Periodic orbits for the Kuramoto-Sivashinski PDE on the line

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Some related work

- "Classical" PDEs setting, use of Schauder or Banach fixed points in suitable function spaces : Nakao, Yamamoto, Plum, McKenna, Watanabe and others. restricted to static problems, no dynamics

- functional analytic approach: Arioli and Koch - results on fixed points and bifurcations for Kuramoto-Sivashinski PDE , dynamics ? work in progress (see previous talk by H. Koch)

- self-consistent bounds - good for dynamics of dissipative PDEs (and static problems too).

  – fixed points and connections for Cahn-Hillard (gradient system): Maier-Papper, Mischaikow, Wanner

- periodic orbits for Kuramoto-Sivashinski PDE in 1D - P. Z.

- others: Swift-Hohenberg eq. - steady states : Hiraoka, Ogawa, Mischaikow, Day

- bifurcations of steady states for KS eq. - P.Z.

Outline of this talk

1. Kuramoto-Sivashinsky Eq. and our results about the dynamics

2. About the method

3. Some data from the proofs

A Model Problem - Kuramoto-Sivashinsky PDE

Consider the Kuramoto-Sivashinsky (KS) ej.

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x, \qquad \nu > 0$$

where $(t, x) \in [0, \infty) \times \mathbf{R}$ subject to periodic and odd boundary conditions

$$\begin{aligned} u(t, 0) &= u(t, 2\pi) \\ u(t, -x) &= -u(t, x) \end{aligned}$$

For various values of ν a variety of dynamics,

fixed points,
periodic orbits,
heteroclinic orbits,
chaotic dynamics,

have been observed numerically.

Goal: A rigorous means of proving these numerical results.

Kuramoto-Sivashinsky PDE, Fourier expansion

Fourier expansion is: $u(t,x) = \sum_{k=-\infty}^{\infty} b_k(t) e^{ikx}$

Substituting in KS and applying boundary conditions gives:

$$\dot{a}_k = k^2(1-\nu k^2)a_k - k\sum_{n=1}^{k-1} a_n a_{k-n} + 2k\sum_{n=1}^{\infty} a_n a_{n+k}$$

where $b_k = i a_k$ and $k = 1, 2, 3, \ldots$.

Linearization: $\dot{a}_k = k^2(1-\nu k^2)a_k$

- k-th mode is unstable for $k < \frac{1}{\sqrt{\nu}}$
- k-th mode is stable for $k > \frac{1}{\sqrt{\nu}}$
- the modes with $k >> \frac{1}{\sqrt{\nu}}$ should be irrelevant for the dynamics

Periodic orbits for $\nu = 0.1212$, possibly chaotic behavior

Numerical fact: As ν decreases from 0.1217 to 0.1212 the system system undergoes a sequence of period doubling bifurcation with possible creation of chaotic attractor.

Our rigorous results: The existence of two periodic orbits, with periods one and two (in terms of the Poincare map).

Numerical fact: we have found the heteroclinic connections between the above orbits in both directions.

Next step rigorous: build a chain of covering relations, which will guarantee the existence of of the symbolic dynamics.

Tools: covering relations, self-consistent bounds, rigorous integration of differential inclusions

Periodic orbits for $\nu = 0.1212$ some data from the proofs

the size of the box $4 \cdot 10^{-5}$, $h = 4 \cdot 10^{-4}$, $m = 13$, $M = 3m$, computation time $2 - 3$ minutes per initial condition, each covering relations requires at least 3 initial conditions

Period one orbit - 2 covering relations,

Period one two - 4 covering relations

We consider half-Poincare maps.

The procedure for the construction of h-sets for covering relations is automatic.

Switch to paper, show covering relation, idea of self-consistent bounds

Rigorous integration of dissipative PDEs - the general idea

$$u_t = Lu + N(u, Du, \dots, D^r u) + f(x), \qquad (1)$$

$u \in \mathbf{R}^n$, $x \in \mathbf{T}^d$, L is a linear, N - a polynomial (or analytic), f smooth enough.

L is diagonal in the Fourier basis $\{e^{kx}\}_{k \in \mathbf{Z}^d}$

$$Le^{ikx} = \lambda_k e^{ikx}, \qquad (2)$$

$$\begin{aligned} \lambda_k &= -v(|k|)|k|^p & (3)\\ 0 &< v_0 \leq v(|k|) \leq v_1, \qquad \text{for } |k| > K_- & (4)\\ p &> r. & (5) \end{aligned}$$

1 We replace PDE by an infinite ladder of ODEs for Fourier coefficients of $u(t,x)$.

$$\frac{du_k}{dt} = \lambda_k u_k + N_k(u), \qquad \text{for all } k \in \mathbf{Z}^d. \quad (6)$$

2 we split 'the phase space' for (6) into two parts: the finite dimensional part, X, containing the Fourier modes most relevant for the dynamics of (1) and the tail in $X^\perp$. Now problem (6) is replaced by two problems (7) and (8).

3 The first part consist of a finite dimensional differential inclusion for $p \in X$, given by

$$\frac{dp}{dt} \in P(Lp + N(p+T)), \qquad p \in X \qquad (7)$$

P is a projection onto X. The second part is concerned with the evolution of T

$$\lambda_k u_{k,j} + N^-_{k,j} < \frac{du_{k,j}}{dt} < \lambda_k u_{k,j} + N^+_{k,j}, \quad \text{"}k \text{ not in } X\text{"} \qquad (8)$$

where $N^\pm_{k,j}$ are suitably chosen constants.

Obviously, to infer from (7) and (8) any information on the behavior of solutions of the full system (6) one needs some **consistency conditions** and **fast decay of of Fourier coefficients**.

Our algorithm gives uniform and compact bounds for all Galerkin projections of PDE. The solution of PDE is obtained through passing to the limit with the dimension of Galerkin projection.

Why it is a easy to find a good tail = self-consistent bounds

$$u_t = Lu + N(u, Du, \dots, D^r u)$$

$x \in \mathbf{T}^n$ (periodic boundary conditions),
L - linear, diagonal, N - polynomial

Fourier expansion $u(t) = \sum_{k \in \mathbf{Z}^n} a_k(t) e^{ik \cdot x}$

Lemma. Let $s > s_0$. If $|a_k| \leq C/|k^s|$, $|a_0| \leq C$, then there exists $D = D(C, s)$

$$|N_k| \leq \frac{D}{|k|^{s-r}}, \qquad |N_0| \leq D$$

Isolation. Assume $L(a)_k = -|k|^p a_k$, $p > r$.

Assume $|a_k| \leq \frac{C}{|k|^s}$, $|a_{k_0}| = \frac{C}{|k_0|^s}$, then

$$\frac{d|a_{k_0}|}{dt} \leq -|k_0|^p |a_{k_0}| + |N_{k_0}(a)| \leq$$
$$-C|k_0|^{p-s} + D|k_0|^{r-s}$$
$$\frac{d|a_{k_0}|}{dt} < 0, \qquad |k_0| > M$$

Representation used for KS equation

We look for solutions in

$$W \oplus T = W \oplus \Pi_{k=m+1}^{k \leq M} [a_k^-, a_k^+] \oplus \Pi_{k>M} \left[\frac{-C}{k^s}, \frac{C}{k^s} \right] \tag{9}$$

where $W \subset X_m$.

$$N_k(W \oplus T) \subset [N_k^-, N_k^+], \qquad k = m+1, \ldots, M$$

$$N_k(W \oplus T) \subset \left[\frac{-D(W \oplus T)}{k^{s-2}}, \frac{D(W \oplus T)}{k^{s-2}} \right], \qquad k > M$$

We solve (estimate rigorously) the solutions of the following system of differential inclusions

$$\begin{aligned} x' &\in P_m F(x) + \Gamma, \qquad && x \in W \subset X_m \\ x'_k &\in \lambda_k x_k + [N_k^-, N_k^+], \qquad && k = m+1, \ldots, \end{aligned}$$

x_k for $k > M$ are given by a single formula.

Tail evolution

Our problem $a_k' = \lambda_k a_k + N_k(a)$,
$\lambda_k \to -\infty$, for $|k| \to \infty$

W, $T([0,h])$ - the rough enclosure for $Z \oplus T(0)$ for $t \in [0,h]$

For $k > m$ we have

$$N_k^{\pm} = N_k^{\pm}(W, T([0,h]))$$
$$\lambda_k a_k + N_k^{-} < \frac{da_k}{dt} < \lambda_k a_k + N_k^{+},$$

hence

$$b_k^{\pm} = \frac{N_k^{\pm}}{-\lambda_k}, \qquad (10)$$
$$T(h)_k^{\pm} = \left(T(0)_k^{\pm} - b_k^{\pm}\right) e^{\lambda_k h} + b_k^{\pm} \qquad (11)$$

It remains to put $T(h)$ for $k > M$ in the form

$$T(h)_k^{\pm} = \frac{\pm C(T(h))}{k^{s(T(h))}}$$

For $k > M$ we have

$$
\begin{aligned}
0 < b_k^+ &\le \frac{C(b)}{k^{s(b)}} \\
T(0)_k^+ &= \frac{C(T(0))}{k^{s(T(0))}} \\
T(h)_k^+ &\le T(0)_k^\pm e^{\lambda_k h} + b_k^\pm .
\end{aligned}
$$

$$
T(h)_k^+ \le \frac{C(T(0))}{k^{s(T(0))}} e^{\lambda_k h} + \frac{C(b)}{k^{s(b)}}.
$$

Let

$$
E = e^{h\lambda_{M+1}} (M+1)^{s(b)-s(T(0))}.
$$

then (modulo some conditions on M, h)

$$
e^{\lambda_k h} \le \frac{E}{k^{s(b)-s(T(0))}}, \quad k > M
$$

and finally we can set

$$
T_k^\pm(h) = \pm \frac{C(T(0))E + C(b)}{k^{s(b)}}.
$$

About the computations

- gnu C++
- interval arithmetic - from CAPD package developed in Kraków, Poland
- we use the Lohner algorithm to integrate differential inclusions

Some computation data

On 3GHz machine, Linux, gnu C++

- $\nu = 0.127$, $m = 10$, $M = 3 * m$, $h = 1e - 3$, $order = 4$, $T/2 \approx 1.12$, computation time around 10 sec

- $\nu = 0.1215$, $m = 13$, $M = 3*m$, $h = 4e-4$, $order = 6$, $T \approx 3.07$, computation time around 240 sec

- $\nu = 0.032$, $m = 23$, $M = 3*m$, $h = 1.5e-4$, $order = 5$, $T/2 \approx 0.41$, computation time around 300 sec

- $\nu = 0.02991$, an unstable orbit on chaotic attractor, $m = 25$, $M = 3 * m$, $h = 1e - 4$, $order = 5$, $T/2 \approx 0.449$, computation time around 760 sec