Non-Smooth Invariant Tori for Analytic Hamiltonians and Computer-Assisted Proofs

or

(A) **Computer-assisted analysis in infinite dimensions**
(B) **A renormalization fixed point for area-preserving maps (with G. Arioli)**
(C) **Non-smooth invariant tori for analytic Hamiltonians**

Main issue: Discretization

- Need to work with sets in function spaces.
- Sets in a space X are described by finitely many constraints. Need compactness.
- Constraints: At each step of the proof, the relevant information has to be available. Not less! Not much more!

Trivial: $X = \mathbb{R}$ or $\mathbb{C}$ (constant functions)

Let $\mathbb{U}_{\mathbb{C}} = \{z \in \mathbb{C} : |z| \leq 1\}$.

`Rep`: Set of representable numbers, containing 0 and 1.
In our proofs: IEEE 754 extended precision (80 bit) floats.
Exact comparisons. Operations $\{+, -, *, /\}$ exact or rounded up .
Raise `Error` if overflow etc.

`Radius`: Set of nonnegative `Rep`.

`Disk`: $\mathcal{B}(\mathtt{S}) = \mathtt{S.C} + (\mathtt{S.R})\mathbb{U}_{\mathbb{C}}$, defined by a pair $\mathtt{S} = (\mathtt{S.C}, \mathtt{S.R}) \in$ `Rep` $\times$ `Radius`.

```
S := S1-S2 :
    S.C := S1.C-S2.C;
    S.R := (S1.R+S2.R)+(S.C+(S2.C-S1.C));
```

Property: if $s_1 \in \mathcal{B}(\mathtt{S1})$ and $s_2 \in \mathcal{B}(\mathtt{S2})$, then $s_1 - s_2 \in \mathcal{B}(\mathtt{S})$.

Other elementary operations are implemented similarly.

Other spaces X depend on the problem.

Feigenbaum equation. Solve $\mathcal{N}(f) = f$ for real analytic functions f on $[-1,1]$, with $f(0) = 1$.

$$\mathcal{N}(f) = \lambda^{-1} f(f(\lambda\,\boldsymbol{.}\,))\,, \qquad \lambda = f(1)\,.$$

↖ $|\lambda| < 1$ gives compactness

Computer-assisted methods pioneered ($\approx$ 1980) by O.E. Lanford III .

Dyson's hierarchical model. Fixed point $h > 0$ for $\mathcal{R}$,

$$\mathcal{R}(h) = \int e^{-s^2} h(\alpha t + s) h(\alpha t - s)\, ds\,, \qquad \alpha = 2^{-1/6}\,.$$

Joint work (1986–1996) with P. Wittwer .

Orbits for parabolic equations

$$\dot{u} + Lu + H(u, \nabla u, \ldots) = 0\,, \qquad L = (-\Delta)^m\,.$$

Solve $\widetilde{u} = u$,

$$\widetilde{u}(t) = e^{-tL} u_0 + \int_0^t e^{-(t-s)L} H(u, \nabla u, \ldots)(s)\, ds\,.$$

Joint work with G. Arioli (see later).

MacKay RG for pairs of area-preserving maps. Solve $\widetilde{F} = F$ and $\widetilde{G} = G$,

$$\widetilde{F} = \Lambda^{-1} G \Lambda \,, \qquad \widetilde{G} = \Lambda^{-1} F G \Lambda \,, \qquad \Lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \alpha \end{bmatrix},$$

Joint work with G. Arioli (mp_arc 09-67).
Earlier partial results (1997) by A. Stirnemann .

RG for Hamiltonians on $\mathbb{T}^2 \times \mathbb{R}^2$. Let $T = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. Then

$$\mathcal{R}(H) = \frac{\vartheta}{\mu} H \circ \mathcal{T}_\mu \circ \mathcal{U}_H \,, \qquad \mathcal{T}_\mu(q,p) = \bigl(Tq, \mu T^{-1} p\bigr) \,.$$

The definition of $\mathcal{U}_H$ involves implicit equations of Nash-Moser type (see later).

For every space X involved, choose a collection $\Sigma(X)$ of representable subsets. Normally,

$$\Sigma(X_1 \times X_2) = \bigl\{ S_1 \times S_2 : \; S_j \in \Sigma(X_j) \bigr\} \,.$$

A **bound** on $f : X \to Y$ is a function

$$F : \Sigma(X) \to \Sigma(Y) \cup \{\texttt{Error}\} \,,$$

such that for $S \in \Sigma(X)$,

$$x \in S \implies f(x) \in F(S) \quad \text{or} \quad F(S) = \texttt{Error} \,.$$

Convenient spaces in all of these examples: **Banach algebra $\mathcal{A}$ with unit 1**.
Let $\mathbb{U}_{\mathcal{A}} = \{a \in \mathcal{A} : \|a\| \le 1\}$. The basic object is a

`Ball`: $\mathcal{B}(\mathtt{S}) = [\mathtt{S.C} + (\mathtt{S.R})\mathbb{U}_{\mathbb{C}}]\mathbf{1} + (\mathtt{S.B})\mathbb{U}_{\mathcal{A}}$, defined by $\mathtt{S} = (\mathtt{S.C}, \mathtt{S.R}, \mathtt{S.B}) \in \mathtt{Rep} \times \mathtt{Radius}^2$.

Consider e.g. the space $\mathcal{A}_R$ of analytic functions on $\Omega = \{z \in \mathbb{C} : |z| < R\}$,

$$f(z) = \sum_n f_n z^n \,, \qquad \|f\| \stackrel{\text{def}}{=} \sum_n |f_n| R^n < \infty \,.$$

Let $Z(z) = z$.

`Taylor`: a set $\mathcal{B}(\mathtt{F}) = \sum_{K=1}^{D} \mathcal{B}(\mathtt{F.C(K)}) Z^K$, defined by a pair `F=(F.R,F.C)`, with

```
F.R: Rep;
F.C: array [0..D] of Ball;
```

Define $\Sigma(\mathcal{A}_R)$ to be the collection of all these `Taylor` sets.

```
F := F1-F2:
   F.R := Min(F1.R,F2.R);
   for K in 0..D loop F.C(K) := F1.C(K)-F2.C(K); end loop;
```

In the same spirit, we can define bounds `+`, `*`, `Norm`, `Der`, `Comp`, ...
Useful as well: procedures like `NumInv`, ... that compute approximate solutions.

The result: a collection of “basic” bounds (package `Taylors`).

More generally,

`Taylor` coefficients `F.C` can be a generic type `Scalar` that can be instantiated e.g. with

- `Ball` to get the functions described above.
- `Numeric` (some floating point type) to get purely numerical procedures.
- an instantiation of `Taylor, Laurent, Fourier, ...` for functions of several variables.

In particular, at the next level, we can simultaneously use different instantiations, as in the following example:

```
package NTaylors is new Taylors (Scalar => Numeric);
package STaylors is new Taylors (Scalar => Scalar);

subtype NTaylor is NTaylors.Taylor;
subtype STaylor is STaylors.Taylor;

function Approx(F: STaylor) return NTaylor;
function Convert(F: NTaylor) return STaylor;
```

This is useful for a-posteriori estimates:

Solve **implicit equations** first numerically, then estimate the error.

Example: Given $F \in \mathcal{A}$, and $G \approx 1/F$, estimate the norm of $g_* = G - 1/F$.

g_* is the fixed point of the map $\mathcal{M}(g) = (G + 2g)[1 - FG] - Fg^2$.

Proposition. *Assume* $\|1 - FG\| \leq \delta$ *and* $\|G\|\delta \leq \varepsilon$. *Let* $\mathtt{R} = \frac{9}{8}\varepsilon$. *If*

$$2\delta + 2\mathtt{R}\|F\| < \tfrac{1}{9}\,, \qquad (✡)$$

then $\|g_*\| \leq \mathtt{R}$.

Note: (✡) is a bound on $D\mathcal{M}(g)$ on the ball $\|g\| \leq \mathtt{R}$.

```
function Inv(F: STaylor) return STaylor is
  Del, Eps, R: Scalar;
  G: STaylor;
begin
  Relax_Checks;
  G := Convert(NumInv(Approx(F)));
  Restore_Checks;
  .
  .     compute R, check inequality (✡)
  .
  AddBall(R,G);
  return G;
end Inv;
```

Use a similar strategy at higher levels, e.g. for

$$\mathcal{N}(f) = \lambda^{-1} f(f(\lambda\,.))\,, \qquad \lambda = f(1)\,.$$

Quasi-Newton map $\mathcal{M}$ near an approximate fixed point f_0:

$$\mathcal{M}(\phi) = \phi + \mathcal{N}(f_0 + M\phi) - (f_0 + M\phi)\,,$$
$$D\mathcal{M}(\phi) = \mathbb{I} - \big[\mathbb{I} - D\mathcal{N}(f_0 + M\phi)\big]M\,.$$

M is an approximate inverse of $[\ldots]$. Typically, $M = \mathrm{I} + A$ with A a finite rank "matrix".

Notice: what makes $\mathcal{M}$ a contraction are cancelations .
$\mathcal{M}$ will not map any set $\mathcal{B}(\mathtt{F})$ into itself. Need to bound $\mathbb{L} = D\mathcal{M}(\phi)$ the "hard way".

Fortunately: In a space $\mathcal{A}$ where

$$\phi = \sum_n c_n Z_n\,, \qquad \|\phi\| = \sum_n |c_n|\,,$$

we simply have

$$\|\mathbb{L}\| = \sup_n \|\mathbb{L}Z_n\|\,.$$

In the cases considered, it suffices to estimate $\|\mathbb{L}Z_n\|$ explicitly for $n \le N$.
That is, there exists a set $\mathcal{B}$ in $\Sigma(\mathcal{A})$, containing Z_n for all $n > N$, such that

$$\|\mathbb{L}\mathcal{B}\| \le K < 1\,.$$

Before starting to program, may need to reformulate the problem.

Example: Hierarchical model

$$\mathcal{R}(h) = \int e^{-s^2} h(\alpha t + s) h(\alpha t - s)\, ds\,, \qquad \alpha = 2^{-1/6}\,.$$

$$\downarrow \quad \text{Laplace transform and ...}$$

$$\mathcal{N}(f)(t) = \int e^{-s^2} f(s + \beta t)^2\, ds\,, \qquad \beta = 2^{-5/6}\,.$$

$\mathcal{N}$ can be controlled in a space of entire functions.

Example: Steady states ($\dot{u} = 0$) for

$$\dot{u} + (-\Delta)^m u + H(u, \nabla u, \ldots) = 0\,.$$

$$\downarrow \quad \text{gives compactness}$$

$$\mathcal{N}(u) = -\,(-\Delta)^{-m} H(u, \nabla u, \ldots)\,.$$

After this: Choice of functions spaces, domains, parameters, . . .

MacKay RG for pairs $P = (F, G)$ of (commuting) area-preserving maps,

$$\mathfrak{R}(P) = (\widetilde{F}, \widetilde{G}), \quad \widetilde{F} = \Lambda^{-1} G \Lambda, \quad \widetilde{G} = \Lambda^{-1} F G \Lambda, \quad \Lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \alpha \end{bmatrix}.$$

where λ and α are determined by the normalization condition $\widetilde{G}(0,0) = (-1,-1)$.

Desirable, but not preserved by $\mathfrak{R}$: the reversibility property

$$G = S G^{-1} S, \qquad S(x,z) = (-x, z).$$

Notice: for a reversible **fixed point** of $\mathfrak{R}$, one has

$$\Lambda^{-1} G F \Lambda = \Lambda^{-1} S G^{-1} F^{-1} S \Lambda = S G^{-1} S = G = \Lambda^{-1} F G \Lambda,$$

implying that F and G commute: $GF = FG$.

First reformulation: Solve $\mathfrak{N}(G) = G$, where

$$\mathfrak{N}(G) = \Lambda^{-2} G F G \Lambda^{2}, \qquad F = \Lambda^{-1} G \Lambda,$$

on a space of reversible maps G (preserved by $\mathfrak{N}$).

But we need an additional condition . . .

Write

$$\mathfrak{N}(G) = \Lambda^{-1} F \widetilde{G} \Lambda \,, \qquad \widetilde{G} = \Lambda^{-1} F G \Lambda \,, \qquad F = \Lambda^{-1} G \Lambda \,.$$

Condition: $\widetilde{G} = G$.

This condition is satified almost automagically!
Namely, assume that $\mathfrak{N}(G) = G$. Then the map $J = G^{-1}\widetilde{G}$ satisfies

$$J(0,0) = (0,0) \,, \qquad \Lambda^{-2} J \Lambda^{2} = J \,.$$

For an analytic J, this implies $J = \pm \mathrm{I}$. So we only need to exclude $J = -\mathrm{I}$ to get $\widetilde{G} = G$.

Second reformulation uses generating function g,

$$G(x,z) = (y,w) \,, \qquad z = -g_1(x,y) \,, \qquad w = g_2(x,y) \,,$$

where $g_j = \partial_j g$. Consider the scaling operator $h \mapsto h'$, defined by

$$h'(x,y) = (\lambda\alpha)^{-1} h(\lambda x, \lambda y) \,.$$

Then the generating function of $\mathfrak{N}(G)$ is given by

$$\mathcal{N}(g) = h'' \,,$$
$$h(x,y) = g(x,\mathcal{V}) + g'(\mathcal{V},\mathcal{W}) + g(\mathcal{W},y) \,,$$

with the “midpoint functions” $\mathcal{V} = \mathcal{V}(x,y)$ and $\mathcal{W}$ defined by

$$g_2(x,\mathcal{V}) + g_1'(\mathcal{V},\mathcal{W}) = 0 \,, \qquad \mathcal{W}(x,y) = -\mathcal{V}(-y,-x) \,.$$

So $\mathfrak{R} \longrightarrow \mathfrak{N} \longrightarrow \mathcal{N}$, and later to $\mathcal{N} \longrightarrow \mathcal{M}$. But why study $\mathfrak{R}$ in the first place?

Motivation: Observation of critical phenomena (self-similarity, universal scaling) during the breakup of invariant circles with rotation number $\vartheta^{-1} = \vartheta - 1$,

$$\vartheta = \frac{\sqrt{5}+1}{2} = 1.618033988\ldots \qquad \text{(golden mean)},$$

in many families $\beta \mapsto G_\beta$, including the standard map

$$G_\beta(x,z) = (x+w,w)\,, \qquad w = z - \beta\sin(2\pi x)\,.$$

$\beta = 0$: smooth golden circle at $w = \vartheta^{-1}$.
$\beta > 0$ *small*: smooth golden circle persists, by KAM theory.

$\beta = \beta_*$: golden circle turns non-smooth and breaks up.
In suitable coordinates, the golden circle and nearby orbits are asymptotically invariant under a scaling $\Lambda = \left[\begin{smallmatrix}\lambda_* & 0\\ 0 & \alpha_*\end{smallmatrix}\right]$,

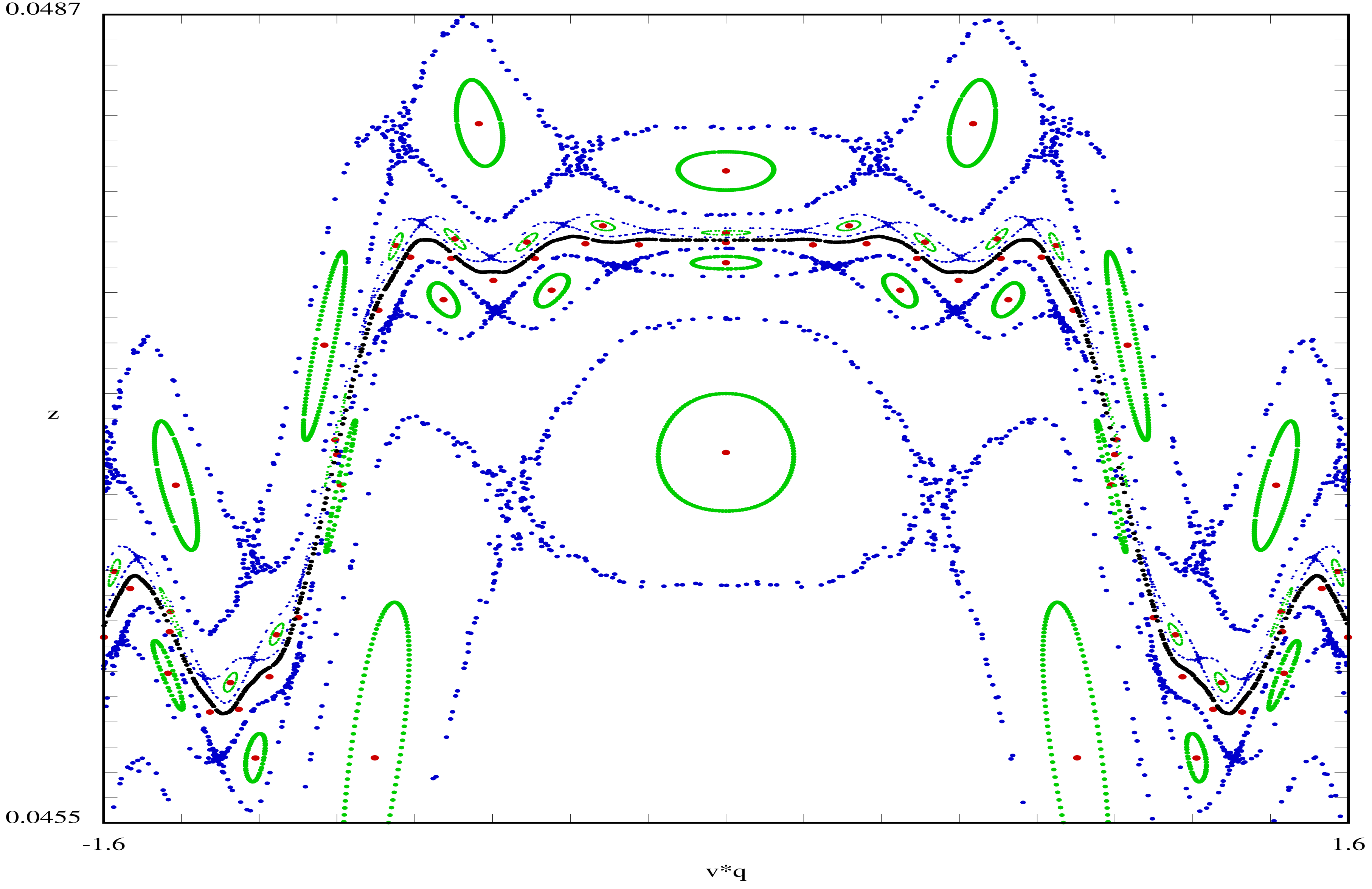
$$\lambda_* = -0.7067956691\ldots\,, \qquad \alpha_* = -0.3260633966\ldots\,, \tag{$\circledast$}$$

with λ_* and μ_* universal (independent of the family). This could be explained if

$$\boldsymbol{\mathfrak{R}^n(P_{\beta_*}) \to P_*\,, \qquad \mathfrak{R}(P_*) = P_*\,,}$$

where $P_\beta = (F_\beta, G_\beta)$ and $F_\beta(x,z) = (x-1,z)$.

Orbits for a **critical Hamiltonian**, from [J. Abad, H.K., P. Wittwer '98]



Theorem [G. Arioli, H.K, '09]. $\mathfrak{R}$ *has a fixed point* $P_* = (F_*, G_*)$. *The associated scaling* Λ_* *satisfies* (✻). *The maps* F_* *and* G_* *are analytic, area-preserving, reversible, and they commute.*

The proof follows the strategy outlined before: Reconstruct P_* from a fixed point g_* for $\mathcal{N}$. (This is done "pointwise" on small rectangles covering the domains of F_* and G_*. Proving that $J = \mathrm{I}$ requires some extra work.) The fixed point g_* is obtained by proving that

$$\mathcal{M}(\phi) = \phi + \mathcal{N}(g_0 + M\phi) - (g_0 + M\phi)\,,$$

is a contraction near $\phi = 0$, where g_0 is an approximate fixed point of $\mathcal{N}$.

The two **main issues** in this problem were:

- Finding an approach that yields reversible and commuting maps (the transformation $\mathfrak{N}$, and the argument involving J).
- Finding an appropriate function space and domains!

$$f = \sum_{m,n} a_{m,n} \mathbf{u}^m \mathbf{v}^n + \mathbf{t} \sum_{m,n} b_{m,n} \mathbf{u}^m \mathbf{v}^n$$

($b_{m,n} = 0$ if f is the generating function of a reversible map), where

$$\mathbf{u} = \left[\mathbf{t}^2 - t_0^2\right] + b\mathbf{v}\,, \qquad \mathbf{v} = \mathbf{s} - s_0\,,$$

and

$$\mathbf{t}(x,y) = x + y\,, \qquad \mathbf{s}(x,y) = x - y\,.$$

A "corresponding" **renormalization of Hamiltonians** on $\mathbb{T}^2 \times \mathbb{R}^2$ is of the form

$$\mathcal{R}(H) \equiv H \circ \mathcal{T}_1 \quad \text{(modulo "equivalences")},$$

where $\mathcal{T}_\mu(q,p) = \big(Tq, \mu T^{-1}p\big)$ and $T = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. Specifically, $\mathcal{R} = \mathcal{N} \circ \mathcal{L} \circ \mathcal{S}$, where

$$\mathcal{S}(H)(q,p) = \frac{\mu_0}{\mu} H\Big(q, \frac{\mu}{\mu_0} p\Big), \qquad \text{(trivial)}$$

$$\mathcal{L}H = \frac{\vartheta}{\mu_0} H \circ \mathcal{T}_{\mu_0} \circ U_0, \qquad \text{(linear)}$$

$$\mathcal{N}(K) = K \circ U_1 \circ U_2 \circ U_3 \circ \ldots \qquad \text{(close to the identity)}$$

U_0: A fixed change of coordinates, making $K = \mathcal{L}\mathcal{S}(H)$ almost "resonant".
$U_1, U_2, \ldots$: Small coordinate changes (depending on K), eliminating remaining nonresonant part.
A $\mathcal{N}$ash-Moser type procedure. $U_1 - \mathrm{I} = \text{small}$, $U_2 - \mathrm{I} = \mathcal{O}(\text{small}^2)$, ...

Task: Find the critical fixed point of $\mathcal{R}$. It suffices to consider "special" Hamiltonians

$$H(q,p) = \omega \cdot p + \sum_{\nu,k} H_{\nu,k} \cos(\nu \cdot q)(\Omega \cdot p)^k,$$

where ω and Ω are the expanding and contracting eigenvectors, respectively, of T.

Difficulty: $\mathcal{N}$ is quite nontrivial. Estimating the corresponding $\mathcal{M}$ "as usual" is prohibitive.

$$K \circ U_1 = \underbrace{K + \{K, \psi_K\}} + \mathcal{O}(\text{small}^2)\,.$$

Way out: Use that $\mathcal{N}(K) \approx \mathcal{N}_1(K)$, define $\mathcal{R}_1 = \mathcal{N}_1 \circ \mathcal{L} \circ \mathcal{S}$, and decompose $\mathcal{M}$ as in

$$\begin{aligned} \mathcal{M}(h) = &\big(\mathcal{R}_1(H_1) - H_1\big)\,, && \text{(tiny)} \\ &+ \big(\mathbb{I} - [\mathbb{I} - D\mathcal{R}_1(H_1)]M\big)h && \text{(linear contraction)} \\ &+ Q(h)\,. && \text{(order small}^2\text{)} \end{aligned}$$

where H_1 is an approximate fixed point for $\mathcal{R}$. For the derivative of Q use a Cauchy estimate.

Theorem [H.K. '04]. *Existence of an analytic fixed point H_* for $\mathcal{R}$, with*

$$\mu_* = 0.230460196\ldots$$

Note: $\mu_* = \lambda_* \alpha_*$

To get the proof to run in an acceptable amount of time (110 days CPU time)

- program optimized parameters in first 5 Nash-Moser steps.
- order of operations chosen carefully.
- used variable "degrees".
- parallelized composition estimates.
- started using many "higher order" terms.
- started avoiding FPU mode switching.

By a golden **invariant torus** for H we mean the map $\Gamma : \mathbb{T}^2 \times \{0\} \to \mathbb{T}^2 \times \mathbb{R}^2$ that semiconjugates the flow Φ^t for H to the linear flow $\Psi^t(q,0) = (q + t\omega, 0)$, that is,

$$\Phi^t \circ \Gamma = \Gamma \circ \Psi^t \,.$$

Then a golden invariant torus Γ' for the renormalized Hamiltonian

$$H' = \tfrac{\vartheta}{\mu} H \circ \mathcal{T}_\mu \circ \mathcal{U} \,, \qquad \mathcal{U} = U_0 \circ U_1 \circ U_2 \circ \ldots \,,$$

yields a golden invariant torus for H,

$$\Gamma = \mathcal{T}_\mu \circ \mathcal{U} \circ \Gamma' \circ \mathcal{T}_\mu^{-1} \,.$$

In particular, the golden invariant torus for $H = H_*$ is a fixed point of the map $\mathfrak{M}$,

$$\mathfrak{M}(F) = \mathcal{T}_\mu \circ \mathcal{U} \circ F \circ \mathcal{T}_\mu^{-1} \,.$$

Useful: $\mathcal{U}$ and Γ are trivial in the expanding direction of $\mathcal{T}_\mu$, due to the "special" form of our Hamiltonians. So we can expect $\mathfrak{M}$ to be a contraction on a space of low regularity functions, say

$$\Gamma(q,0) = (q,0) + \sum_\nu \gamma_\nu \sin(\nu \cdot q) \,,$$

with finite norm

$$\|\gamma\| = \sum_\nu |\gamma_\nu| e^{\varepsilon|\omega\cdot\nu|} \big(1 + |\Omega \cdot \nu|\big)^r \,,$$

if $\varepsilon, r > 0$ are small.

Theorem [H.K. '07]. *If* H *is near* H_* *and* $\mathcal{R}^n(H) \to H_*$ *exponentially, then* H *has a non-differentiable golden invariant torus. The (asymptotic) scaling is given by*

$$\begin{aligned} \vartheta &= 1.618033988\ldots , \\ \lambda_* &= -0.70695\ldots , \\ \alpha_* &= -0.326063\ldots , \\ \vartheta^{-1}\mu_* &= 0.142432234\ldots . \end{aligned}$$

Remarks

- The scaling constants are the eigenvalues of $V = \mathcal{T}_\mu \circ \mathcal{U}$ at a fixed point of this map V.
- The computer-assisted part of the proof "only" deals with $H = H_*$. It consists in
- estimating $\mathcal{U} = U_0 \circ U_1 \circ U_2 \circ \ldots$, and then
- proving that $\mathfrak{M}$ contracts near an approximate fixed point Γ_0.
- Fortunately, the domain and contraction of $\mathfrak{M}$ are very generous, so the lack of precision (slow convergence of the series for the function Γ) is not a problem.
- The critical torus looks quite nice ...

The golden invariant torus for H_*