(Re–)Definition of Connection Matrices

by

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Goal of the talk:

The definition of connection matrices is reconsidered, simplified, and presented in a self-contained manner in the language of CONLEY index theory.

Contents:

See also:

Mohamed Barakat and Stanislaus Maier-Paape:

Computation of connection matrices using the software package `conley`,

Internat. J. Bifur. Chaos Appl. Sci. Energy., to appear.

1. Introduction

Connection matrices, a central concept in the Conley index theory, enables one to investigate and prove the existence of heteroclinic connections between (isolated) invariant sets.

Typical Problems:

A. The concept of connection matrices relies heavily on homological algebra.

B. Even the definition or the existence of connection matrices is difficult to comprehend without a solid algebraic background.

C. The full power of the connection matrix concept can only be reached, when non-trivial intrinsic dynamical arguments are used.

Achievements of our paper:

A. Simplification of the definition of connection matrices (no use of braids necessary (this talk)).

B. Automatization of the computation process (Mohameds talk).

To start with, a brief introduction of the main terms is given, which we need to state our (re–)definition result.

2. Definitions from Conley Index Theory

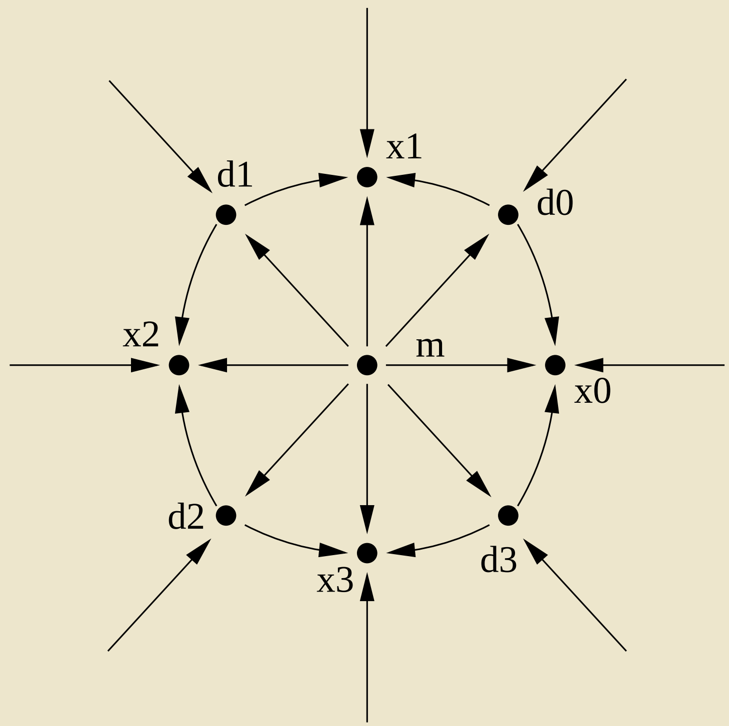
Let (X, ϕ) be a dynamical system an a locally compact metric space.
Let $S \subset X$ be an isolated invariant set and $(P, >)$ be a poset (partially ordered set).
Consider a Morse decomposition of S, i.e. a finite collection

$$\mathcal{M}(S) \; = \; \big\{ M(p) \mid p \in P \big\}$$

of disjoint isolated invariant subsets $M(p)$, such that for every $x \in S \setminus \bigcup_{p \in P} M(p)$ there exist $p, q \in P$, such that $q > p$ and $x \in \mathrm{Con}\big(M(q), M(p)\big)$ $\big($ = set of heteroclinic connections from $M(q)$ to $M(p)\big)$.

Example from the Cahn–Hilliard equation

The **Figure 1**



shows 9 equilibria of a flow from the Cahn–Hilliard equation. The almost circle–like center part is a global attractor $\mathcal{A}$.

We use

$$S := \mathcal{A} \quad \text{(isolated invariant set)}$$

$$M(p) \;=\; \text{Equilibrium } p, \;\text{ with } p \in \{x0, x1, x2, x3, d0, d1, d2, d3, m\} \;=:\; P$$

Partial ordering:

A. Flow induced order ">$_\varphi$": $q >_\varphi p \iff Con\big(M(q), M(p)\big) \neq \emptyset$

B. Energy induced order ">$_E$": let $E\colon X \to \mathbb{R}$ be monotone decreasing on trajectories

$$q >_E p\,, \;\text{ iff } E(y) > E(x) \text{ for all } y \in M(q) \text{ and } x \in M(p)\;.$$

Therefore $x_i < d_j < m$ for $<_\varphi$ as well as for $<_E$.

Homology Conley index

For an isolated invariant set $M \subset X$ the homology CONLEY index of M is defined by

$$CH_*(M) = H_*(N,L) \;=\; H_*\Big(N/L\,,\,[L]\Big)\;, \tag{1}$$

where $H_*(N,L) = \big(H_k(N,L)\big)_{k\in\mathbb{Z}_{\geq 0}}$ denotes the relative homology groups (we usually take coefficients in $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$), where N is an isolating neighborhood of M and L is an exit set (therefore (N,L) is an index pair for M).

If S is a hyperbolic fixed point with an unstable manifold of dimension n (i.e. MORSE index n), then

$$CH_k(S) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = n \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

In our example:

$$CH_*\big(M(x_i)\big) \;=\; \big(\mathbb{Z}_2, 0, 0, 0, \dots\big) \ , \qquad i \;=\; 0, 1, 2, 3$$

$$CH_*\big(M(d_i)\big) \;=\; \big(0, \mathbb{Z}_2, 0, 0, \dots\big) \ , \qquad j \;=\; 0, 1, 2, 3$$

$$CH_*\big(M(m)\big) \;=\; \big(0, 0, \mathbb{Z}_2, 0, \dots\big) \ .$$

Intervals

A subset $I \subset P$ is called an interval in $(P, >)$ if for all $p, q \in I$ and $r \in P$ the following implication holds:

$$q > r > p \quad \Longrightarrow \quad r \in I \ .$$

The set of all intervals in $(P, >)$ is denoted by $\mathcal{I}(P, >)$.
An n–tuple $(I_1, \ldots, I_n)\,,\; n \geq 2\,$, of intervals in $(P, >)$ is called adjacent if these intervals are mutually disjoint, $\bigcup_{i=1}^{n} I_i$ is an interval in $(P, >)$ and for all $p \in I_j\,,\; q \in I_k$ the following implication holds:

$$j < k \quad \Longrightarrow \quad p \not> q \ .$$

The set of all adjacent n–tuples of intervals in $(P, >)$ is denoted by $\mathcal{I}_n(P, >)$. If $(I_1, \ldots, I_n)$ is an adjacent n-tuple of intervals in $(P, >)$, then denote $I_1 I_2 \ldots I_n := \bigcup_{i=1}^{n} I_i$, which by definition is again an interval.

For an interval I define the set

$$M(I) := \bigcup_{p \in I} M(p) \cup \bigcup_{p,q \in I} \mathrm{Con}\big(M(q), M(p)\big) .$$

$M(I)$ is again an isolated invariant set. If $(I, J) \in \mathcal{I}_2(P, >)$, then $\big(M(I), M(J)\big)$ is an attractor–repeller pair in $M(IJ)$

In the **example:**

$$\begin{aligned} I &= \{x_i, i = 0, \ldots, 3\} \\ J &= \{d_j, j = 0, \ldots, 3\} \\ K &= \{m\} \end{aligned}$$

are all intervals for $(P, >_\varphi)$ as well as $(P, >_E)$. (I, J) and (J, K) are adjacent, but (I, K) is not adjacent. $M(IJ)$ is the boundary of the circle in Figure 1.

Connection matrices–algebraic part

Let $\mathcal{M}(S) = \{M(p) \mid p \in (P,>)\}$ be a Morse decomposition of S, with Conley indices $CH_*(M(p))$. We consider a group homomorphism

$$\Delta\colon \bigoplus_{p\in P} CH_*(M(p)) \to \bigoplus_{p\in P} CH_*(M(p)) \,.$$

For an interval I in $(P,>)$ set $C_*(I) := \bigoplus_{p\in I} CH_*(M(p))$ and denote by $\Delta(I)\colon C_*(I) \to C_*(I)$ the canonical restriction of Δ to $C_*(I)$, which may be represented as

$$\Delta(I) \;=\; \big(\Delta(p_1,p_2)\big)_{p_1,p_2\in I} : \bigoplus_{p\in I} CH_*(M(p)) \to \bigoplus_{p\in I} CH_*(M(p)) \,,$$

where $\Delta(p_1,p_2)\colon C_*(p_2) \to C_*(p_1)$.

Definition 1: [Franzosa 1988, Def. 1.3].

Δ being as above:

(1) Δ is said to be upper triangular, if $\Delta(p_1, p_2) \neq 0$ implies $p_2 > p_1$ or $p_1 = p_2$.

(2) Δ is called a boundary map if it is a homomorphism of degree -1, i.e. it maps $C_n(P)$ to $C_{n-1}(P)$, and $\Delta \circ \Delta = 0$.

In our example $C_*(P) = (\mathbb{Z}_2^4, \mathbb{Z}_2^4, \mathbb{Z}_2, 0, 0, \ldots)$ and

$$\Delta = \begin{array}{|cccc|cccc|c|l}
\mathbf{x}_0 & \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{d}_0 & \mathbf{d}_1 & \mathbf{d}_2 & \mathbf{d}_3 & m & \\
\hline
 & & & & * & * & * & * & & \mathbf{x}_0 \\
 & & & & * & * & * & * & & \mathbf{x}_1 \\
 & & & & * & * & * & * & & \mathbf{x}_2 \\
 & & & & * & * & * & * & & \mathbf{x}_3 \\
\cline{5-9}
 & & & & & & & & * & \mathbf{d}_0 \\
 & & & & & & & & * & \mathbf{d}_1 \\
 & & & & & & & & * & \mathbf{d}_2 \\
 & & & & & & & & * & \mathbf{d}_3 \\
\cline{9-9}
 & & & & & & & & & m \\
\hline
\end{array}$$

all other entries are zero because Δ is a boundary map.

Proposition: [Franzosa 1989, Prop. 3.3].
Let $\Delta\colon \bigoplus_{p\in P} CH_*\big(M(p)\big) \to \bigoplus_{p\in P} CH_*\big(M(p)\big)$ be an upper triangular boundary map. Then:

(1) $C_*(I)$ and $\Delta(I)$ form a chain complex denoted by $C_*^\Delta(I)$ for all $I \in \mathcal{I}(P,>)$ and therefore the homology groups

$$H_*\Delta(I) \;=\; H_*\big(C_*^\Delta(I)\big) \;:=\; \ker\Delta(I)/\operatorname{im}\Delta(I)$$

are well defined.

(2) Furthermore, for each $(I,J) \in \mathcal{I}_2(P,>)$ there is a long exact homology sequence

$$\cdots \to H_n\Delta(I) \to H_n\Delta(IJ) \to H_n\Delta(J) \xrightarrow{\delta_n} H_{n-1}\Delta(I) \to \cdots,$$

where δ_* are the connecting homomorphisms constructed by the snake lemma.

Connection matrices–dynamical part

For a pair (I, J) of adjacent intervals, $(M(I), M(J))$ is an attractor–repeller pair for the isolated invariant set $M(IJ)$, which has an index triple (N_2, N_1, N_0).
This index triple (N_2, N_1, N_0) is always guaranteed and provides a short exact sequence of chain complexes

$$0 \to \mathcal{C}_*(N_1, N_0) \to \mathcal{C}_*(N_2, N_0) \to \mathcal{C}_*(N_2, N_1) \to 0 \ ,$$

where $\mathcal{C}_*(N_i, N_j)$ is the complex of relative chains. This short exact sequence induces a long exact homology sequence

$$\cdots \to H_n(N_1, N_0) \to H_n(N_2, N_0) \to H_n(N_2, N_1) \xrightarrow{\partial_n} H_{n-1}(N_1, N_0) \to \cdots \ .$$

In other words the last long exact sequence is by definition (cf. (1))

$$\cdots \to CH_n\big(M(I)\big) \to CH_n\big(M(IJ)\big) \to CH_n\big(M(J)\big) \xrightarrow{\partial_n} CH_{n-1}\big(M(I)\big) \to \cdots \ .$$

3. The main result: the Re–Definition

Definition 2: (Connection matrix).

Let $\Delta\colon \bigoplus_{p\in P} CH_*(M(p)) \to \bigoplus_{p\in P} CH_*\big(M(p)\big)$ be an upper triangular boundary map. Δ is called a connection matrix if for each interval $K \in \mathcal{I}(P,>)$ there exits an isomorphism $\theta(K)\colon H_*\Delta(K) \to CH_*\big(M(K)\big)$ such that for all pairs $(I,J) \in \mathcal{I}_2(P,>)$ of adjacent intervals the following diagram

$$
\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\delta_{n+1}} & H_n\Delta(I) & \longrightarrow & H_n\Delta(IJ) & \longrightarrow & H_n\Delta(J) & \xrightarrow{\delta_n} & H_{n-1}\Delta(I) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \theta(I)} & & \downarrow{\scriptstyle \theta(IJ)} & & \downarrow{\scriptstyle \theta(J)} & & \downarrow{\scriptstyle \theta(I)} & & \\
\cdots & \xrightarrow{\partial_{n+1}} & CH_n(M(I)) & \longrightarrow & CH_n(M(IJ)) & \longrightarrow & CH_n(M(J)) & \xrightarrow{\partial_n} & CH_{n-1}(M(I)) & \longrightarrow & \cdots
\end{array}
$$

is an isomorphism of long exact sequences, i.e. that additionally all the squares commute.

Remarks:

A. We want to emphasize the importance of first choosing a fixed isomorphism $\theta(K)$ for each interval K. Notably, in Franzosa's definition of connection matrices also a fixed isomorphism $\theta(K)$ for each interval K has to be chosen a priori.

B. We show that this braid free definition coincides with Franzosa's definition of connection matrices:
In contrast to the above definition of connection matrices, Franzosa's definition requires the isomorphism of two graded module braids.

first braid: homology of a chain complex braid in the setup of the upper triangular boundary map Δ (purely algebraic)

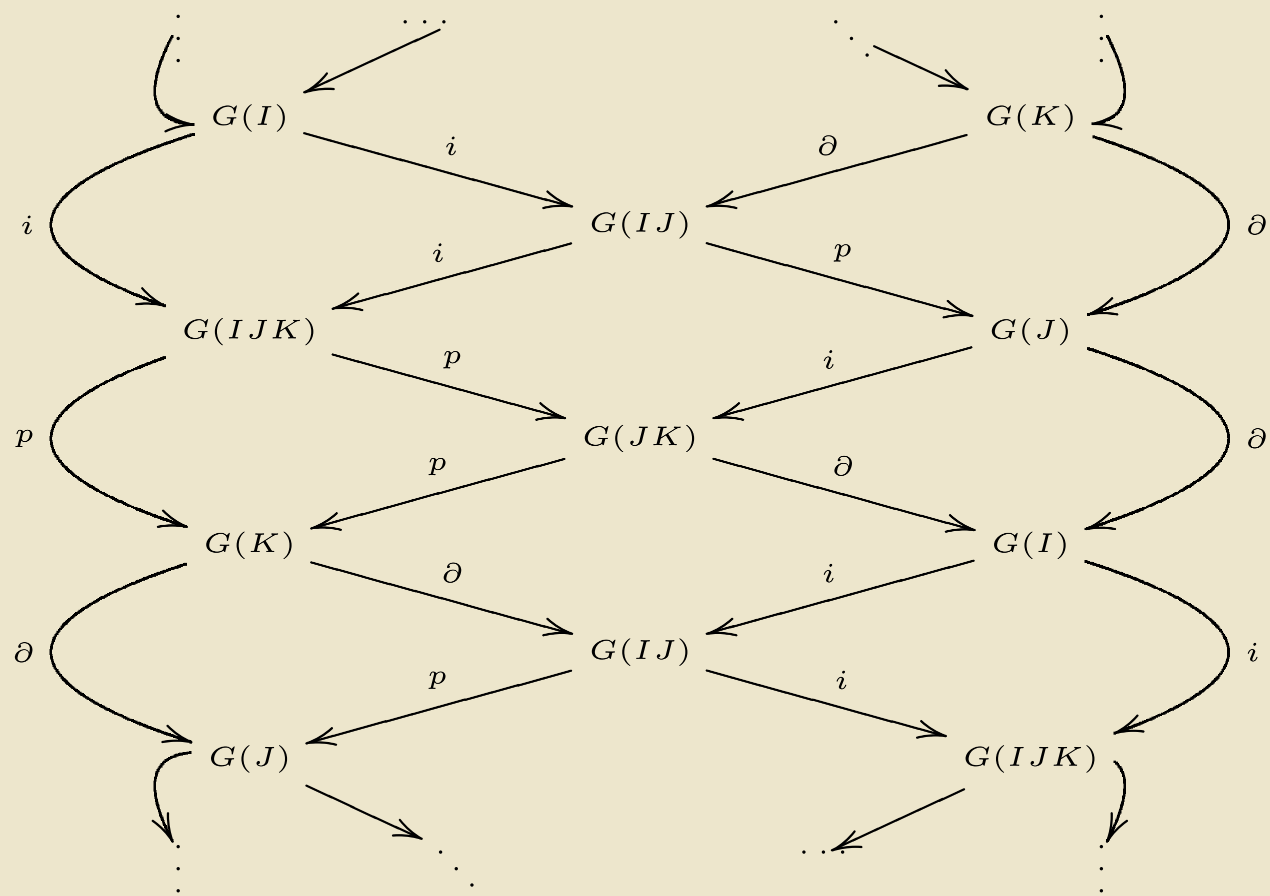
second braid: homology of the chain complex braid of an index filtration, which in turn generalizes our index triples (dynamical info)

Both braids are well known to exists (due to Franzosa, Salomon, Mischaikow)

Clearly, and because of the a priori chosen isomorphisms $\theta(K)$, the isomorphism of the long exact sequences in Definition 2 gives rise to the isomorphism of the graded module braids, as required by Franzosa.

Corollary: The definition of connection matrices following Franzosa [Franzosa, 1988, Def. 1.4] is equivalent to Definition 2 above.

For all $(I, J, K) \in \mathcal{I}_3(P, >)$ the following braid diagram commutes:



with $G(I) = H_n \Delta(I)$ or $G(I) = CH_n(M(I))$ and both braids are isomorph.