Chaos and semiconjugacy arguments

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Factor maps

1. Factor map
   1. (X, f), (Y, g) - continuous self-maps of compact metric spaces
   2. $\pi \colon X \to Y$ - onto, $\pi \circ f = g \circ \pi$
2. Some possible properties of π:
   1. π is a homeomorphism (some special situations...)
   2. π is almost one to one, i.e. $Y_1 = \{y \; : \; \#\pi^{-1}(\{y\}) = 1\}$ is residual (horseshoes in dimension one, Toeplitz shifts, extensions over co-cycles...)
   3. one point $y_0 \in Y$ has unique preimage, possibly with some extra knowledge about y_0 (snap back repellers, isolating segments, covering relations, horseshoes)
   4. π is finite to one (horseshoes in dimension one, hyperbolic maps, ...)
   5. ...
3. Some possible Y:
   1. full shift on $n \geq 2$ symbols
   2. shift of finite type
   3. minimal system (e.g. Odometer, rotation, ...)
   4. other well studied spaces/maps

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Factor maps and entropy

$$\pi\colon (X, f) \to (Y, g)$$

1. Topological entropy: $h_{top}(f) \in [0, +\infty]$,
2. $h_{top}(f) > 0 \quad \Longrightarrow \quad$ chaos,
3. Positive topological entropy:
   1. $h_{top}(f) \geq h_{top}(g)$.
   2. if π is at most k-to-1 for some $k > 0$ then $h_{top}(f) = h_{top}(g)$.
4. Chaos in the sense of Li and Yorke:
   1. there is an uncountable set $S \subset X$ such that for any $x \neq y$, $x, y \in S$:

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \quad , \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0 \quad (> \varepsilon).$$

   2. In $h_{top}(f) > 0$ then f is LY-chaotic.

   Blanchard, Glasner, Kolyada, Maass, *J. Reine Angew. Math.* 547 (2002) 5168

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Extensions of L-Y definition

$$\pi\colon (X,f) \to (Y,g)$$

1. distributionally chaotic pair:
   1. $\Phi_{xy}^{(n)}(t) = \frac{1}{n}\#\left\{i \;:\; d(f^i(x), f^i(y)) < t \quad, 0 \le i < n\right\}$,
   2. $\Phi_{xy}(t) = \liminf_{n\to\infty} \Phi_{xy}^{(n)}(t)$
   3. $\Phi_{xy}^*(t) = \limsup_{n\to\infty} \Phi_{xy}^{(n)}(t)$
   4. a pair is DC if
      - $\Phi_{xy}^*(t) = 1$, for all $t > 0$,
      - $\Phi_{xy}(s) = 0$ for some $s > 0$.
2. ω-chaotic pair
   1. $\omega_f(x) \setminus \omega_f(y)$ is uncountable,
   2. $\omega_f(x) \cap \omega_f(y) \neq \emptyset$ and
   3. $\omega_f(x) \setminus Per(f) \neq \emptyset$.
3. The above properties are independent of PTE and of each other...

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How to get π? A cookbook…

$$\pi\colon (\Lambda, f|_\Lambda) \to (\Sigma_2^+, \sigma)$$

1. Fix two disjoint closed sets N_0, N_1.
2. For every sequence $\alpha = \{a_0, a_1, \ldots, a_{n-1}\}$, $a_i \in \{0,1\}$ construct p_α (periodic ?) such that $f^j(p_\alpha) \in N_{a_{j(\bmod n)}}$,
3. $\Lambda = \overline{\left\{f^j(p_\alpha)\ :\ \alpha \in \{0,1\}^+, j \in \mathbb{N}\right\}}$,
4. Assign $\pi(f^j(p_\alpha)) = \sigma^j(\alpha\alpha\ldots)$ and extend continuously on Λ

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One-to-one covering

$$\pi\colon (\Lambda, f^p|_\Lambda) \to (\Sigma_2^+, \sigma)$$

1. $z \in X$ is a repelling fixed point for f in U if
   - $U \subset f(U)$
   - $\bigcap_{n=0}^{\infty} f^{-n}(U) = \{z\}$
2. z is a snap-back repeller for f in U if
   - z is a repelling fixed point
   - there is $y \in U \setminus \{z\}$ such that $f^m(y) = z$ for some $m > 0$ and f^m is open at y.

(Boyarski, Góra, Lioubimov, *Nonlinear Analysis*, 43 (2001) 591–604)

(Marotto, *J. Math. Anal. Appl.* 63 (1978) 199–223)

A little more on π...

$$\pi\colon (X,f) \to (\Sigma_2^+, \sigma)$$

1. if there is $y \in \Sigma_2^+$ such that:
   - $\overline{\mathrm{Orb}^+(y)} \neq \Sigma_2^+$
   - $\#\Phi^{-1}(\{y\}) = 1$ (2, more...?)

   then f is distributionally chaotic

   (joint work with P. Wilczyński)

2. if there is $y \in \Sigma_2^+$ ($\overline{\mathrm{Orb}^+(y)} \neq \Sigma_2^+$)such that:
   - $\pi^{-1}(\{y\})$ is at most countable

   or

   - $\Omega = \bigcup_{x\in\pi^{-1}(\{y\})} \omega_f(x)$ contains at most countably many minimal subsets

   then f is ω-chaotic

   (joint work with M. Lampart)

Problems with thick fibers...

$$\pi\colon (\Lambda, f|_\Lambda) \to (\Sigma_2^+, \sigma)$$

$$\Lambda = \overline{\left\{f^j(p_\alpha) \;:\; \alpha \in \{0,1\}^+, j \in \mathbb{N}\right\}}$$

1. is there $x \in \Lambda$ such that periodic points are dense in $\overline{\mathrm{Orb}^+(x)}$? (chaos in the sense of Devaney)