{\alpha} & B & \xleftarrow{\beta} & C \\ \circlearrowleft & & \circlearrowleft & (=) & \uparrow \rho? \\ 0 & \longleftarrow & \ker \alpha & \dashrightarrow & \end{array}$$

$\Rightarrow$ **Organizational algebraic problems:**

$$0 \longleftarrow \mathbb{Z}/2\mathbb{Z} \xleftarrow{\text{pr}} \mathbb{Z} \qquad \rho?: \mathbb{Z}/2\mathbb{Z} \dashrightarrow \mathbb{Z}$$

where ρ cannot be a group homomorphism.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Functional Programming:

The art of handling and creating functional objects.

Examples of functional objects:

$$(\mathbb{Z}, +, -, \times) \quad (\mathbb{Z}[X], +, -, \times)$$

Other example:

Kan model for the loop space $\Omega S^3 := \mathcal{C}(S^1, S^3)$:

$$(\mathcal{S}_{\Omega S^3}, \{\partial_i^n\}_{n \geq 1, 0 \leq i \leq n}, \{\eta_i^n\}_{n \geq 0, 0 \leq i \leq n})$$

with $\mathcal{S}_{\Omega S^3}$ = the simplex set of the Kan model.

= "Locally" effective objects.

Main problem:

Designing programs $(f_1, \ldots, f_n) \mapsto f$.

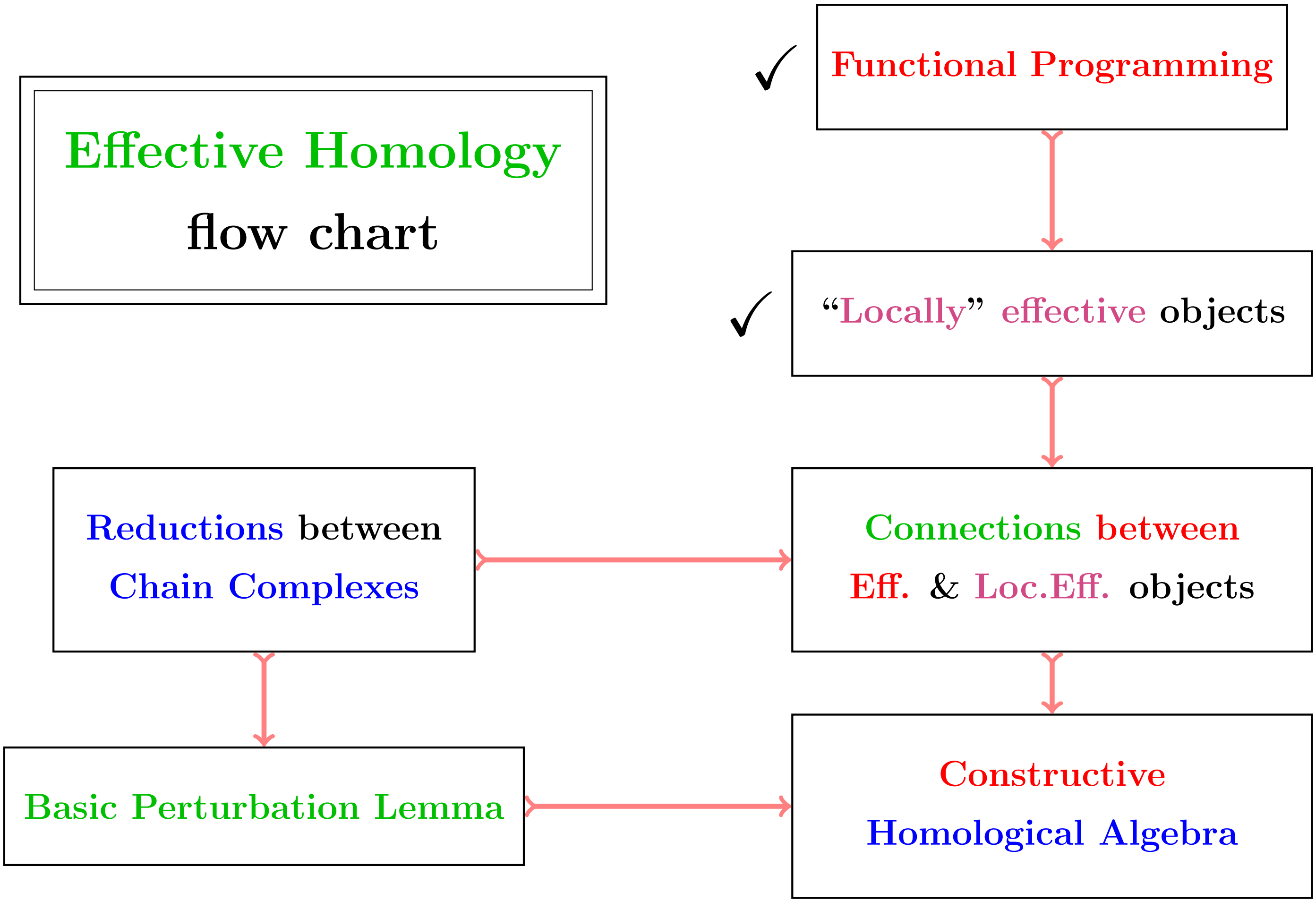
Example:

$$(\mathfrak{R}, +_{\mathfrak{R}}, -_{\mathfrak{R}}, \times_{\mathfrak{R}}) \mapsto (\mathfrak{R}[X], +_{\mathfrak{R}[X]}, -_{\mathfrak{R}[X]}, \times_{\mathfrak{R}[X]})$$

Topological example. X = **topological space.**

$$(\mathcal{S}_X, \{\partial(X)_i^n\}_{n\geq 1, 0\leq i\leq n}, \{\eta(X)_i^n\}_{n\geq 0, 0\leq i\leq n})$$
$$\mapsto (\mathcal{S}_{\Omega X}, \{\partial(\Omega X)_i^n\}_{n\geq 1, 0\leq i\leq n}, \{\eta(\Omega X)_i^n\}_{n\geq 0, 0\leq i\leq n})$$

Solution = **λ-calculus**, **Lisp**, **ML**, **Axiom**, **Haskell**...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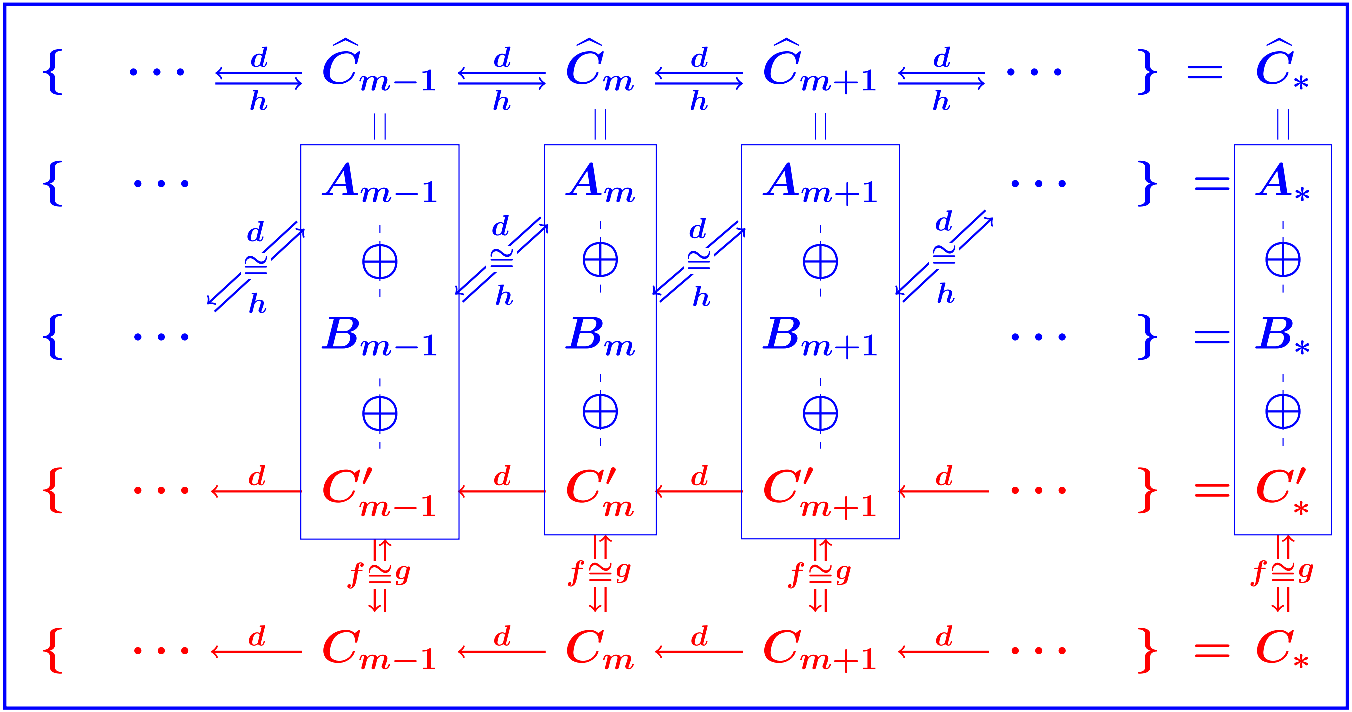
Definition: A (homological) reduction is a diagram:

$$\rho: \boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*}$$

with:

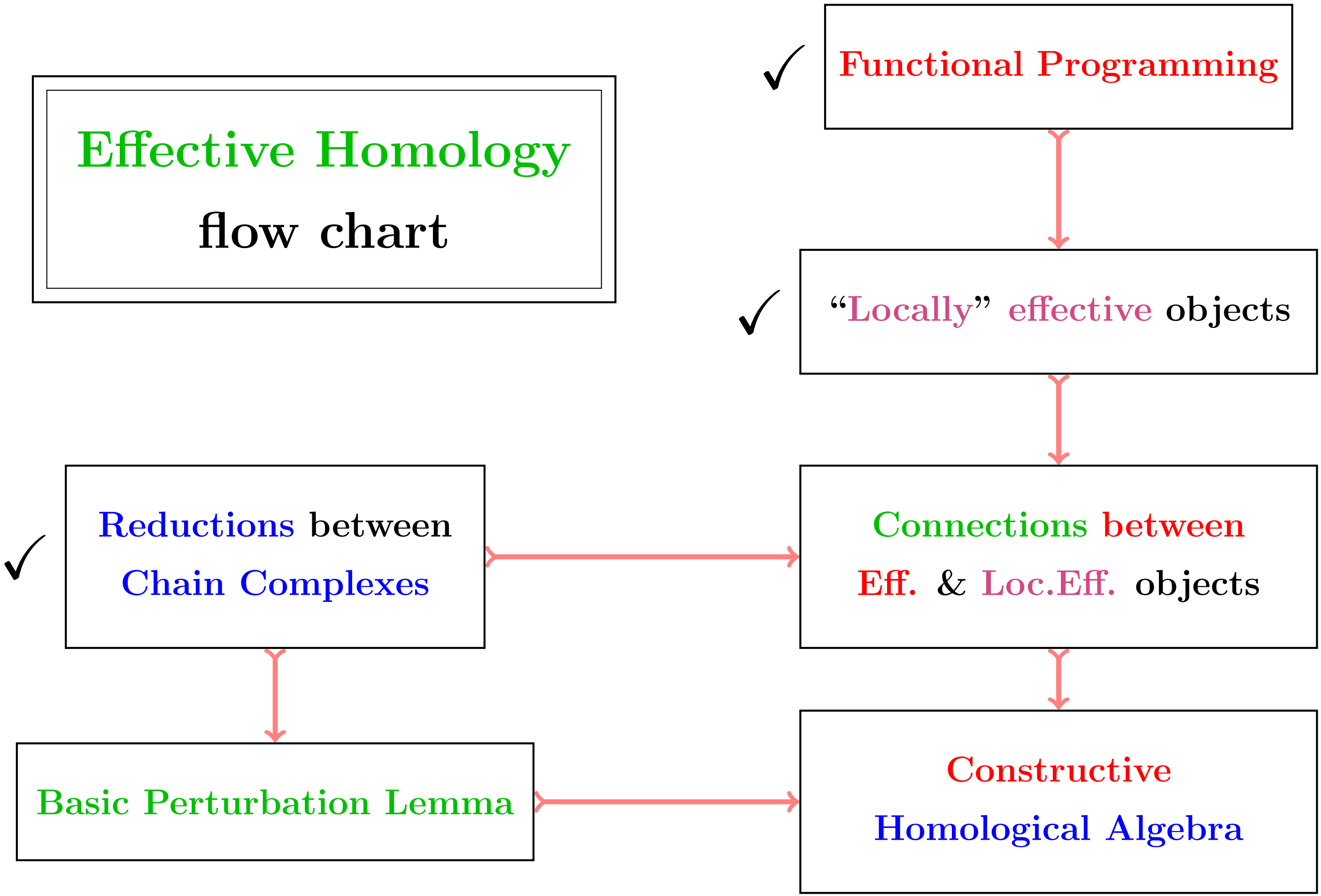
1. $\widehat{C}_*$ and C_* = chain complexes.
2. f and g = chain complex morphisms.
3. h = homotopy operator (degree $+1$).
4. $fg = \mathrm{id}_{C_*}$ and $d_{\widehat{C}} h + h d_{\widehat{C}} + gf = \mathrm{id}_{\widehat{C}_*}$.
5. $fh = 0$, $hg = 0$ and $hh = 0$.



$\boxed{A_* = \ker f \cap \ker h} \quad \boxed{B_* = \ker f \cap \ker d} \quad \boxed{C'_* = \operatorname{im} g}$

$$\widehat{C}_* = \boxed{A_* \oplus B_* \text{ exact}} \oplus \boxed{C'_* \cong C_*}$$

Effective Homology flow chart



Let ρ: $\boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\leftleftarrows}} C_*}$ be a reduction.

Frequently:

1. **$\widehat{C}_*$ is a locally effective chain complex:**
   its homology groups are unreachable.

2. **C_* is an effective chain complex:**
   its homology groups are computable.

3. **The reduction ρ is an entire description of**
   the homological nature of $\widehat{C}_*$.

4. **Any homological problem in $\widehat{C}_*$ is solvable**
   thanks to the information provided by ρ.

ρ: $\boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\rightleftarrows}} C_*}$

1. **What is** $H_n(\widehat{C}_*)$**?** **Solution: Compute** $H_n(C_*)$.

2. **Let** $x \in \widehat{C}_n$**. Is** x **a cycle?** **Solution: Compute** $d_{\widehat{C}_*}(x)$.

3. **Let** $x, x' \in \widehat{C}_n$ **be cycles. Are they homologous?**

   Solution: Look whether $f(x)$ **and** $f(x')$ **are homologous.**

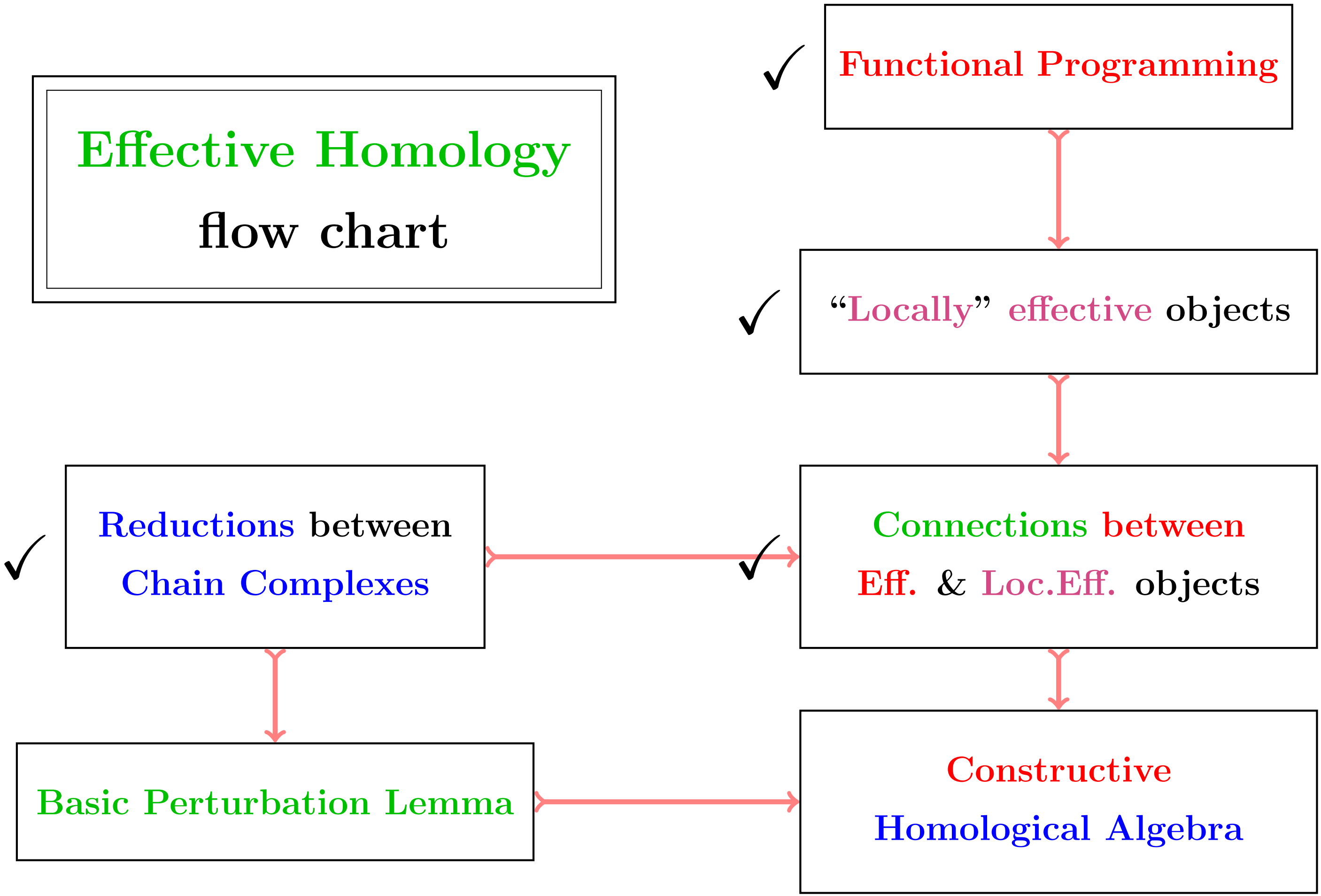
4. **Let** $x, x' \in \widehat{C}_n$ **be homologous cycles.**

   Find $y \in \widehat{C}_{n+1}$ **satisfying** $dy = x - x'$**?**

   Solution:

   (a) **Find** $z \in C_{n+1}$ **satisfying** $dz = f(x) - f(x')$.

   (b) $y = g(z) + h(x - x')$.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<u>Definition</u>: (C_*, d) = given chain complex.

A perturbation $\delta : C_* \to C_{*-1}$ is an operator of degree -1

satisfying $(d + \delta)^2 = 0$ ($\Leftrightarrow$ $(d\delta + \delta d + \delta^2) = 0$):

$$(C_*, d) + (\delta) \mapsto (C_*, d{+}\delta).$$

<u>Problem</u>: Let ρ: $\boxed{h \circlearrowright (\widehat{C}_*, \widehat{d}) \underset{f}{\overset{g}{\rightleftarrows}} (C_*, d)}$ be a given reduction and $\widehat{\delta}$ a perturbation of $\widehat{d}$.

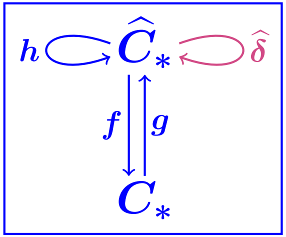
How to determine a new reduction:

$$???: \boxed{h{+}? \circlearrowright (\widehat{C}_*, \widehat{d}{+}\widehat{\delta}) \underset{f+?}{\overset{g+?}{\rightleftarrows}} (C_*, d{+}?)}$$

describing in the same way the homology of

the chain complex with the perturbed differential?

Basic Perturbation "Lemma" (BPL):

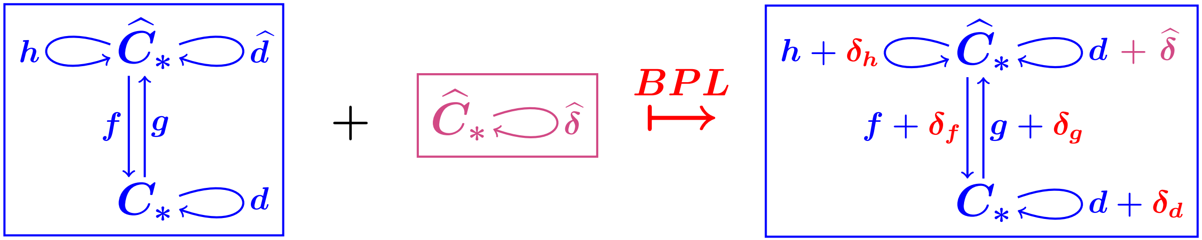
Given: 

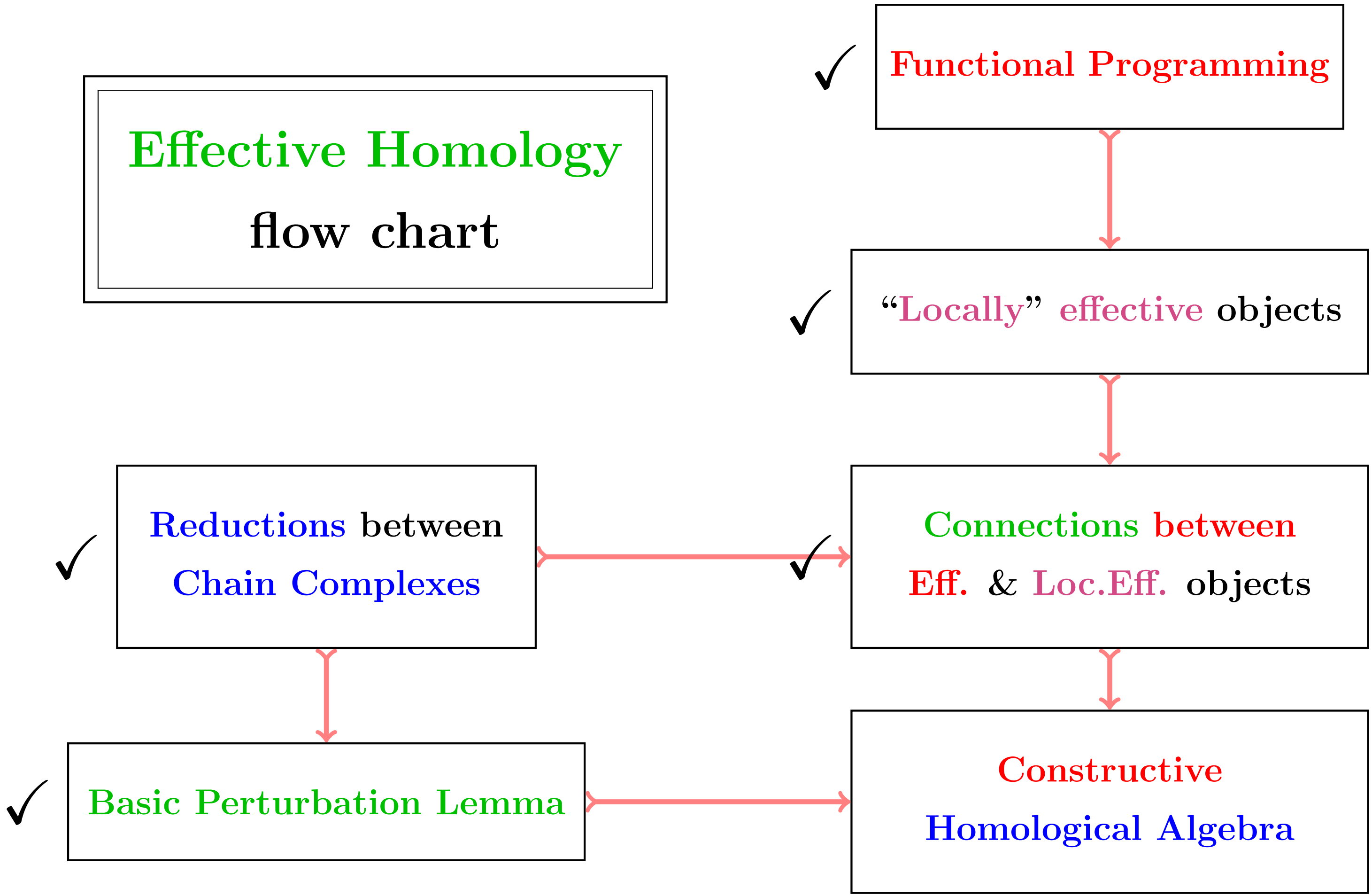
satisfying:

1. $\widehat{\delta}$ is a perturbation of the differential $\widehat{d}$ of $\widehat{C}_*$;

2. The operator $h \circ \widehat{\delta}$ is pointwise nilpotent.

Then a general algorithm BPL constructs:

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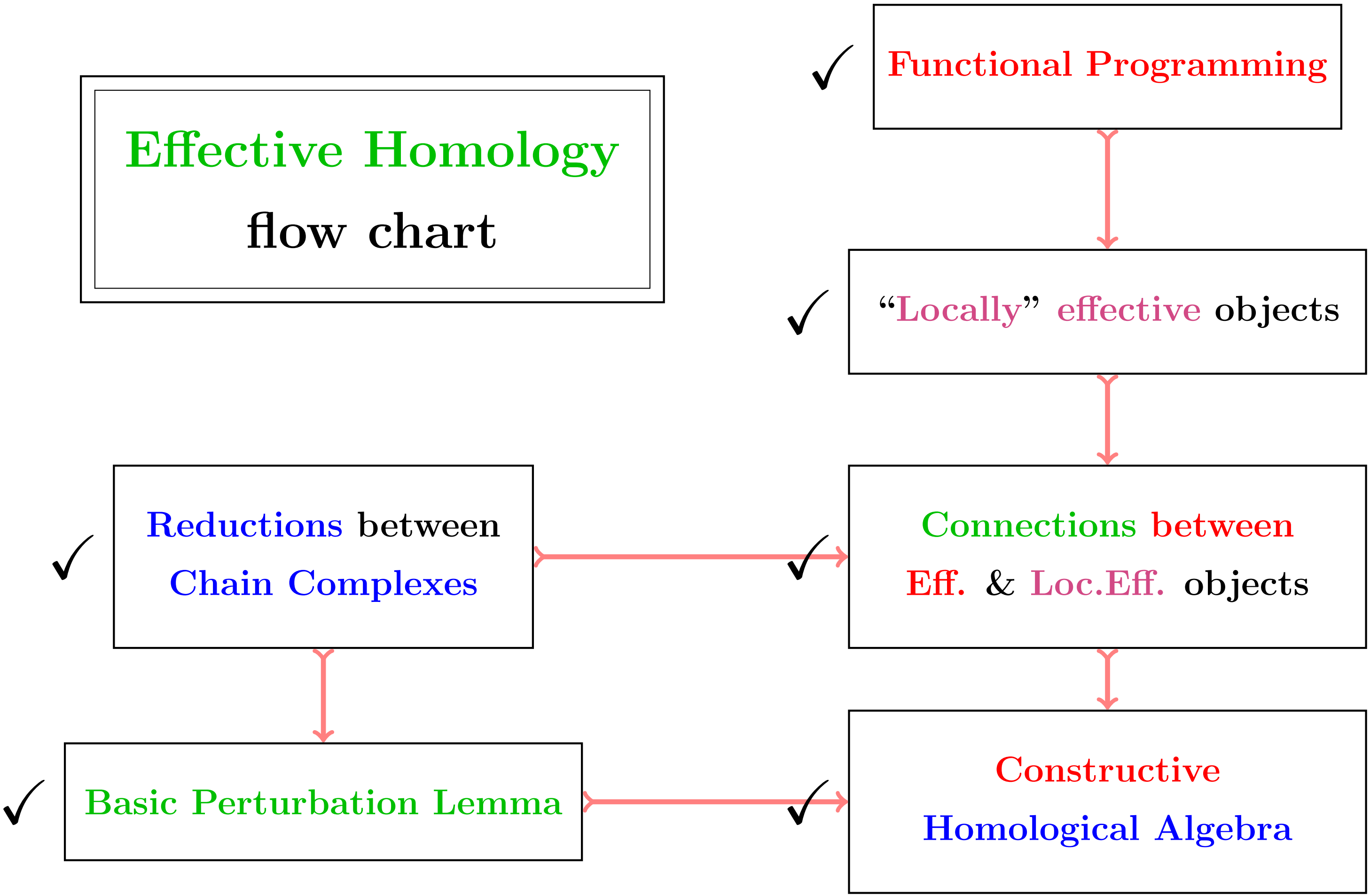
Serre: "Everything" in Algebraic Topology
can be reduced to Fibration problems.

Examples: Loop spaces, Classifying spaces, Homogeneous spaces, Whitehead tower, Postnikov tower, ...

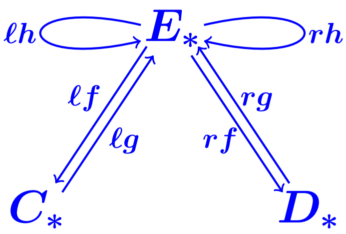
Remark: Fibration = Twisted Product
= Perturbation of Trivial Product.

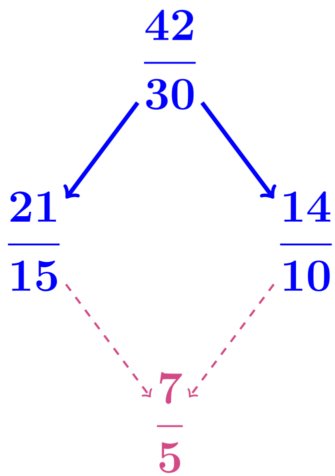
Corollary: BPL is effective
+ Fibration = Perturbation of Trivial Product
+ Everything is Fibration
$\Rightarrow$ Alg. Topology becomes Constructive.

Effective Homology flow chart



Definition: A (strong chain-) equivalence $\varepsilon : C_* \Longleftarrow\!\!\!\Longrightarrow D_*$ is a pair of reductions $C_* \overset{\ell\rho}{\Longleftarrow} E_* \overset{r\rho}{\Longrightarrow} D_*$:





Normal form problem ??

More structure often necessary in C_*.

Most often: no possible choice for C_*.

Definition: An object with effective homology X is a 4-tuple:

$$X = \boxed{X, C_*(X), EC_*, \varepsilon}$$

with:

1. X = an arbitrary object (simplicial set, simplicial group, differential graded algebra, ...)
2. $C_*(X)$ = "the" chain complex "traditionally" associated with X to define the homology groups $H_*(X)$.
3. EC_* = some effective chain complex.
4. ε = some equivalence $C_*(X) \stackrel{\varepsilon}{\Longleftrightarrow} EC_*$.

Main result of effective homology:

Meta-theorem: Let $X_1, \ldots, X_n$ be a collection of objects with effective homology and ϕ be a reasonable construction process:

$$\phi : (X_1, \ldots, X_n) \mapsto X.$$

Then there exists a version with effective homology ϕ_{EH}:

$$\phi_{EH}: (\boxed{X_1, C_*(X_1), EC_{1*}, \varepsilon_1}, \ldots, \boxed{X_n, C_*(X_n), EC_{n*}, \varepsilon_n}) \mapsto \boxed{X, C_*(X), EC_*, \varepsilon}$$

The process is **perfectly stable** and can be again used with X for further calculations.

Example:

Julio Rubio's solution of Adams' problem.

$$\boxed{\boldsymbol{X} = (X,\ C_*(X),\ EC_*^X,\ \varepsilon^X)}$$

$$\Downarrow\Downarrow\Downarrow \qquad \textbf{Eil.-Moore}_{EH}$$

$$\boxed{\boldsymbol{\Omega X} = (\Omega X,\ C_*(\Omega X),\ EC_*^{\Omega X},\ \varepsilon^{\Omega X})}$$

$\Longrightarrow$ **Trivial iteration now available.**

$\Rightarrow$ Very simple solution of Adam's problem :

Indefinite iteration of the Cobar construction ???

$$\begin{aligned}
X &= (X, C_*(X), EC_*^X, \varepsilon^X) \\
&\Downarrow \Omega_{EH} \\
\Omega X &= (\Omega X, C_*(\Omega X), EC_*^{\Omega X}, \varepsilon^{\Omega X}) \\
&\Downarrow \Omega_{EH} \\
\Omega^2 X &= (\Omega^2 X, C_*(\Omega^2 X), EC_*^{\Omega^2 X}, \varepsilon^{\Omega^2 X}) \\
&\Downarrow \Omega_{EH} \\
\Omega^3 X &= (\Omega^3 X, C_*(\Omega^3 X), EC_*^{\Omega^3 X}, \varepsilon^{\Omega^3 X}) \\
&\Downarrow \Omega_{EH} \\
\Omega^4 X &= \ldots
\end{aligned}$$

"Cobar" $\boxed{3}$ (EC_*^X)

Example: **Effective homology version of the Serre spectral sequence.**

$$\boldsymbol{F} = (F,\ C_*(F),\ \boldsymbol{EC}_*^{\boldsymbol{F}},\ \varepsilon^F)$$
$$+\ \ \boldsymbol{B} = (B,\ C_*(B),\ \boldsymbol{EC}_*^{\boldsymbol{B}},\ \varepsilon^B)$$
$$+\ \ \boldsymbol{\tau} : B \longrightarrow F$$
$$\Downarrow \Downarrow \Downarrow \Downarrow \Downarrow \ \text{Serre}_{EH}$$
$$\boldsymbol{E} = \boldsymbol{F} \times_{\boldsymbol{\tau}} \boldsymbol{B} = (E,\ C_*(E),\ \boldsymbol{EC}^{\boldsymbol{E}},\ \varepsilon^E)$$

(Serre + G. Hirsch + H. Cartan + Shih W. + Szczarba + Ronnie Brown + J. Rubio + FS)

Proof.

$$C_*(F \times B) \overset{\text{id}}{\Lleftarrow} C_*(F \times B) \overset{EZ}{\Rrightarrow} C_*F \otimes C_*B$$

$$C_*F \otimes C_*B \overset{\otimes}{\Lleftarrow} \widehat{C}^F \otimes \widehat{C}^B \overset{\otimes}{\Rrightarrow} EC^F \otimes EC^B$$

$$\Downarrow\Downarrow\Downarrow\Downarrow\Downarrow \ \text{Serre}_{EH}$$

$$C_*(F \times_{\boxed{\tau}} B) \overset{\text{id}}{\Lleftarrow} C_*(F \times_{\boxed{\tau}} B) \overset{\text{Shih}}{\Rrightarrow} C_*F \otimes_{\boxed{t}} C_*B$$

$$C_*F \otimes_{\boxed{t}} C_*B \overset{EPL}{\Lleftarrow} \widehat{C}^F \otimes_{\boxed{t'}} \widehat{C}^B \overset{BPL}{\Rrightarrow} EC^F \otimes_{\boxed{t''}} EC^B$$

+ **Composition** of **equivalences** $\Longrightarrow$ **O.K.**

Combining these ingredients $\Rightarrow$

Homological Algebra becomes constructive.

Corollary: The "standard" exact and spectral sequences of Homological Algebra really become computational tools.

$\Rightarrow$ Concrete computer programs (EAT, Kenzo).

Example of computation.

$$P^2\mathbb{R} \subset P^3\mathbb{R} \subset P^4\mathbb{R} \subset \cdots \subset P^\infty\mathbb{R}$$

$\Rightarrow \boxed{\mathtt{P}} = P^\infty\mathbb{R}/P^3\mathbb{R}$ **is defined.**

$$\boxed{\mathtt{OOP}} = \Omega^2 \boxed{\mathtt{P}} = \mathcal{C}(S^2, P^\infty\mathbb{R}/P^3\mathbb{R})$$

$$\pi_2(\boxed{\mathtt{OOP}}) = H_2(\boxed{\mathtt{OOP}}) = \mathbb{Z}$$

$\Rightarrow f : S^2 \longrightarrow \boxed{\mathtt{OOP}}$ **of degree 2 defined.**

$\Rightarrow \boxed{\mathtt{DOOP}} = D^3 {}_2\!\cup \boxed{\mathtt{OOP}}$ **defined.**

$$\boxed{\mathtt{ODOOP}} = \Omega \boxed{\mathtt{DOOP}} = \mathcal{C}(S^1, \quad D^3 {}_2\!\cup \mathcal{C}(S^2, P^\infty\mathbb{R}/P^3\mathbb{R})).$$

Exercise: $H_4(\boxed{\mathtt{ODOOP}}) = ??$

Example of computation.

$$P^2\mathbb{R} \subset P^3\mathbb{R} \subset P^4\mathbb{R} \subset \cdots \subset P^\infty\mathbb{R}$$

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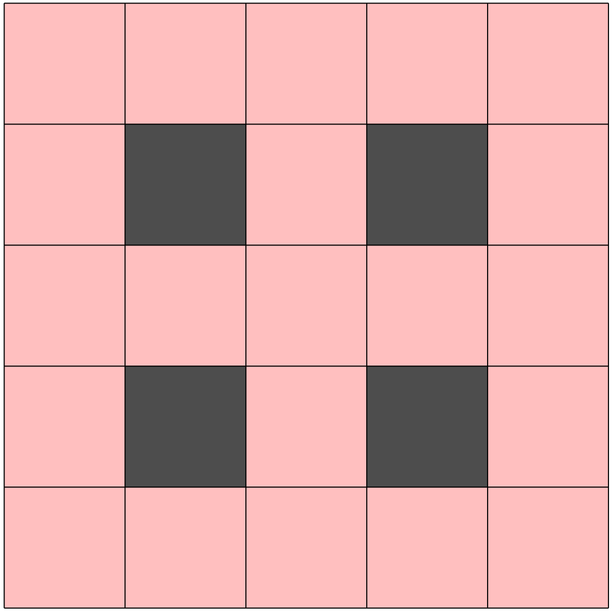
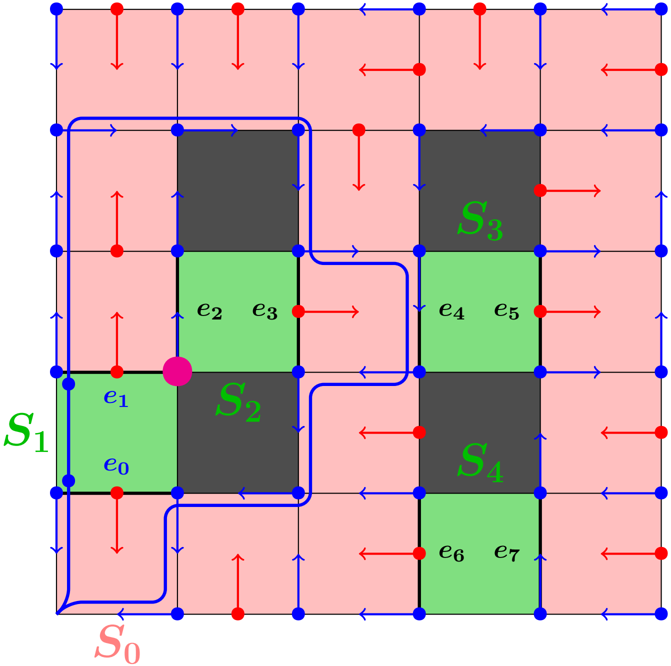
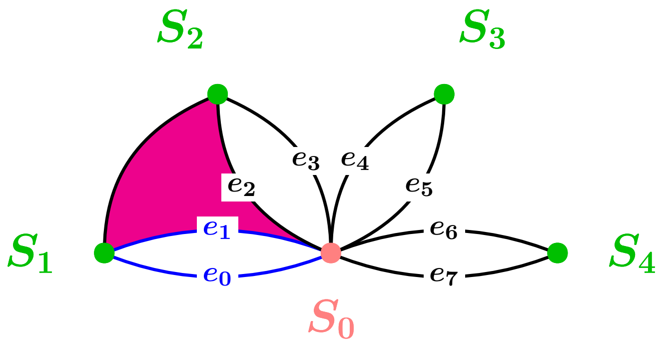
Solution: $H_4(\boxed{\texttt{ODOOP}}) = (\mathbb{Z}/2)^8 + \mathbb{Z}$

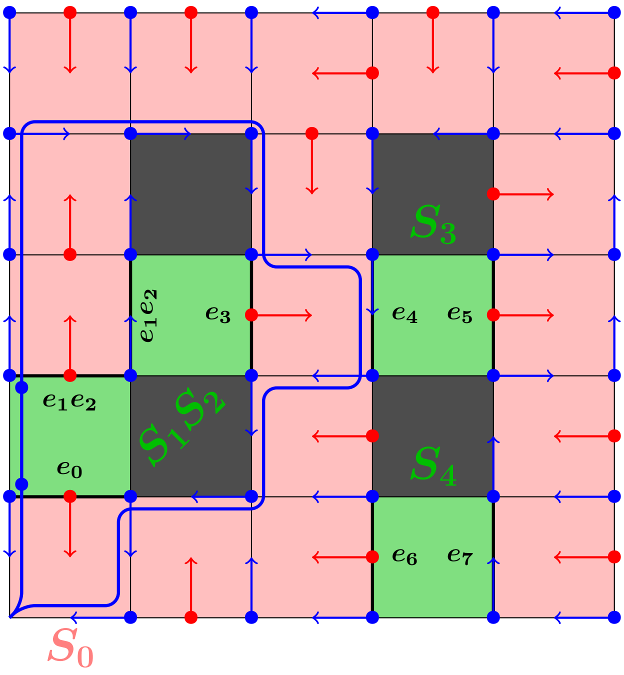
Application to Analysis of Digitalized Images:

Example $=$ 

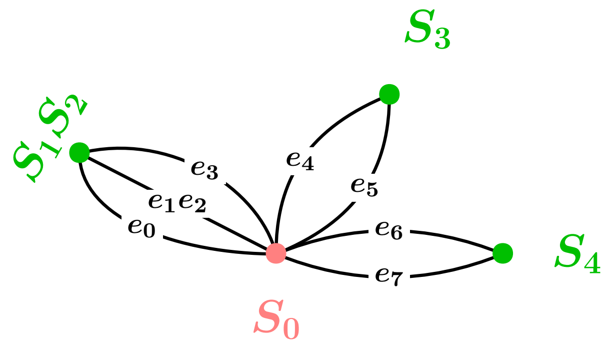


$$
\begin{array}{ccccccccc}
\mathbf{0} & \longleftarrow & \mathbb{Z}^{36} & \underset{d}{\overset{h}{\rightleftarrows}} & \mathbb{Z}^{60} & \underset{d}{\overset{h}{\rightleftarrows}} & \mathbb{Z}^{21} & \longleftarrow & \mathbf{0} \\
 & & f\,\downarrow\uparrow\,g & & f\,\downarrow\uparrow\,g & & f\,\downarrow\uparrow\,g & & \\
\mathbf{0} & \longleftarrow & \mathbb{Z}^{5} & \underset{d}{\longleftarrow} & \mathbb{Z}^{9} & \underset{d}{\longleftarrow} & \mathbb{Z} & \longleftarrow & \mathbf{0}
\end{array}
$$





$$
\begin{array}{ccccccccc}
\mathbf{0} & \longleftarrow & \mathbb{Z}^{36} & \underset{d}{\overset{h}{\rightleftarrows}} & \mathbb{Z}^{60} & \underset{d}{\overset{h}{\rightleftarrows}} & \mathbb{Z}^{21} & \longleftarrow & \mathbf{0} \\
 & & f \downarrow \uparrow g & & f \downarrow \uparrow g & & f \downarrow \uparrow g & & \\
\mathbf{0} & \longleftarrow & \mathbb{Z}^{5} & \underset{d}{\longleftarrow} & \mathbb{Z}^{8} & \underset{d}{\longleftarrow} & \mathbf{0} & \longleftarrow & \mathbf{0}
\end{array}
$$



Most general result: X = combinatorial space

$(X_i)_i$ = finite covering by subspaces.

$X_{i_0 \cdots i_m} := X_{i_0} \cap \cdots \cap X_{i_m}$

Algorithm:

Input: $\{(X_{i_0 \cdots i_m}, C_*(X_{i_0 \cdots i_m}), EC_{i_0 \cdots i_m}, \varepsilon_{i_0 \cdots i_m})\}_{i_0, \ldots, i_m}$

Output: $(X, C_*(X), EC_*, \varepsilon)$

Effective Homology of all $X_{i_0 \cdots i_m}$'s

$\longmapsto$ **Effective Homology of X**

$\Rightarrow$ **Many efficient possible recursive processes**

computing the Effective Homology of X

Other example:

Combinatorial object X

provided with a Morse function $f : X \to \mathbb{Z}$.

The critical values $(c_1, \ldots, c_n)$ define a filtration:

$$\emptyset = X_1' \subset X_1 \subset X_2' \subset X_2 \subset \cdots \subset X_n' \subset X_n = X$$

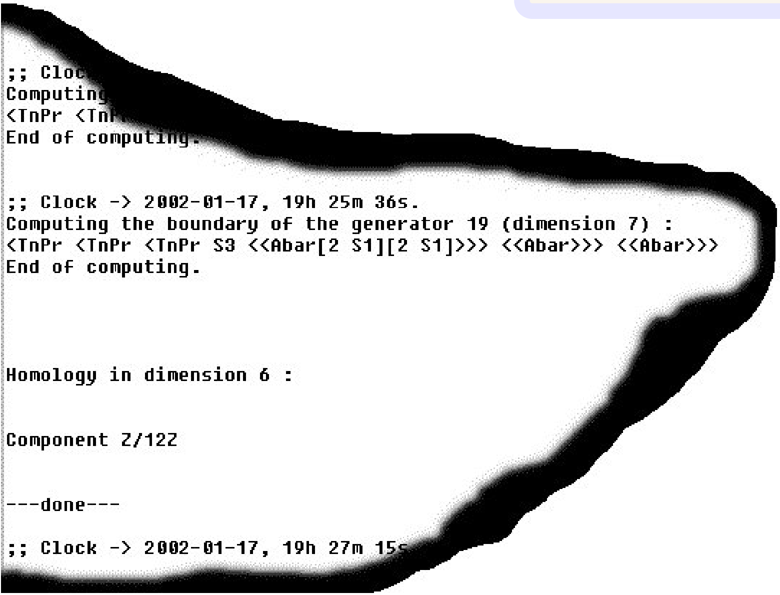
with:

1) Reductions $C_*(X_n')/C_*(X_{n-1}) \ggg 0$

2) Reductions $C_*(X_n)/C_*(X_n') \ggg \mathbb{Z}_{\mathrm{index}(c_n)}$ (Morse lemma)

$\Rightarrow$ Simple recursive method giving $(X, C_*X, EC_*, \varepsilon)$.

The END



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