AASP - a new kind of average shadowing

Piotr Oprocha (AGH Univ.) and Marcin Kulczycki

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General setting

- (X, d) - compact metric space

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- $f : X \to X$ - continuous map

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- f has limit shadowing if every limit pseudo-orbit of f is shadowed in limit by some $z \in X$

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- f has average shadowing if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-average pseudo-orbit of f is ε-shadowed in average by some $z \in X$

Asymptotic average shadowing property

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- AASP = limit shadowing + average shadowing

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- f has AASP if every asymptotic-average pseudo-orbit of f is asymptotically shadowed in average by some $z \in X$

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- 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, . . .

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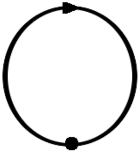
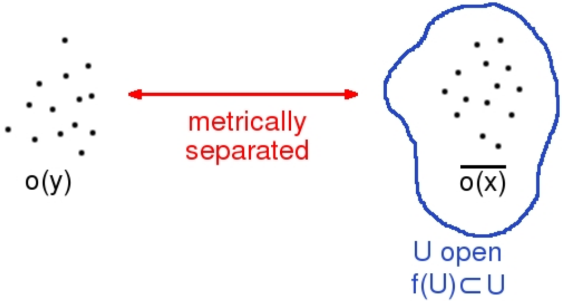
- Constant maps have the AASP
- What about this map?



When there can't be AASP?

- There is no AASP if you can find:



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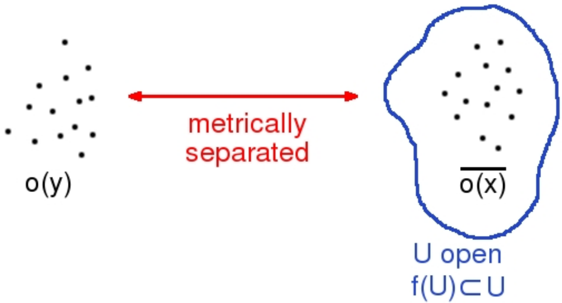

- In particular, the following map doesn't have the AASP:

Interesting properties

$$\text{AASP} + \text{density of minimal points of } f$$

$$\Downarrow$$

$$f \text{ is totally transitive}$$

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$$\forall \varepsilon > 0 \, \exists M > 0 \, \forall n \in \mathbb{N} \, \forall y_1, \dots y_n \in X$$

$$\forall a_1 \leq b_1 < a_2 \leq b_2 < a_3 \leq \dots \leq b_n \text{ such that}$$

$$\forall \, 2 \leq i \leq n \; : \; a_i - b_{i-1} \geq M \; \exists z \in X$$

$$\forall 1 \leq i \leq n \; \forall a_i \leq k \leq b_i \; : \; d(f^k(y_i), f^k(z)) < \varepsilon$$

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- In particular all mixing maps on $[0, 1]$ have the AASP

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- Surjectivity + AASP $\Rightarrow$ chain mixing

A very interesting property

- If $A \subset X$, $f(A) \subset A$, $f|_A$ has AASP, and $\forall \varepsilon > 0\, \exists n \in \mathbb{N}$

$$\forall x \in X \ \frac{1}{n} \#\{0 \leq i < n : d(f^i(x), A) < \varepsilon\} > 1 - \varepsilon$$

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  then f has AASP
- (The last assumption is in particular satisfied when $\forall x \in X\; \omega(x) \subset A$)

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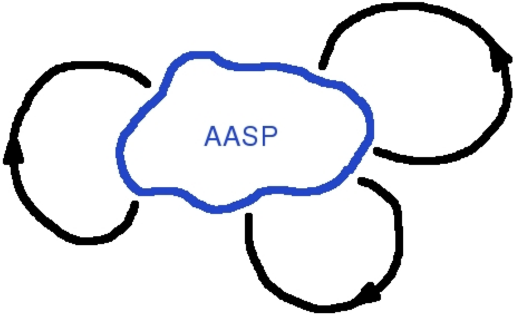
These kinds of maps have the AASP:



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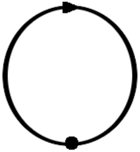
- In particular, this map has the AASP:



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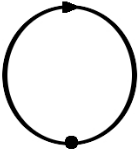

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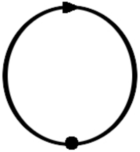


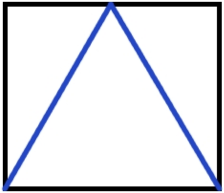
- Which proves that shadowing $\not\Leftarrow$ AASP ...
- ...and specification property $\not\Leftarrow$ AASP

Building AASP on $[0, 1]$

It is possible to obtain a nontransitive map on $[0, 1]$ with AASP (credits go to prof. M. Misiurewicz)

Building AASP on $[0, 1]$

- Start with a standard tent map...



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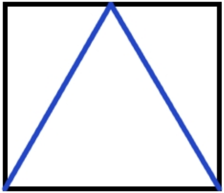
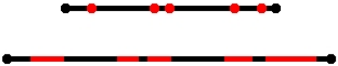


- ...pick a dense orbit and blow up each point to an interval as in the construction of the Denjoy flow...



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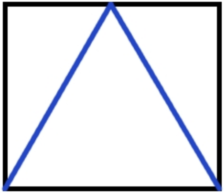

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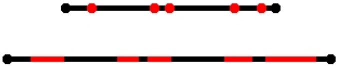

- ...it is not transitive, but it has AASP.

Building AASP on $[0, 1]$

The proof that this map has AASP is not an application

of the 

theorem!

Main tool

The main tool for working with AASP is a lemma from *"Introduction to ergodic theory"* by Walters:

Let $\{a_i\}_{i=0}^{\infty}$ be a bounded sequence of nonnegative real numbers. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} a_i = 0$$

$$\Updownarrow$$

$\exists J \subset \mathbb{N}$ of density 0 such that $\lim_{n\to\infty} a_n = 0$ provided that $n \notin J$.

The End