Constructing planar vector fields with many limit cycles

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DyToComp09

Outline

The aim of this talk is to :

- Present an accurate method to rigorously integrate over level curves
- Apply the method to compute lower bounds on the number of limit cycles that can bifurcate from a given planar polynomial Hamiltonian vector field under polynomial perturbation

Study perturbations of a Hamiltonian system of the form:

$$\begin{cases} \dot{x} &= -H_y(x,y) + \epsilon f(x,y) \\ \dot{y} &= H_x(x,y) + \epsilon g(x,y), \end{cases}$$

The weak formulation of Hilbert's 16th problem, asks for an upper bound on the number of limit cycles that can bifurcate as $\epsilon \to 0$.

For, $n = \deg(H) - 1 = \max(\deg(f), \deg(g))$
denote the upper bound $Z(n)$.
Only $Z(2) = 2$ is known.

- Many lower bounds on $Z(n)$ exist.
  Most of these use the *detection function* method by Li and Huang.
  Some of these are:
- $Z(3) \geq 12$ *(Yu and Han)*
- $Z(4) \geq 15$ *(Zhang)*
- $Z(5) \geq 24$ *(Chan)*
- $Z(6) \geq 35$ *(Wang and Yu)*
- $Z(7) \geq 49$ *(Li and Zhang)*
- $Z(9) \geq 80$ *(Wang et al)*
- $Z(11) \geq 121$ *(Wang and Yu)*.

- We describe a method to compute lower bounds on $Z(n)$, as an application we show that

$$Z(5) \geq 27.$$

Let,

$$\omega = f(x,y)\,dy - g(x,y)\,dx.$$

An *oval* is a closed level curve of H.
Let γ_h and D_h be such that

$$\begin{aligned} \gamma_h &= \{H = h\} \\ \partial D_h &= \gamma_h \end{aligned}$$

The Abelian integral, in general multi-valued, is defined as

$$I(h) = \int_{\gamma_h} \omega$$

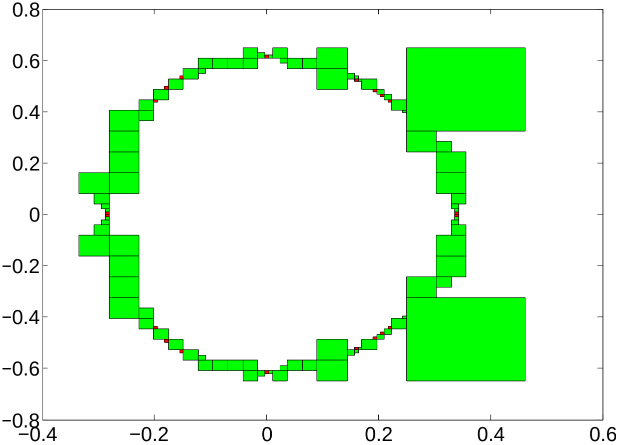
The connection between Abelian integrals and bifurcations is given by the Poincáre-Pontryagin theorem: Let P be the return map defined on some section transversal to the ovals of H.
$I(h)$ is a first order (in ϵ) approximation to $P - Id$.

It follows that a simple zero of $I(h)$ corresponds to a unique limit cycle bifurcating from the Hamiltonian system as $\epsilon \longrightarrow 0$.
To prove existence of a limit cycle, it suffices to have a zero of odd order.

Our approach to integrating monomials over implicitly defined closed curves in the plane is:

(i) Rewrite as surface integrals
(ii) Fill the domain with boxes
(iii) Cover the boundary with (relatively) large boxes
(iv) Contract the boxes on the boundary to piecewise linear tubular neighbourhoods
(v) On the boxes inside of the closed curve the integration is carried out exactly (up to roundoff)
(vi) The error depends on the area of the tubular neighbourhood

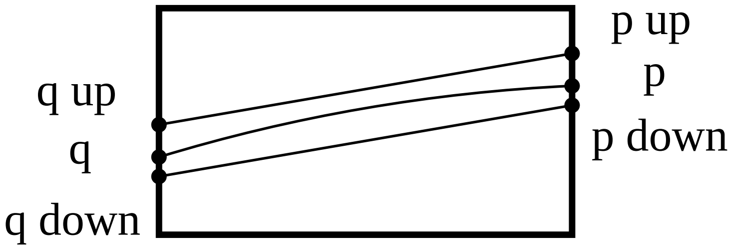
Enclosing an oval:



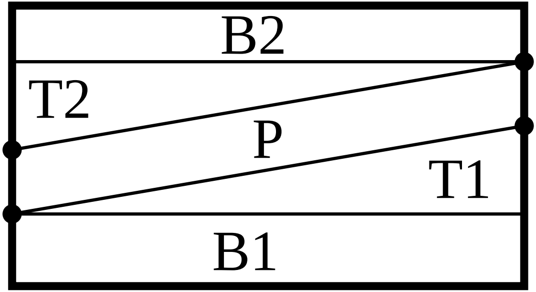
We split the cover into four parts:

- Boxes inside of the oval
- Tubular neighbourhoods of the oval (green)
- Boxes on the oval (red)
- Boxes outside of the oval

Enclosing an oval:



The change of variables splitting:



The value of the Abelian integral is enclosed by summing over all the computed integrals that are labelled as either `inside`, `fail`, or `on`.

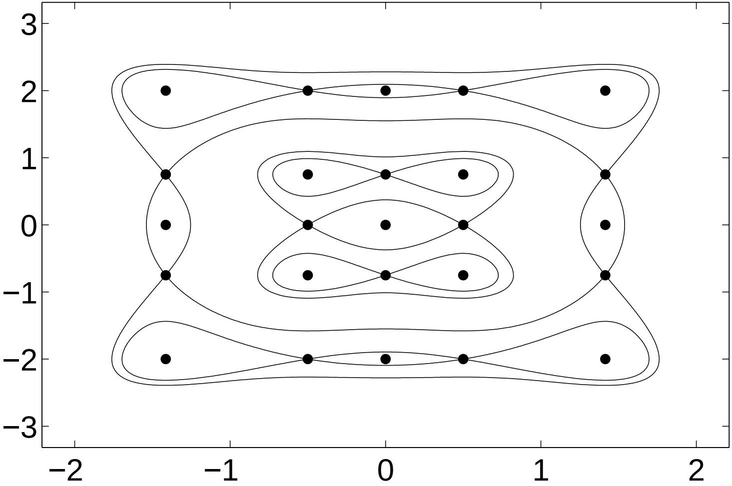
$$
\begin{aligned}
I(h) \in & \sum_{B \in \texttt{inside}} \int_B x^i y^j \, dx \wedge dy \\
+ & \sum_{T \in \texttt{inside}} \Box T_1^i \Box T_2^j |T| \\
+ & \sum_{B \in \texttt{fail}} \mathrm{Hull}(0, \int_B x^i y^j \, dx \wedge dy) \\
+ & \sum_{P \in \texttt{on}} \mathrm{Hull}\left(0, \Box P_1^i \Box P_2^j |P|\right)
\end{aligned}
$$

We study,

$$H(x,y) = \frac{x^2}{2} - \frac{9x^4}{8} + \frac{x^6}{3} + \frac{y^2}{2} - \frac{73y^4}{144} + \frac{2y^6}{27}$$

$$\begin{cases} \dot{x} & = -y\left(1 - \frac{16y^2}{9}\right)\left(1 - \frac{y^2}{4}\right) \\ \dot{y} & = x\left(1 - 4x^2\right)\left(1 - \frac{x^2}{2}\right) \end{cases}$$

Phase portrait of the Hamiltonian:

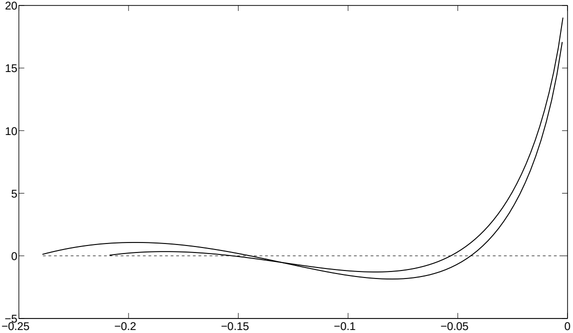


We study the following $\mathbf{Z}_2$ equivariant perturbation of the Hamiltonian system,

$$
\begin{aligned}
p(x,y) &:= \frac{\alpha_{00}}{2} + \frac{\alpha_{20}}{4}x^2 + \frac{\alpha_{02}}{4}y^2 + \frac{\alpha_{40}}{6}x^4 + \frac{\alpha_{22}}{6}x^2y^2 + \frac{\alpha_{04}}{6}y^4 \\
f(x,y) &:= xp(x,y) \\
g(x,y) &:= yp(x,y)
\end{aligned}
$$

We generate heuristically
$\alpha = (\alpha_{00}, \alpha_{20}, \alpha_{02}, \alpha_{40}, \alpha_{22}, \alpha_{04})$ by:

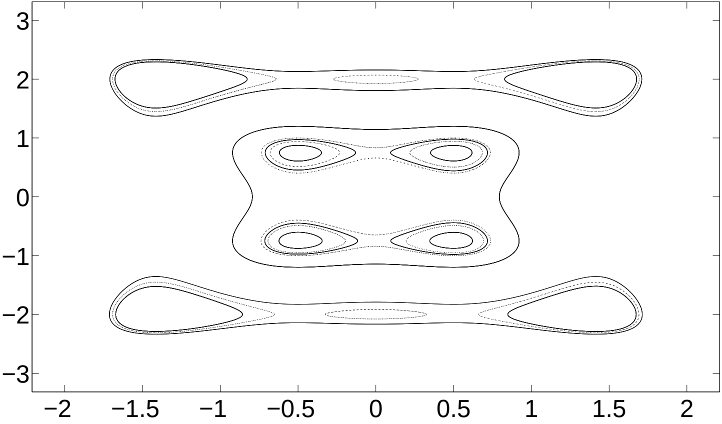
- Joint oscillation
- Local-Global attraction
- Separation of higher order zeroes


20
15
10
5
0
−5
−0.25
−0.2
−0.15
−0.1
−0.05
0

The generated coefficients:

α_{00}	2.176832745375219
α_{20}	0.203687169951339
α_{02}	-4.663680776344302
α_{40}	-8.410822908376025
α_{22}	4.313536179874701
α_{04}	1.000000000000000

The ovals, from which the limit cycles bifurcate:



Performance:

- Total number of integrals computed: 15
- Total run-time: 50 minutes
- Number of boxes containing a tubular neighbourhood $= 54863$
- Number of boxes with C^0-enclosure (`failed`) $= 1691 (\approx 3\%)$

Programs and papers are available at:

www.math.uu.se/∼johnson