Topological entropy for local processes

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DYNAMICS, TOPOLOGY AND COMPUTATIONS
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- (Pieniążek & Wójcik 2003) The equation

$$\dot{z} = \left[1 + (\cos(t^2) + 2)e^{i\kappa t}|z|^2\right]\overline{z}$$

has complicated dynamics provided that $\kappa > 0$ is small enough.

Let (Y, d) be a metric space and $\Omega \subset \mathbb{R} \times \mathbb{R} \times Y$ be an open set. By a *local process* on Y we mean a continuous map $\varphi : \Omega \longrightarrow Y$, such that the following three conditions are satisfied:

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i) $\forall \sigma \in \mathbb{R}$, $x \in Y$, $(t^-_{(\sigma,x)}, t^+_{(\sigma,x)}) = \{t \in \mathbb{R} : \ (\sigma, t, x) \in \Omega\}$ is an open interval containing 0, Let (Y,d) be a metric space and $\Omega \subset \mathbb{R} \times \mathbb{R} \times Y$ be an open set. By a *local process* on Y we mean a continuous map $\varphi : \Omega \longrightarrow Y$, such that the following three conditions are satisfied:

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iii) $\forall x \in Y, \sigma, s \in \mathbb{R},\ t \in \mathbb{R}$ if $(\sigma, x, s) \in \Omega$, $(\sigma + s, t, \varphi(\sigma, x, s)) \in \Omega$ then $(\sigma, s + t, x) \in \Omega$ and $\varphi(\sigma, s + t, x) = \varphi(\sigma + s, t, \varphi(\sigma, s, x))$.

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For abbreviation, we write $\varphi_{(\sigma,t)}(x)$ instead of $\varphi(\sigma, t, x)$.

Let M be a smooth manifold and let $v : \mathbb{R} \times M \longrightarrow TM$ be a time-dependent vector field. We assume that v is regular enough to guarantee that for every $(t_0, x_0) \in \mathbb{R} \times M$ the Cauchy problem

$$\dot{x} = v(t, x), \tag{2}$$

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hence there is a one-to-one correspondence between T-periodic solutions of (2) and fixed points of the Poincaré map $\varphi_{(0,T)}$.

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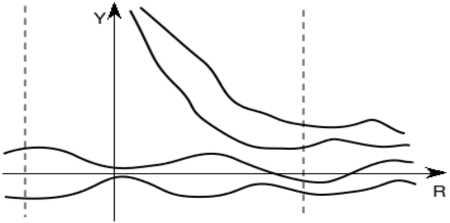

Let φ be a local process acting on a metric space (X, d), $s \in \mathbb{R}$ and K be a compact subset of X. We define sets

$$\Lambda_K^+(\varphi, s) = \left\{ x \in K \ : \ t_{(s,x)}^+ = +\infty \text{ and } \varphi(s,t,x) \in K \text{ for every } t \geq 0 \right\},$$

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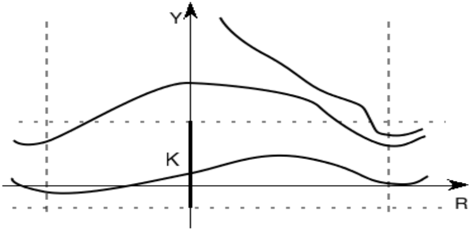

Fix $\varepsilon > 0$, $s \in \mathbb{R}$, $T > 0$. We say that a subset $E \subset K$ is a $(s, T, \varepsilon, K, \varphi)$*-spanning set (with respect to the set* K*)* if for every $y \in \Lambda_K(\varphi, s)$ there is $x \in E$ such that

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The topological entropy of the process φ *at* s is the number $h(\varphi, s) \in [0, +\infty]$ defined by

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(Cánovas & Rodríguez 2005)

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- We call (X_∞, f_∞) a *nonautonomous discrete dynamical system (NDDS)*.
- We say that a NDDS (X_∞, f_∞) is *proper (with respect to a metric space* (Y, d)*) (denoted PNDDS)* iff for every $i \in \mathbb{Z}$

$$X_i \subset Y \text{ and } d_i = d\,|_{X_i \times X_i}\ .$$

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The *Υ-discretisation of φ* is the nonautonomous discrete dynamical system $\varphi_\Upsilon = (X_\infty, f_\infty)$ given by

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- Υ-discretisation of φ is a proper nonautonomous discrete dynamical system with respect to Y.

Let $i \in \mathbb{Z}$, (X_∞, f_∞) be a PNDDS with respect to Y and let K be a compact subset of Y. We define sets

$$\Lambda_K^+(f_\infty, i) = \{x \in X_i \cap K \ : \ \text{the trajectory of } x \text{ is a subset of } K\}$$
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$$h_K(f_\infty, -\infty) = \lim_{\varepsilon \to 0^+} \limsup_{N \to \infty} \frac{\log S_{f_\infty}(0, N, \varepsilon, K)}{N}$$

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Let us fix $i \in \mathbb{Z}$.

$$h_K(f_\infty, -\infty) = \lim_{\varepsilon\to 0^+} \limsup_{N\to\infty} \frac{\log S_{f_\infty}(0, N, \varepsilon, K)}{N}$$

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The topological entropy of (X_∞, f_∞) *at* i is the number $h(f_\infty, i) \in [0, +\infty]$ defined by

$$h(f_\infty, i) = \sup\left\{h_K(f_\infty, i) \ : \ K \text{ is a compact subset of } Y\right\}.$$

We say that a local process φ on a metric space (Y, d) is *locally equicontinuous* if for every compact set $K \subset Y$, every $\varepsilon > 0$ and every $T > 0$ there is $\delta = \delta(K, \varepsilon, T) > 0$ such that for every $\sigma \in \mathbb{R}$ and every $x, y \in \Lambda_K^+(\varphi, \sigma)$ the following implication holds:

$$d(x, y) < \delta \Longrightarrow \{d(\varphi(\sigma, t, x), \varphi(\sigma, t, y)) < \varepsilon \text{ for every } t \in [0, T]\} .$$

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Let $n \geq 1$, $n \in \mathbb{N}$ and $v \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. If for every compact set $K \in \mathbb{R}^n$ there exists constant $0 \leq L_v(K) < \infty$ such that

$$|D_x v(t, x)| \leq L_v(K) \text{ for every } (t, x) \in \mathbb{R} \times K,$$

then the local process φ generated by

$$\dot{x} = v(t, x)$$

is locally equicontinuous.

We say that a strictly increasing and unbounded sequence $\Upsilon = \{t_i\}_{i\in\mathbb{Z}} \subset \mathbb{R}$ is *forward syndetic* if there exist $k \in \mathbb{Z}$ and $N > 0$ such that $t_{m+1} - t_m < N$ for every $m > k$.

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Theorem 1

Let φ be a locally equicontinuous local process on a metric space (Y, d) and $\Upsilon = \{t_i\}_{i\in\mathbb{Z}}$ be a forward syndetic sequence such that

$$\liminf_{n\to\infty} \frac{t_n}{n} \geq \alpha$$

holds for some $\alpha \in [0, \infty]$. Then the inequality

$$h(\varphi_\Upsilon, i) \geq \alpha\, h(\varphi, t_i)$$

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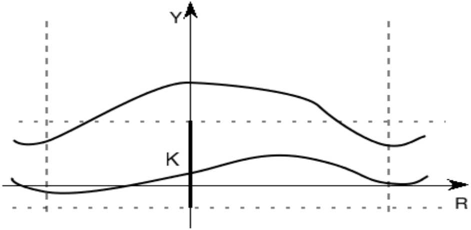

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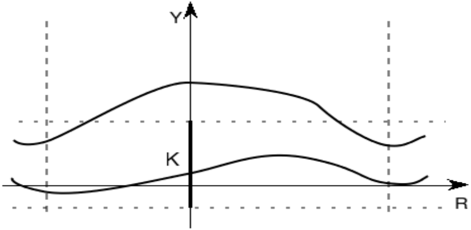

For a given K we need a compact set M such that

$$\Lambda^+_K(\varphi_\Upsilon, i) \subset \Lambda^+_M(\varphi, t_i),$$
$$\Lambda_K(\varphi_\Upsilon, i) \subset \Lambda_M(\varphi, t_i).$$

We say that a local process φ on a metric space (Y, d) *has bounded oscillations* if there exists a sequence of compact subsets $\{M_j\}_{j=1}^{\infty}$ of Y such that

$$M_j \subset M_{j+1} \text{ for every } j \text{ and } Y = \bigcup_{i=0}^{\infty} M_i$$

and for every compact set $K \subset Y$ there exists j such that $K \subset M_j$ and for every $\sigma \in \mathbb{R}$ and $x \in K$ such that $t^{+}_{(\sigma,x)} = +\infty$ one of the following conditions hold:

$\varphi(\sigma, s, x) \in M_j$ for all $s > 0$ or

if $\varphi(\sigma, t, x) \notin M_j$ for some $t > 0$ then $\varphi(\sigma, s, x) \notin M_j$ for all $s > t$.

Theorem 2

Let φ be a local process with bounded oscillations (defined on a metric space (Y, d)) and let $\Upsilon = \{t_i\}_{i\in\mathbb{Z}} \subset \mathbb{R}$ be a strictly increasing unbounded sequence such that the condition

$$\limsup_{n\to\infty} \frac{t_n}{n} \leq \alpha$$

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Thank You for Your attention