# AASP - a new kind of average shadowing

#### Piotr Oprocha (AGH Univ.) and Marcin Kulczycki

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# General setting

#### ▶ (X, d) - compact metric space

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# General setting

- ► (X, d) compact metric space
- $f: X \to X$  continuous map

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# Limit shadowing

▶ Y. Pilyugin et al, 1997

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# Limit shadowing

- ▶ Y. Pilyugin et al, 1997
- ►  $\{x_i\}_{i=1}^{\infty}$  is a limit pseudo-orbit if  $\lim_{i\to\infty} d(f(x_i), x_{i+1}) = 0$

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- ►  $\{x_i\}_{i=1}^{\infty}$  is shadowed in limit by  $z \in X$  if  $\lim_{i \to \infty} d(f^i(z), x_i) = 0$

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# Limit shadowing

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- ►  $\{x_i\}_{i=1}^{\infty}$  is shadowed in limit by  $z \in X$  if  $\lim_{i \to \infty} d(f^i(z), x_i) = 0$
- ► f has limit shadowing if every limit pseudo-orbit of f is shadowed in limit by some z ∈ X

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# Average shadowing

M. Blank, 1988

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# Average shadowing

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- ►  $\{x_i\}_{i=1}^{\infty}$  is a  $\delta$ -average pseudo-orbit if  $\exists N \forall n > N \forall k \ge 0$  $\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta$

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- ►  $\{x_i\}_{i=1}^{\infty}$  is  $\varepsilon$ -shadowed in average by  $z \in X$  if  $\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} d(f^i(z), x_i) < \varepsilon$

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# Average shadowing

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- F has average shadowing if for every ε > 0 there exists δ > 0 such that every δ-average pseudo-orbit of f is ε-shadowed in average by some z ∈ X

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# Asymptotic average shadowing property

▶ R. Gu, 2007

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Asymptotic average shadowing property

- R. Gu, 2007
- AASP = limit shadowing + average shadowing

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# Asymptotic average shadowing property

► 
$$\{x_i\}_{i=1}^{\infty}$$
 is an asymptotic-average pseudo-orbit if  
lim sup\_{n\to\infty}  $\frac{1}{n} \sum_{i=1}^{n} d(f(x_i), x_{i+1}) = 0$ 

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# Asymptotic average shadowing property

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- ▶  $\{x_i\}_{i=1}^{\infty}$  is asymptotically shadowed in average by  $z \in X$  if  $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} d(f^i(z), x_{i+1}) = 0$

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# Asymptotic average shadowing property

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- ► f has AASP if every asymptotic-average pseudo-orbit of f is asymptotically shadowed in average by some z ∈ X

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# Size of errors

There can be infinitely many large errors in a pseudo-orbit, but they have to be sparse enough

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# Size of errors

- There can be infinitely many large errors in a pseudo-orbit, but they have to be sparse enough
- ▶ 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, ...

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# Interesting examples are hard to find

▶ An identity map on [0,1] does not have the AASP

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# Interesting examples are hard to find

- ▶ An identity map on [0,1] does not have the AASP
- Constant maps have the AASP

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# Interesting examples are hard to find

- ▶ An identity map on [0,1] does not have the AASP
- Constant maps have the AASP
- What about this map?



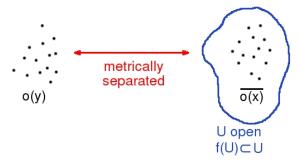
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# When there can't be AASP?

▶ There is no AASP if you can find:

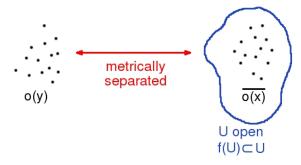


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# When there can't be AASP?

▶ There is no AASP if you can find:



▶ In particular, the following map doesn't have the AASP:



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# Interesting properties

#### AASP + density of minimal points of f

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f is totally transitive

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# Interesting properties

• f surjective with specification property  $\Rightarrow$  AASP

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# Interesting properties

- f surjective with specification property  $\Rightarrow$  AASP
- Specification property:

 $\forall \varepsilon > 0 \ \exists M > 0 \ \forall n \in \mathbb{N} \ \forall y_1, \dots y_n \in X$  $\forall a_1 \le b_1 < a_2 \le b_2 < a_3 \le \dots \le b_n \text{ such that}$  $\forall 2 \le i \le n : a_i - b_{i-1} \ge M \ \exists z \in X$  $\forall 1 \le i \le n \ \forall a_i \le k \le b_i : d(f^k(y_i), f^k(z)) < \varepsilon$ 

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# Interesting properties

- f surjective with specification property  $\Rightarrow$  AASP
- Specification property:

 $\forall \varepsilon > 0 \ \exists M > 0 \ \forall n \in \mathbb{N} \ \forall y_1, \dots y_n \in X$  $\forall a_1 \le b_1 < a_2 \le b_2 < a_3 \le \dots \le b_n \text{ such that}$  $\forall 2 \le i \le n : a_i - b_{i-1} \ge M \ \exists z \in X$  $\forall 1 \le i \le n \ \forall a_i \le k \le b_i : d(f^k(y_i), f^k(z)) < \varepsilon$  $\blacktriangleright \text{ In particular all mixing maps on } [0, 1] \text{ have the AASP}$ 

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### Even more properties

► For c-expansive maps with shadowing:

 $\mathsf{mixing} \Leftrightarrow \mathsf{AASP}$ 

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# Even more properties

• For c-expansive maps with shadowing:

 $\mathsf{mixing} \Leftrightarrow \mathsf{AASP}$ 

For positively expansive open surjections:

 $\mathsf{mixing} \Leftrightarrow \mathsf{AASP}$ 

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# Even more properties

► For c-expansive maps with shadowing:

 $\mathsf{mixing} \Leftrightarrow \mathsf{AASP}$ 

► For positively expansive open surjections:

 $\mathsf{mixing} \Leftrightarrow \mathsf{AASP}$ 

• Surjectivity + AASP  $\Rightarrow$  chain mixing

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# A very interesting property

► If 
$$A \subset X$$
,  $f(A) \subset A$ ,  $f|_A$  has AASP, and  $\forall \varepsilon > 0 \exists n \in \mathbb{N}$   
 $\forall x \in X \ \frac{1}{n} \# \{ 0 \le i < n : d(f^i(x), A) < \varepsilon \} > 1 - \varepsilon$ 

then f has AASP

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# A very interesting property

► If 
$$A \subset X$$
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then f has AASP

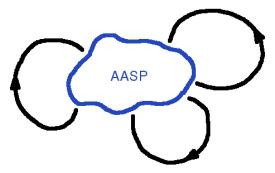
• (The last assumption is in particular satisfied when  $\forall x \in X \ \omega(x) \subset A$ )

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# A very interesting property

These kinds of maps have the AASP:



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# A very interesting property

▶ In particular, this map has the AASP:



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# A very interesting property

In particular, this map has the AASP:



• Which proves that shadowing  $\Leftrightarrow$  AASP ...

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# A very interesting property

In particular, this map has the AASP:



- Which proves that shadowing  $\Leftrightarrow$  AASP ...

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# Building AASP on [0, 1]

# It is possible to obtain a nontransitive map on $\left[0,1\right]$ with AASP (credits go to prof. M. Misiurewicz)

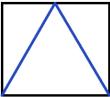
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# Building AASP on [0, 1]

Start with a standard tent map...

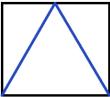


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# Building AASP on [0, 1]

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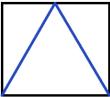
 ...pick a dense orbit and blow up each point to an interval as in the construction of the Denjoy flow...

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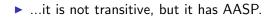
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# Building AASP on [0, 1]

Start with a standard tent map...



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# Building AASP on [0, 1]

# The proof that this map has AASP is not an application of the theorem!

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# Main tool

The main tool for working with AASP is a lemma from *"Introduction to ergodic theory"* by Walters:

Let  $\{a_i\}_{i=0}^\infty$  be a bounded sequence of nonnegative real numbers. Then

 $\exists J \subset \mathbb{N}$  of density 0 such that  $\lim_{n \to \infty} a_n = 0$  provided that  $n \notin J$ .

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Definitions Results	Basic observations
	A handful of results
	A nontrivial example

# The End

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