A rigorous lower bound for the stability regions of the quadratic map

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DyToComp09

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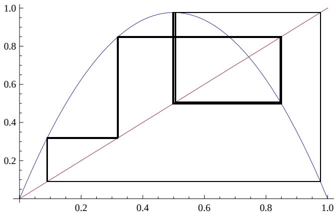
The quadratic map

The quadratic (logistic) map

$$Q_a\colon [0,1] \to [0,1] \qquad x \mapsto ax(1-x) \qquad a \in [0,4]$$



This map is arguably the most studied object in the theory of dynamical systems, beginning with the famous article by Robert May: *Simple mathematical models with very complicated dynamics*. Nature **261:459**, 1976.

Basic definitions

Definition

A parameter a is called *regular* if Q_a has an attracting cycle.

In this case the cycle is unique, and attracts almost all orbits in $[0, 1]$.

Definition

A parameter a is called *stochastic* if Q_a has an absolutely continuous invariant measure.

In this case, the measure is unique, and almost all orbits in $[0, 1]$ are asymptotically equidistributed with respect to it.

For convenience, let us denote the set of regular parameters by $\mathcal{R}$, and the stochastic parameters by $\mathcal{S}$.

Known facts about Q_a

Let $|\cdot|$ denote Lebesgue measure.

Known facts

(1) The set of regular parameters has positive measure: $|\mathcal{R}| > 0$;

- This follows immediately by openess of conditions.

(2) The set of stochastic parameters has positive measure: $|\mathcal{S}| > 0$;

- First result 1981 by Jakobson; improved 1985 by Benedicks and Carleson. Uses parameter exclusion principle.

(3) Almost every parameter $a \in [0, 4]$ is either *regular* or *stochastic*: $|\mathcal{R}| + |\mathcal{S}| = 4$.

- Established 2002 by Lyubich. Uses complex techniques.

Quantative questions

What are the sizes of $\mathcal{R}$ and $\mathcal{S}$?

The measure of $\mathcal{R}$ and $\mathcal{S}$

The stochastic parameters

The only non-trivial result is the (partial) lower bound by Luzzatto and Takahashi from 2006 (for $1 - bx^2$):

$$|\mathcal{S} \cap [2-\epsilon, 2]|/\epsilon > 0.97 \quad \text{with} \quad \epsilon = 10^{-4990}.$$

The regular parameters

Computationally much easier due to openess. A lower bound for $|\mathcal{R}|$ translates to an upper bound for $|\mathcal{S}|$ (and vice versa).

Theorem (Main Theorem)

The set of regular parameters for the quadratic map satisfies the lower bound: $|\mathcal{R} \cap [2, 4]| \geq 1.613942108535606042\,22.$

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The measure of $\mathcal{R}$ and $\mathcal{S}$

An upper bound for $|\mathcal{S}|$

We know that also $(0,1) \cup (1,2] \subset \mathcal{R}$, so we get the upper bound

$$|\mathcal{S}| < 4 - 3.61394210853560604222 < 0.38605789146439395778$$

Question

How good is this bound, really?

The first period doubling cascade ends at $a^* \approx 3.56994567187083$. Beyond this, we show that ca 10.23% of the parameters are regular.

According to non-rigorous numerical experiments (Simó and Tatjer 1991) the regular parameters in $[a^*, 4]$ make up no more than 10.66%. The comparison is non-trivial since they use a different map: $x \to 1 - bx^2$.

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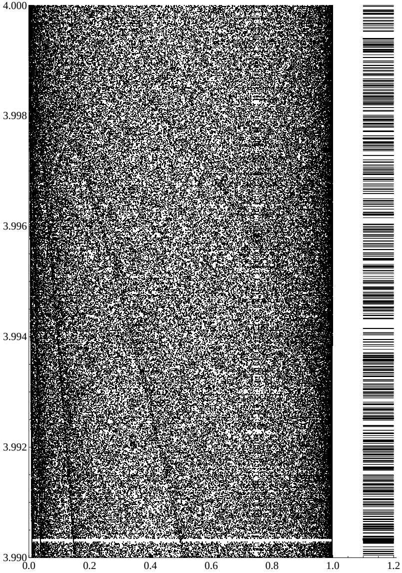
The measure of $\mathcal{R}$



We also prove the existence of period doubling bifurcations.

Although $|PD| = 0$, we spend effort on this step.

This produces larger connected parameter sets within each period doubling cascade, and thus adds measure to our final bound.

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Strategy of the proof

We use the following three methods for verifying the existence of stable orbits:

- the Brouwer theorem,
- the method of backward shooting,
- the modified interval Krawczyk operator.

The three methods are increasing in computational complexity, and therefore, when prove the existence of a stable orbit, we first use the Brouwer theorem. If this method fails, we apply the method of backward shooting. If we still have no success, we switch to the modified interval Krawczyk method, provided the assumptions of this method are satisfied.

Computational approach

Where and why

- Brouwer's method is very fast, but has some disadvantages: the condition $Q_{\mathbb{A}}^{p}(\mathbb{X}_1) \subset \mathbb{X}_1$ might fail due to the width of $\mathbb{A}$. This also forces $\mathbb{X}_1$ to be "large", which makes stability hard to establish close to a bifurcation parameter.
- The backward shooting relies upon Newton's method, and thus requires C^1-computations. It breaks down on/near superstable periodic orbits.
- In general, Krawczyk's method has a cubic (in period) complexity, which is prohibitive. But using special structures from the superstable setting, we can obtain quadratic complexity only.
- Period doublings can be established/isolated by verifying some inequalities involving the derivatives of $Q_a(x)$.

Backward shooting

Given a smooth function $F : \mathbb{R}^n \to \mathbb{R}^n$, an interval vector $\mathbb{X}$, and a point $x \in \mathbb{X}$, we define the *interval Newton operator* by

$$N(F, x, \mathbb{X}) = x - [DF]^{-1}(\mathbb{X})F(x). \tag{1}$$

Theorem

Let $\mathbb{X}$ be an interval vector, $x \in \mathbb{X}$, and $F : \mathbb{R}^n \to \mathbb{R}^n$ be smooth. Assume that $DF(\mathbb{X})$ is invertible as an interval matrix. If the interval Newton operator satisfies

$$N(F, x, \mathbb{X}) \subset \mathbb{X}$$

then the map F has a unique zero x^* in the box $\mathbb{X}$. Moreover, $x^* \in N(F, x, \mathbb{X})$.

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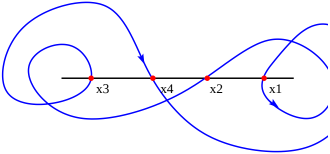
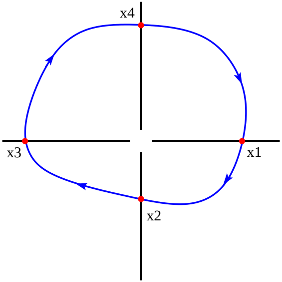
Backward shooting

Main idea: Trade iterations for dimension.

Instead of solving the scalar problem $f^p(x) = x$, we solve the $p-$dimensional problem $F(x) = 0$, where

$$F(x_1, \ldots, x_p) = (x_2 - f(x_1), \ldots, x_p - f(x_{p-1}), x_1 - f(x_p)). \quad (2)$$





A period$-p$ orbit of f corresponds to a zero of F. We use $f = Q_a$.

Backward shooting

Theorem (Galias'02)

Let $\mathbb{X} = (\mathbb{X}_1, \ldots, \mathbb{X}_p)$ *be an interval vector,* $x = (x_1, \ldots, x_p) \in \mathbb{X}$*, and let* $f : \mathbb{R} \to \mathbb{R}$ *be a smooth map. Define* F *as in (2). Assume that (for* $k = 1, \ldots, p$*)* $\mathbb{S}_k$*,* $\mathbb{G}_k$*, and* $\mathbb{H}_k$ *are intervals such that*

$$f'(\mathbb{X}_k) \subset \mathbb{S}_k \ \textit{and}\ 0 \notin \mathbb{S}_k$$

$$f(x_k) - x_{(k \mod p)+1} \in \mathbb{G}_k$$

$$\left(1 - \mathbb{S}_1^{-1} \cdots \mathbb{S}_p^{-1}\right) \sum_{i=1}^{p} \mathbb{S}_1^{-1} \cdots \mathbb{S}_i^{-1} \mathbb{G}_i \subset \mathbb{H}_1$$

$$\mathbb{S}_k^{-1} \left(\mathbb{H}_{(k \mod p)+1} + \mathbb{G}_k\right) \subset \mathbb{H}_k, \quad (k = 2, \ldots, p)$$

Then $[DF]^{-1}(\mathbb{X})F(x) \subset \mathbb{H}$*, and* $N(F, x, \mathbb{X}) \subset x - \mathbb{H}$*.*

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Superstable orbits

For superstable orbits, the backward shooting approach fails since $0 \in \mathbb{S}_i$ for some i. Assuming that $Q_a'(x_1) = 0$, we have

$$DF(x_1, \ldots, x_p) = \begin{bmatrix} \mathbf{0} & 1 & 0 & \ldots & 0 \\ 0 & -Q_a'(x_2) & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -Q_a'(x_{p-1}) & 1 \\ 1 & 0 & \cdots & 0 & -Q_a'(x_p) \end{bmatrix}.$$

This means that the linear equation $DF(x) \cdot y = z$ has the solution

$$\begin{cases} y_2 &= z_1 \\ y_3 &= z_2 + Q_a'(x_2)y_2 \\ \vdots & \\ y_p &= z_{p-1} + Q_a'(x_{p-1})y_{p-1} \\ y_1 &= z_p + Q_a'(x_p)y_p \end{cases} \tag{3}$$

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Superstable orbits - Krawczyk's method

But for intervals $\mathbb{A}$ and $\mathbb{X}_1$, we only have inclusion: $0 \in Q'_{\mathbb{A}}(\mathbb{X}_1)$. Thus we cannot solve explicitly for the Newton correction term, and Gaussian elimination has high ($\mathcal{O}(p^3)$) complexity.

A modified Krawczyk operator

For superstable orbits, we use Galias' approach applied to the interval Krawczyk operator.

The interval Krawczyk operator is defined by

$$K(F, x, \mathbb{X}, C) = x - C \cdot F(x) + (\mathrm{Id} - C \cdot DF(\mathbb{X}))(\mathbb{X} - x). \quad (4)$$

Theorem

Let C be an invertible matrix, and let $x \in \mathbb{X} \in \mathbb{IR}^p$. If the interval Krawczyk operator (4) *satisfies*

$$K(F, x, \mathbb{X}, C) \subset \operatorname{int} \mathbb{X}$$

then F has a unique zero $x^* \in \mathbb{X}$. Moreover, $x^* \in K(F, x, \mathbb{X}, C)$.

Superstable orbits - Krawczyk's method

Our strategy is to form the matrix $J \approx DF(x)$ with $J_{11} = 0$.

Taking $C = J^{-1}$, the value of $C \cdot F(x)$ can be computed via (3).

What remains for us to show is the algorithm for the computation of $(\mathrm{Id} - C \cdot DF(\mathbb{X}))(\mathbb{X} - x)$.

Lemma

Assume that Algorithm 1 is called with its arguments, and assume that $s_1 = 0$. Then the algorithm always stops and returns an interval vector $\mathbb{Y}$ which is an enclosure for the interval vector $(\mathrm{Id} - C \cdot D)(\mathbb{X} - x)$, where

$$D = \begin{bmatrix} -\mathbb{S}_1 & 1 & 0 & \dots & 0 \\ 0 & -\mathbb{S}_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\mathbb{S}_{p-1} & 1 \\ 1 & 0 & \dots & 0 & -\mathbb{S}_p \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & -s_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -s_{p-1} & 1 \\ 1 & 0 & \dots & 0 & -s_p \end{bmatrix}^{-1}$$

Superstable orbits - Krawczyk's method

Algorithm 1: Modified interval Krawczyk method ($\mathcal{O}(p^2)$)

Data: double $s_1, \ldots, s_p$;
interval $\mathbb{S}_1, \ldots, \mathbb{S}_p$;
vector $x = (x_1, \ldots, x_p)$;
box $X = (\mathbb{X}_1, \ldots, \mathbb{X}_p)$;

begin
  box $\mathbb{Y} = (\mathbb{Y}_1, \ldots, \mathbb{Y}_p)$;
  $\mathbb{Y} \leftarrow 0$;
  for $i \leftarrow 1$ **to** p **do**
    interval $\sigma \leftarrow \mathbb{S}_i - s_i$;
    $\mathbb{Y}_{(i \mod p)+1} \leftarrow \mathbb{Y}_{(i \mod p)+1} + \sigma \times (\mathbb{X}_i - x_i)$;
    for $j \leftarrow i + 1$ **to** p **do**
      $\sigma \leftarrow \sigma \times s_j$;
      $\mathbb{Y}_{(j \mod p)+1} \leftarrow \mathbb{Y}_{(j \mod p)+1} + \sigma \times (\mathbb{X}_j - x_j)$;
  return $\mathbb{Y}$;
end

Putting it all together

Our method results in a non-uniform partition of the original search domain $\mathbb{A} = \cup_{i \in \mathcal{I}} \mathbb{A}_i$.

Brief strategy

- Search for superstable orbits of some maximal length N, using the midpoint of $\mathbb{A}_i$ as seed.
- Once found, estimate the size of the corresponding periodic window.
- Try to verify that the endpoints have periodic orbits with the correct period.
- Try to fill in the entire window.
- Scan for possible period doublings.
- Merge compatible parameter domains.

Computational results

Implementation

The algorithm was coded in C++ using the CAPD library. The program was run on *the machine* - a 32 core HP DL785 G5 (8 AMD Opteron 8354 processors) equipped with 32 GB DDR2 RAM.

level	N	measure	wall time (h:m:s)	per. doublings
1	256	1.60620127942955014935	2 : 05 : 05	14
2	256	1.61118596303518551971	3 : 05 : 08	78
3	256	1.61287812977207803823	1 : 43 : 51	335
4	256	1.61349027421762439933	2 : 39 : 44	1381
5	256	1.61372268390076635341	5 : 18 : 22	4019
6	256	1.61381346643292467144	8 : 47 : 48	9075
7	512	1.61389940246044893686	67 : 26 : 04	20128
8	512	1.61391413966146151119	95 : 46 : 06	41692

Table: Here, the reported wall time corresponds to the current subdivision level only – not the accumulated time. The listed values of the measure and period doublings, however, correspond to the accumulated amount from all previous levels.

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Computational results

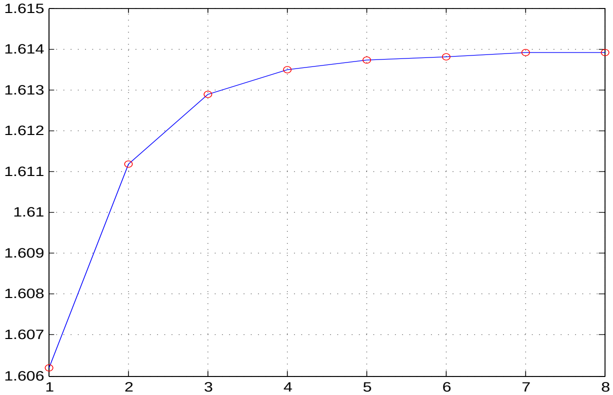

Figure: A plot of the search level versus the verified measure.

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Computational results - verified periods

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37
38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71
72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104
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544 546 550 552 556 558 560 561 564 567 570 572 574 575 576 578 580 585 588 592 594 595 598 600 605 608
609 612 616 620 621 624 625 627 630 637 638 640 644 646 648 650 656 660 665 666 672 675 676 680 682 684
686 688 690 693 696 700 702 704 710 714 715 720 722 726 728 729 730 735 736 740 744 748 750 752 754 756
759 760 765 768 770 780 782 784 792 798 800 805 810 812 816 819 820 825 828 832 833 836 840 850 855 858
864 868 870 875 880 882 884 891 896 900 910 912 918 920 924 928 930 931 935 936 938 940 945 950 952 960
966 968 972 975 980 984 988 990 992 1000 1001 1008 1012 1014 1020 1024 1040 1050 1056 1072 1080 1088
1092 1100 1104 1120 1125 1134 1140 1144 1152 1170 1176 1184 1188 1200 1215 1216 1224 1232 1248 1260 1280
1296 1300 1320 1344 1350 1352 1360 1368 1372 1386 1392 1400 1408 1428 1440 1456 1458 1500 1512 1520 1536
1540 1560 1568 1584 1600 1620 1632 1664 1680 1728 1744 1760 1764 1792 1800 1824 1848 1872 1904 1920 1944
1960 1980 2000 2016 2040 2048 2080 2100 2112 2156 2160 2176 2184 2200 2240 2268 2304 2340 2352 2376 2400
2464 2496 2520 2560 2592 2640 2688 2700 2720 2800 2816 2880 2912 2940 3000 3024 3040 3072 3120 3136 3168
3200 3240 3328 3360 3456 3520 3528 3584 3600 3640 3696 3780 3840 3888 3920 3960 4000 4032 4096 4160 4200
4224 4320 4416 4480 4500 4608 4704 4752 4800 4928 4992 5000 5040 5120 5184 5280 5376 5400 5600 5632 5760
6000 6048 6144 6272 6300 6336 6400 6480 6720 6912 7040 7168 7200 7680 7776 7840 8000 8064 8192 8400 8448
8640 8960 9216 9408 9600 10080 10240 10368 10752 11200 11264 11520 12288 12544 12800 13440 13824 14336
15360 16000 16128 16384 17280 17920 18432 19200 20480 21504 23040 24576 25600 27648 28672

Computational results

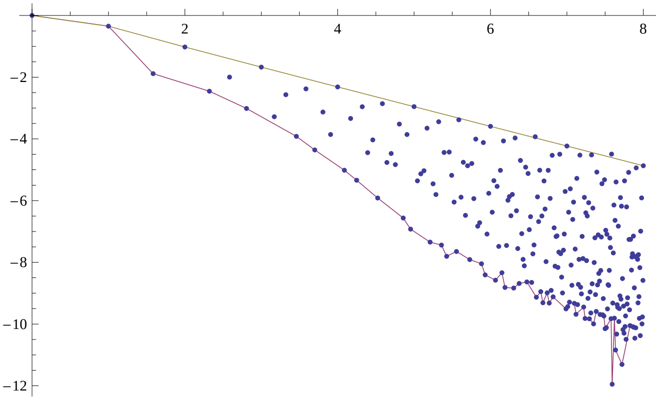

Figure: A plot of $\log_2$ of the period versus $\log_{10}$ of the verified measure. Note how the prime periods form the lower line, whereas periods of the form 2^k form the upper line. Also note the lack of some primes after period 222.

Computational results

Tweaking the measure

- In order to obtain the stability measure reported in the Main Theorem, we first compute according to Table 1.
- We also compute through 18 subdivision levels with the restriction on the maximal period set to 33000, but with no searching for superstable orbits. (ca one year of CPU time.)
- The gain is not impressive: $\approx 3 \times 10^{-5}$.

CONJECTURE:

The set $\mathcal{S}$ has measure near 0.3860579.