Methods of detecting periodic orbits: guiding functions and periodic segments

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3.06.2009

Notation and terminology

Consider the non-autonomous ODE of the form

$$x' = f(t, x), \tag{1}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, T-periodic with respect to t and locally Lipschitzian with respect to x.
Let Φ be a T-periodic local process on $\mathbb{R}^n$ generated by (1) and let ϕ be the corresponding local flow on $\mathbb{R} \times \mathbb{R}^n$.
The map $\Phi_{(0,T)}$ is called the *Poincaré map*.
We denote by π_1 and π_2 the projections of $\mathbb{R} \times \mathbb{R}^n$ onto $\mathbb{R}$ and, respectively, $\mathbb{R}^n$.

Notation and terminology

For $W \subset \mathbb{R}^n$ we define the *exit set* of W as

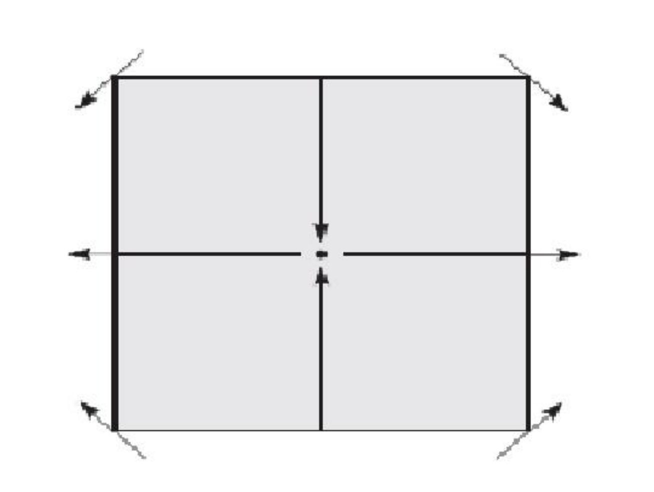
$$W^- := \{x \in W : \phi(x, [0, t]) \not\subset W \, \forall t \in (0, \omega_x)\}.$$

The *entrance set* W^+ of W is defined as the exit set of W with respect to the flow ϕ^* obtained from ϕ by the transformation $t \to -t$ of time.

Isolating block

We call W an *isolating block* for ϕ if W, $W^\pm$ are compact and $\partial W = W^- \cup W^+$.

Notation and terminology



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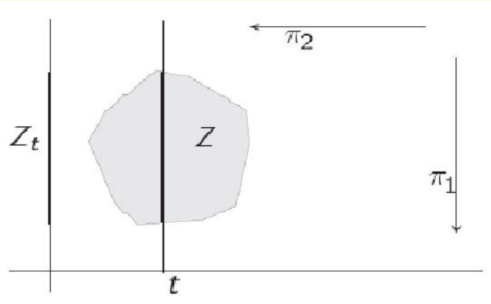
Definitions

If Z is a subset of $\mathbb{R} \times \mathbb{R}^n$ and $t \in \mathbb{R}$ then we put

$$Z_t := \{z \in \mathbb{R}^n : (t, z) \in Z\}.$$

$Z \subset [0, T] \times \mathbb{R}^n$ is T-periodic if $Z_0 = Z_T$.



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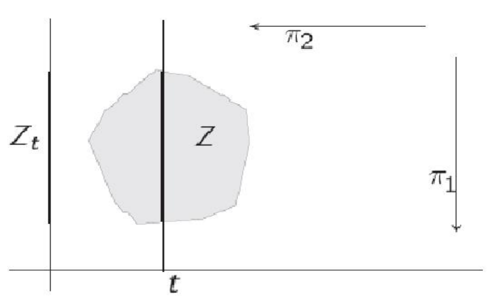

Definitions

Periodic segment

A compact set $W \subset [0, T] \times \mathbb{R}^n$ is called a *periodic isolating segment* over $[0, T]$ if it is an isolating block with respect to ϕ and the following conditions hold:

(a) there exist compact subsets W^{--} and W^{++} of ∂W (called the *essential exit and entrance set*) such that

$$W^- = W^{--} \cup \{T\} \times W_T, \quad W^- \cap ([0, T) \times X) \subset W^{--},$$

$$W^+ = W^{++} \cup \{0\} \times W_0, \quad W^+ \cap ([0, T) \times X) \subset W^{++},$$

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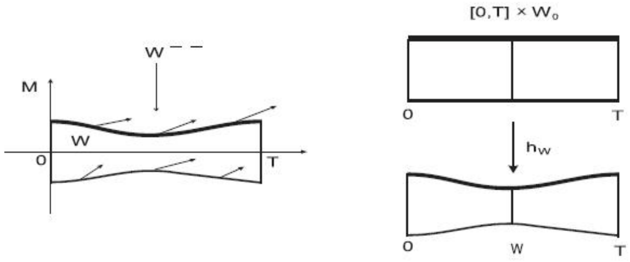


Definitions

Periodic segment

(b) there exists a homeomorphism $h\colon [0,T]\times W_0 \to W$ such that $\pi_1 \circ h = \pi_1$ and

$$h([0,T]\times W_0^{--}) = W^{--}, \quad h([0,T]\times W_0^{++}) = W^{++},$$

(c) W, W^{--}, W^{++} are T -periodic and W_0, W_0^{--}, W_0^{++} are ENR's.

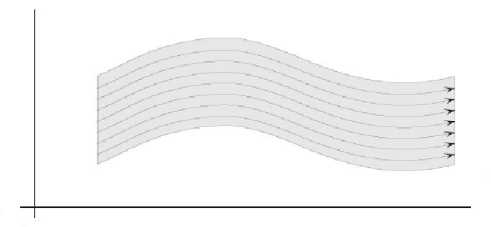
Definitions

A *monodromy homeomorphism*

$$m\colon (W_0, W_0^{--}) \to (W_0, W_0^{--})$$

is defined by

$$m(x) = \pi_2 h(T, \pi_2 h^{-1}(0, x)).$$



$$\mu_W := H(m)\colon H(W_0, W_0^{--}; \mathbb{Q}) \to H(W_0, W_0^{--}; \mathbb{Q})$$

Definitions

Since W_0, W_0^{--} are compact ENR's $\Rightarrow H(W_0, W_0^{--})$ is of finite type and *the Lefschetz number*

$$L(\mu_W) = \sum_{n \geq 0} (-1)^n \operatorname{trace} H_n(m),$$

is correctly defined. Let W be a periodic isolating segment over $[0, T]$. We say that W is *trivial* if

$$W = [0, T] \times W_0, \quad W^{--} = [0, T] \times W_0^{--}, \quad W^{++} = [0, T] \times W_0^{++}.$$

In particular, $h = \mathrm{id}_{[0,T] \times W_0}$ and consequently $L(\mu_W)$ is the Euler-Poincaré characteristic $\chi(W_0, W_0^{--}) = \chi(W_0) - \chi(W_0^{--})$.

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Application

Theorem (Srzednicki,1992)

Let W be an isolating periodic segment over $[0,T]$. Then the set

$$K_W = \{x \in W_0 : \Phi_{(0,T)}(x) = x,\ \Phi_{(0,t)}(x) \in W_t\ \forall t \in [0,T]\}$$

is an isolated set of fixed points of the Poincaré map $\Phi_{(0,T)}$ and

$$\operatorname{ind}(\Phi_{(0,T)}, K_W) = L(\mu_W),$$

where $\operatorname{ind}(\Phi_{(0,T)}, K_W)$ is a fixed point index.
In particular, when $L(\mu_W) \neq 0$ then $K_W \neq \emptyset$, so there exists $x \in W_0$ such that the solution of the problem $x' = f(t,x)$; $x(0) = x_0$ is periodic.

Examples

Theorem

Assume that $A : \mathbb{R}^n \to \mathbb{R}^n$ is linear and $\Re\sigma(A) \neq 0$. If $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is smooth, T-periodic, bounded then $x' = Ax + f(t,x)$ has a T-periodic solution.

Theorem

The equation $z' = z + e^{it}\bar{z}^n$, $n \geq 2$ has a non-trivial 2π-periodic solution.

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Let $h : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ be continuous, $h(\cdot, z)$ — T-periodic, $h(t,z)/|z|^n \to 0$ if $|z| \to \infty$ uniformly with respect to t. If $n \geq 1$ then there exists a T-periodic solution of $\dot{}\ z' = \bar{z}^n + h(t,z)$.

Segment for n=2

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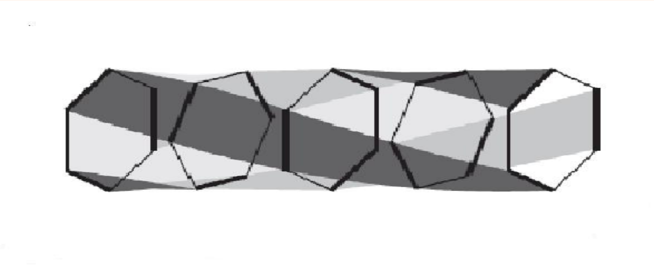

Definitions

Guiding function

We say that a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is a *guiding function for the vector field f* if there is $R > 0$ such that

$$\nabla V(x) \cdot f(t, x) > 0 \tag{2}$$

for $\|x\| \geq R$, $t \in \mathbb{R}$.

Index of a guiding function

We define *the index of V* by

$$\operatorname{Ind} V = \deg(0, \nabla V, D_R), \tag{3}$$

where $D_R = \{x \in \mathbb{R} : \|x\| < R\}$ and $\deg$ is a Brouwer degree.

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Definitions

Let $V_1, \ldots, V_k : \mathbb{R}^n \to \mathbb{R}$ ($k \geq 1$) be a guiding functions for the vector field f that satisfy (2) with $R > 0$.

A complete set of guiding functions

We say that $V_1, \ldots, V_k$ is *a complete set of guiding functions* if

$$\lim_{\|x\| \to \infty} |V_1(x)| + \ldots + |V_k(x)| = \infty. \tag{4}$$

It follows by the condition (2) and the homotopy property of the Brouwer degree that

$$\mathrm{Ind}\, V_i = \mathrm{Ind}\, V_1, \quad i \in \{1, \ldots, k\}.$$

Application

Theorem (Krasnosielski, 1975)

Assume that $V_1, \dots, V_k$ is a complete set of guiding functions for the vector field f with $\mathrm{ind}\, V_1 \neq 0$. Then the equation $x' = f(t,x)$ has at least one T-periodic solution. Moreover, there exists $R > 0$ such that $[0,T] \times D_R$ contains all T-periodic solutions of the equation $x' = f(t,x)$.

Main theorem

Theorem (GK,2007)

Assume that $V_1, \dots, V_k : \mathbb{R}^n \to \mathbb{R}$ *is a complete set of the guiding functions for the vector field* f*. Then there exists a trivial periodic isolating segment* W *over* $[0, T]$ *for the equation* $x' = f(t, x)$ *such that*

$$L(\mu_W) = (-1)^n \mathrm{Ind}\, V_1.$$

Main theorem - Proof

We put

$$M = \max_{i\in\{1,\dots,k\}} \max_{\|x\|\le R} |V_i(x)|,$$

where $R > 0$ is such that (2) and (6) holds. Then for $c > M$ we define

$$B = \{x \in \mathbb{R}^n : \forall i \in \{1, \dots k\} \; -c \le V_i(x) \le c\}.$$

$W = [0, T] \times B$ will be our isolating segment.

Main theorem - Proof

The exit and entrance sets of B are:

$$B^- = \{x \in \mathbb{R}^n : \exists i \in \{1, \dots k\}\ V_i(x) = c\},$$

$$B^+ = \{x \in \mathbb{R}^n : \exists i \in \{1, \dots k\}\ V_i(x) = -c\}.$$

Thus

$$W^{--} = [0, T] \times B^-, \quad W^{++} = [0, T] \times B^+.$$

Main theorem - Proof - 2nd part

We show that $\Lambda(\mu_W) = (-1)^n \mathrm{ind}\, V_1$. By the condition (2) and the homotopy property of the Brouwer degree we get that

$$\mathrm{ind}\, V_1 = \deg\left(0, f(0,\cdot), D_R\right).$$

On the other hand, W is trivial periodic isolating segment, hence

$$L(\mu_W) = \chi(B) - \chi(B^-).$$

Main theorem - Proof - 2nd part

We will need the following lemma

Theorem (Mrozek, Srzednicki)

If B is an isolating block for f such that B, B^- are ENRs, then $\deg(0, f, \operatorname{int} B) = (-1)^n(\chi(B) - \chi(B^-))$.

By that result and the excision property of the Brouwer degree we have:

$$\chi(B) - \chi(B^-) = (-1)^n \deg(0, f(0, \cdot), \operatorname{int} B) = (-1)^n \deg(0, f(0, \cdot), D_R),$$

hence the result follows. □

The converse

2π-periodic planar system

$$z' = e^{it}\overline{z}^n, \quad z \in \mathbb{C}$$

posses a 2π-periodic isolating segment W with $L(\mu_W) = 1$.
The vector field $f(z,t) = e^{it}\overline{z}^n$ does not have a guiding function at all, since for fixed z the vector $f(z,t)$ makes a full rotation as t varies from 0 to 2π.

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Thank You for Your attention.