Computer-assisted methods for the study of dissipative PDEs

(**with G. Arioli**)

($A1$) **Generalities, the Kuramoto-Sivashinsky (KS) equation**
($A2$) **The bifurcation diagram for the KS equation**
($A3$) **Sketch of the proof**

($B1$) **A periodic orbit for the KS equation**
($B2$) **Error estimates**
($B3$) **Boxes**

The methods apply in principle to general **dissipative evolution equations**

$$\dot{u} + (-\Delta)^m u + H(u, \nabla u, \ldots) = 0\,,$$

for sufficiently simple domains and analytic nonlinearities H.

Start with **stationary solutions** $\dot{u} = 0$ and rewrite the resulting equation as

$$F(u) = u\,, \quad \text{where} \quad F(u) = -(-\Delta)^{-m} H(u, \nabla u, \ldots)\,,$$

The idea is to exploit the compactness of $(-\Delta)^{-m}$ to obtain good finite dim approximations.

From now on, restrict to the one-dimensional **Kuramoto-Sivashinsky** (KS) equation $\dot{u} + 4u'''' + \alpha(u'' + 2uu') = 0$ on $[0, \pi]$, with Dirichlet boundary conditions. In this case, solve

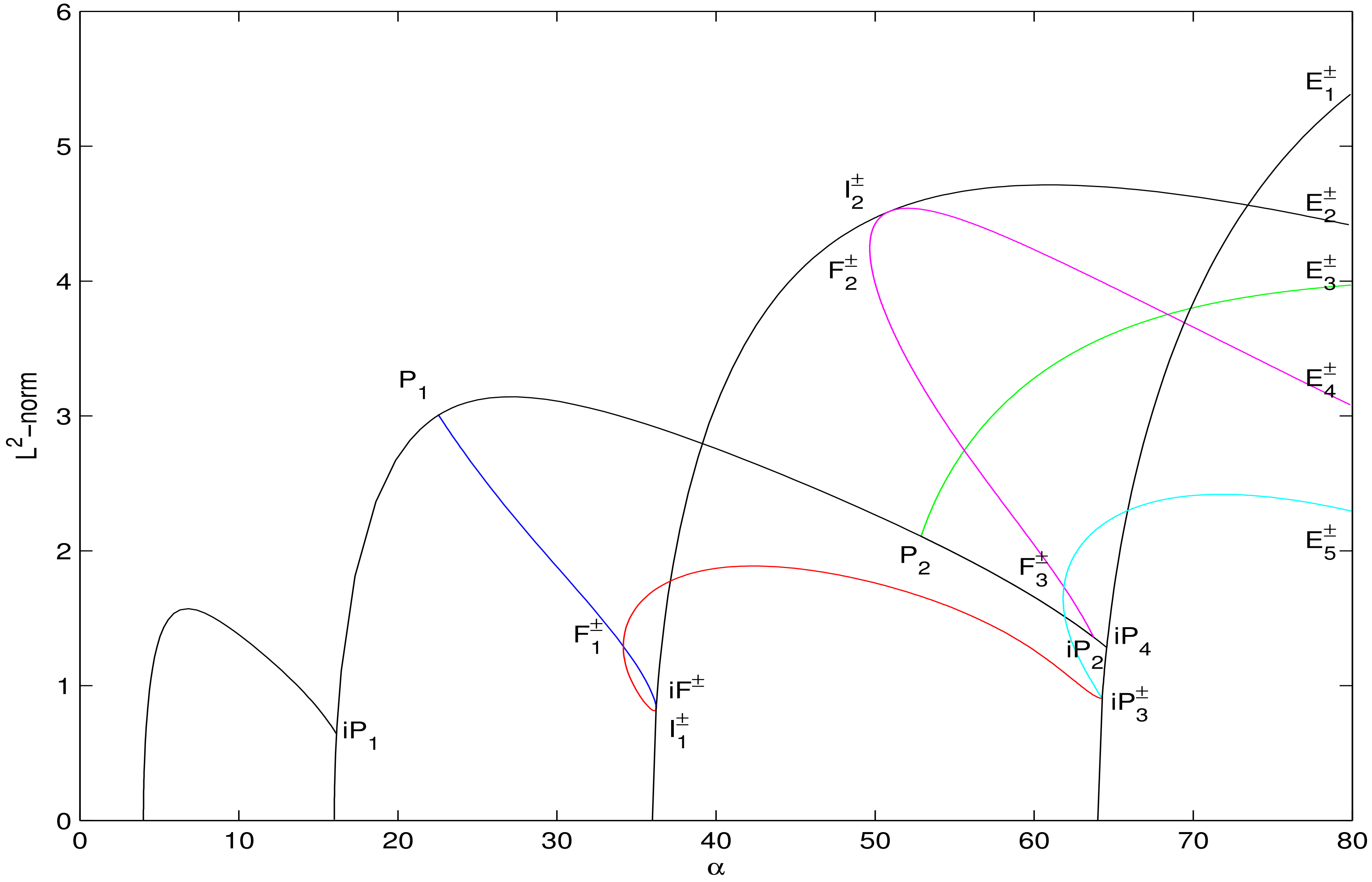
$$\boldsymbol{F_\alpha(u) = u\,,} \quad \text{where} \quad \boldsymbol{F_\alpha(u) = -\tfrac{\alpha}{4}\partial^{-2}u + \tfrac{\alpha}{4}\partial^{-3}(u^2)\,,}$$

with u an odd 2π-periodic function on $\mathbb{R}$.

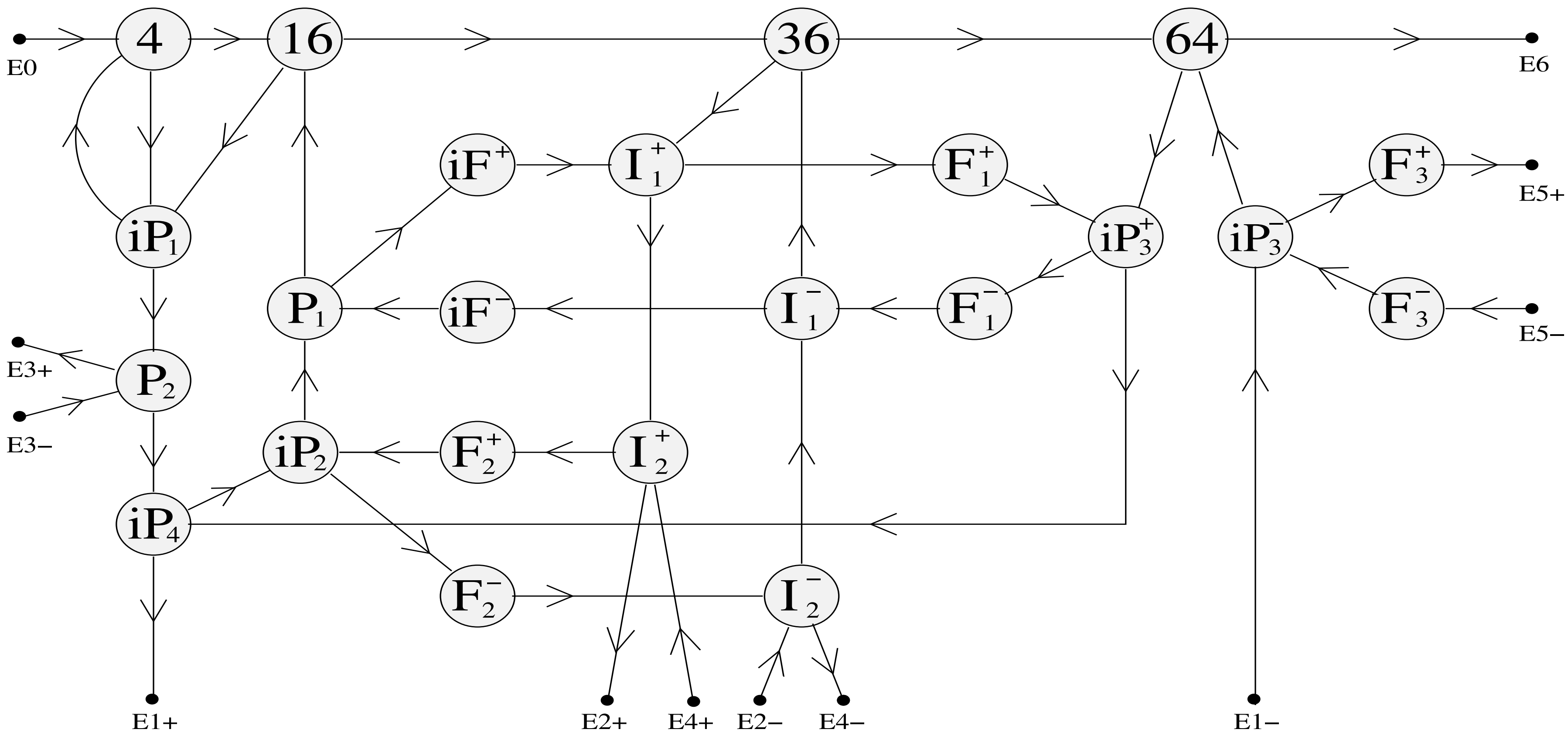
A trivial solution is $u = 0$, for any value of α. It defines a line in the space of pairs (α, u) satisfying $F_\alpha(u) = u$. Other solutions **bifurcate** off this line at $\alpha = 4k^2$, with k a positive integer. The resulting solution curves bifurcate again . . .

Determining the **bifurcation diagram** is simplified by the fact that many bifurcations involve the breaking of some symmetry.

Bifurcation diagram (L^2 norm versus α) for the **Kuramoto-Sivashinsky** equation



Theorem. [G. Arioli, H.K.] *For* $0 \le \alpha \le 80$*, the stationary KS equation equation exhibits eleven pitchfork bifurcations (4, 16, 36, 64,* P_1*,* P_2*,* iP_1*,* iP_2*,* $iP_3^{\pm}$*,* iP_4*), four intersection bifurcations (*$I_1^{\pm}$*,* $I_2^{\pm}$*), and eight fold bifurcations (*$F^{\pm}$*,* $F_1^{\pm}$*,* $F_2^{\pm}$*,* $F_3^{\pm}$*), connected by 44 smooth solution curves, as depicted below. These curves undergo no other bifurcations for* $0 \le \alpha \le 80$*.*



Other results include bounds on the values of α for each bifurcation point, as well as the dimension of the unstable manifold and the L^2 norm of 30 selected solutions.

To solve $F_\alpha(u) = u$ for fixed α, consider a quasi-Newton map

$$\mathcal{M}(u) = F_\alpha(u) - M\big[F_\alpha(u) - u\big]\,,$$

with u_0 an approximate solution, and $M \approx \mathbb{I} - [\mathbb{I} - DF_\alpha(u_0)]^{-1}$ a finite rank "matrix".
For this and similar maps (described later), prove that

$$\|\mathcal{M}_\alpha(u_0) - u_0\| < \varepsilon\,, \qquad \|D\mathcal{M}_\alpha(u)\| < K\,, \qquad \varepsilon + Kr < r\,,$$

whenever $\|u - u_0\| \leq r$. This is done in the space $\mathcal{A}^o$ defined below, with $\rho = \frac{1}{6}$.
Then apply the contraction mapping principle to get a unique solution in the ball $\|u - u_0\| \leq r$.

To get pieces of solution branches, get uniform bounds for α is small intervals.
Then patch the (overlapping) intervals together.
This works as long as the spectrum of $DF_\alpha(u)$ is bounded away from 1.

Given $\rho > 0$, and a nonnegative integer K, define

$\mathcal{A}^o_K$: Space of odd 2π-**periodic real analytic functions** on the strip $|\mathrm{Im}\,x| < \rho$,

$$u(x) = \sum_{k\geq K} u_k \sin(kx)\,, \qquad \|u\| \overset{\text{def}}{=} \sum_{k\geq K} |u_k| e^{\rho k} < \infty\,.$$

$\mathcal{A}^e_K$: Analogous space of even 2π-periodic functions.

Yes, the proof is computer-assisted ...

`Rep`: Representable numbers, containing 0 and 1. Operations $\{+,-,*,/\}$ exact or rounded up .

`Ball`: $\mathtt{S} = (\mathtt{S.C}, \mathtt{S.R}) \in \mathtt{Rep}^2$.
represents intervals in $\mathbb{R}$, or balls in a Banach space $\mathcal{A}$

$$\mathcal{B}(\mathtt{S}) = (\mathtt{S.C}) + (\mathtt{S.R})\mathbb{U}_{\mathbb{R}}\,, \qquad \mathcal{B}(\mathtt{S},\mathcal{A}) = (\mathtt{S.R})\mathbb{U}_{\mathcal{A}}\,.$$

where $\mathbb{U}_{\mathcal{A}} = \{a \in \mathcal{A} : \|a\| \le 1\}$.

The **representable sets** in $\mathcal{A}_1^o$ are taken to be of the form

$$\mathcal{B}(\mathtt{F}) = \sum_{K=1}^{D} \mathcal{B}(\mathtt{F.C(K)}) \sin(\mathtt{K\,.}) + \sum_{K=1}^{2D} \mathcal{B}(\mathtt{F.E(K)}, \mathcal{A}_K^o)\,, \qquad \mathtt{F} \in \mathtt{SFourier}\,.$$

The representable sets in $\mathcal{A}_0^e$ are defined analogously. Both are associated with data of type `SFourier`, which is is an instantiation (`FCoeff` $\Rightarrow$ `Ball`) of

`Fourier`: `F=(F.T,F.C,F.E)`, with `F.T` encoding the type (even or odd, domain ρ), and

```
    F.C: array [0..D] of FCoeff;
    F.E: array [0..2*D] of FCoeff;
```

Implement **bounds** hierarchically, starting with simple and/or generic types, then for more complex types; first for basic operations, then for functions like F_α , $\mathcal{M}$, $D\mathcal{M}$, ...

Our **bifurcations** take place in two-dimensional surfaces $(\alpha, \beta) \mapsto u_{\alpha,\beta}$.

Choose $\hat{u} \approx \partial_\beta u_{\alpha,\beta}$. Let $\mathbb{P}$ be a projection in $\mathcal{A}_\rho^o$, with range $\mathbb{R}\hat{u}$. Consider the equation

$$\boldsymbol{F_{\alpha,\beta}(u) = u\,,} \quad \text{where} \quad \boldsymbol{F_{\alpha,\beta}(u) = (\mathbb{I} - \mathbb{P})F_\alpha(u) + \beta\hat{u}\,,}$$

and a solution $u = u_{\alpha,\beta}$. This solution is a fixed point of F_α if and only if $\mathbb{P}F_\alpha(u) = \beta\hat{u}$.
Thus, the fixed points of F_α are obtained by finding the zeros of the function g, defined by

$$\mathbb{P}F_\alpha(u_{\alpha,\beta}) = \big[g(\alpha, \beta) + \beta\big]\hat{u}\,.$$

So we need to **study the function g** on appropriate rectangles R in parameter space.

Bifurcations are identified by the **nature of the zero-set** of g in R.
Standard conditions involve inequalities on g and its partial derivatives (up to order 3).

Bounds on values of g are obtained by solving $F_{\alpha,\beta}(u) = u$,
using a quasi-Newton map $\mathcal{M}$ of the type $\mathcal{M}(u) = F_{\alpha,\beta}(u) - M[F_{\alpha,\beta}(u) - u]$.

Bounds on the derivatives of g are obtained via implicit differentiation.

Non-stationary solutions for KS. The most interesting are probably chaotic orbits, such as the ones found numerically in [F. Christiansen, P. Cvitanovic, V. Putkaradze, '97] for $\alpha \approx 137$.

For "simplicity", we focus on **periodic orbits**.
Such orbits have also been constructed in [P. Zgliczyński , preprint '08], using different methods.

The eigenvalues of $-L = 4\partial_x^4 + \alpha\partial_x^2$, acting on odd functions that are 2π-periodic in x, are

$$\lambda_k = 4k^4 - \alpha k^2 \,, \qquad k = 1, 2, \ldots$$

Choose κ such that $\lambda_k \geq 0$ for $k > \kappa$. Then define $L^{\pm} = \mathbb{P}^{\pm} L$, where

$$(\mathbb{P}^{-} u)(x,t) = \textstyle\sum_{k \leq \kappa} u_k(t) \sin(kx) \,, \qquad \mathbb{P}^{+} = \mathbb{I} - \mathbb{P}^{-} \,.$$

Then the KS equation can be written as

$$\boldsymbol{\dot{u} = L^{+} u + G(u) \,, \qquad G(u) = L^{-} u - \alpha\partial_x(u^2) \,.}$$

The associated initial value problem, with $u(t_0) = \nu$, can be reformulated as an integral equation

$$u(t) = e^{(t-t_0)L^+}\nu + \int_{t_0}^{t} e^{(t-s)L^+}[G(u)](s)\, ds \,, \qquad t \geq t_0 \,.$$

Equivalently, the function

$$v(t) = u(t) - e^{(t-t_0)L^{+}}\nu$$

is a fixed point of the map $\mathcal{M}_\nu$, defined by

$$\big(\mathcal{M}_\nu(v)\big)(t) = \int_{t_0}^{t} e^{(t-s)L^{+}} \Big[G\big(v + e^{(s-t_0)L^{+}}\nu\big)\Big](s) ds \,.$$

Let $\Delta = [t_0, t_0 + \tau]$.

$\mathcal{M}_\nu$ defines a contraction on $\mathcal{C}(\Delta, \mathcal{A}_1^o)$, for sufficiently small $\tau > 0$.
This fact can be used to implement an "integrator" for the KS equation.
By shadowing an approximate periodic orbit, we obtain the

Theorem. [G. Arioli, H.K.] *The KS equation for $\alpha = 150$ has a hyperbolic periodic orbit with period $T = 0.00204\ldots$. Some associated Poincaré map has a simple eigenvalue > 4.5, and the remaining part of its spectrum lies in the disk $|z| < 0.7$.*

Remarks.

- The derivative of the flow is estimated via the integral operator $\partial_\nu \mathcal{M}_\nu$.
- The spaces used are far from optimal.
- The shadowing procedures uses 4294 rectangular boxes.

Consider a fixed partition of Δ into n subintervals $\Delta_j = [t_{j-1}, t_j]$.
(In our implementation, the leftmost subintervals are much shorter than the interval Δ.)
$\mathcal{C}(\Delta, \mathcal{A}_K^o)$ is the Banach space of all continuous functions $v : \Delta \times \mathbb{T}^2 \to \mathbb{R}$ with ...

$$v(x,t) = \sum_{k \le K} v_k(t) \sin(kx)\,, \quad \|v\| = \max_j \|v\|_j\,, \quad \|v\|_j = \sum_{k \ge K} e^{\rho k} \max_{t \in \Delta_j} |u_k(t)|\,.$$

Simple representable sets for these spaces associated with data of type

`ContFun`: `P=(P.C,P.E)`, where

```
    P.C: array [0..PDeg] of Ball;
    P.E: array [1..NErr] of Ball;
```
(nonnegative)

$\mathcal{B}(\texttt{P.C})$: all polynomials of degree $\le$ `PDeg`, whose `K`-th coefficient belongs to $\mathcal{B}(\texttt{P.C(K)}, \mathbb{R})$.
The polynomials are expanded about the origin , if $\Delta = [-1, 1/2]$.
Otherwise use an affine change of coordinates.

$\mathcal{B}(\texttt{P.E}, \mathcal{A}_K^o)$: all functions $v \in \mathcal{C}(\Delta, \mathcal{A}_K^o)$ such that $\|v\|_J \le \texttt{P.E(J).R}$ for all `J`.

The representable sets for $\mathcal{C}(\Delta, \mathcal{A}_1^o)$ are associated with data of type `TFourier`, which is an instantiation (`FCoeff` $\Rightarrow$ `ContFun`) of `Fourier`. In other words,

$$\mathcal{B}(\texttt{F}) = \sum_{K=0}^{D} \mathcal{B}(\texttt{F.C(K)}) \sin(\texttt{K.}) + \sum_{K=0}^{2D} \mathcal{B}(\texttt{F.E(K)}, \mathcal{A}_K^o)\,, \qquad \texttt{F} \in \texttt{TFourier}\,.$$

The representable sets for $\mathcal{C}(\Delta, \mathcal{A}_0^e)$ are defined analogously.

Now implement bounds `Contr`, `DContr`, `ContrFix`, `DContrFix`, `Phi`, `DPhi`, ...
on the maps $\mathcal{M}_\nu$, $\partial_\nu \mathcal{M}_\nu$, ...

To obtain decent error bounds for `Contr`, we decompose $\mathcal{M}_\nu(v) = \mathcal{P}(\nu + v) + \mathcal{Q}(\nu, v)$, where $\mathcal{P}$ is linear and $\mathcal{Q}$ quadratic,

$$\mathcal{Q}(\nu, v) = -\alpha \int_{t_0}^{t} e^{(t-s)L^+} \partial_x \Big[v + e^{(s-t_0)L^+} \nu \Big]^2 .$$

Then split $\mathcal{Q}$ into terms $\mathcal{Q}^{(n)}$ that are homogeneous of degree n in v. After rewriting the result in terms of Fourier coefficients, we end up with integrals like

$$\big(\mathcal{Q}_m^{(1+)}(\nu, v)\big)(t) = -\alpha m \sum_{k+\ell=m} \nu_k \int_{t_0}^{t} e^{-\lambda_m^+(t-s)} e^{-\lambda_k^+(s-t_0)} v_\ell(s)\, ds \,,$$

and use estimates like

$$\big\|\mathcal{Q}^{(1+)}(\nu, v)\big\|_j \le 2\alpha \|\nu\| \|v\| \sup_{\substack{k\in\mathcal{K}\\ \ell\in\mathcal{L}}} \left[\frac{k+\ell}{b_j + (\lambda_{k+\ell}^+ - \lambda_k^+)} e^{-\lambda_k^+(t_{j-1}-t_0)} \right] , \qquad b_j = \frac{2}{t_j - t_0} .$$

Here, $\mathcal{K}$ and $\mathcal{L}$ are the frequency ranges for ν and v, respectively.
The **sup** is estimated by the program (beforehand), using monotonicity properties of [...].

`ContrFix` first computes an approximate fixed point u for $\mathcal{M}_\nu$. Then it encloses u in successively larger sets $\mathcal{B}(\mathtt{F})$ until one of them is mapped into itself by `Contr`.

The same strategy is applied for `DContr` and `DContrFix`.

Evaluating the result of `ContrFix` at a specified time $t \in \Delta$
yields a bound `Phi` on the flow $\Phi : (t, \nu) \mapsto v(t)$.

As much as possible of the above is kept hidden at the higher "dynamical systems" level.

The package `Boxes` uses data types `Vec` to describe sets in $\mathcal{A}_1^o$.
`V(1..N)` contains bounds on the first `N` Fourier coeffcients, and
`V(N+1)` is a bound on the norm of all "higher order" terms.

Other data types include `LBasis`, `Frame`, `Box`, `TBox`, ...

Roughly speaking, a `Box` represents a set $B = C_0 + L(R_1) \times R_2 \times H$, with
$\boldsymbol{C_0}$: the center of the `Box`.
$\boldsymbol{L(R_1)}$: the image of $R_1 = [-1,1] \times \{0\} \times [-1,1]^{M-2}$ under an linear transformation L on $\mathbb{R}^M$.
$\boldsymbol{R_2}$: a rectangle $R_2 = [-r_{M+1}, r_{M+1}] \times \ldots \times [-r_N, r_N]$.
$\boldsymbol{H}$: a "higher order" ball.

The zero-thickness direction of $L(R_1)$ corresponds to the Poincaré section.
A bound on the **Poincaré map** is obtained by determining a time interval $\mathtt{T} = [t-\varepsilon, t+\varepsilon]$
such that the flow-images of B at the two times $t \pm \varepsilon$
lie on opposite sides of the Poincaré section (at the destination point).

The 4294 boxes used in our shadowing procedure have been determined numerically.

The **box directions** fall into 4 classes.

low: The first 8 directions are roughly eigendirections of the return map (for the entire orbit).
mid-low: The next 12 directions are $(\mathbb{I} - P)\sin(k\,\boldsymbol{.})$, for $k = 9, 10, \ldots, 20 = M$.
Here, P is an approximation to the "low" spectral projection.
mid-high: Simply $\sin(k\,\boldsymbol{.})$ for $k = 21, 22, \ldots, 40 = N$.
high: All higher order modes ($k > N$).

Mapping a box $C_0 + L(R_1) \times R_2 \times H$.

To simplify notation: Omit domain conditions. Write products as direct sums; so our box is $B = C + L(R_1)$, where $C = C_0 + R_2 + H$.

Consider the set-map `P(V)=Phi(T,C)+DPhi(T,B,V)`.
Then Poincaré$(C + V) \subset$ `P(V)`.

This could be applied with $V = L(R_1)$.
A better estimate is obtained by using the following

Property of `P`: If V is contained in the convex hull of X and Y,
then Poincaré$(C + V)$ is contained in the convex hull of `P(X)` and `P(Y)`.

Thus, it suffices to compute the image under `P` of each of the corners of $L(R_1)$.