Rigorous results on short periodic orbits for the Lorenz system

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Finding periodic orbits

- Reduction to the discrete case: the Poincaré map P defined by the section Σ,
- periodic orbits of length n: zeros of $g = \mathrm{id} - P^n$,
- a better choice: use the map $F\colon \Sigma^n \mapsto \Sigma^n$

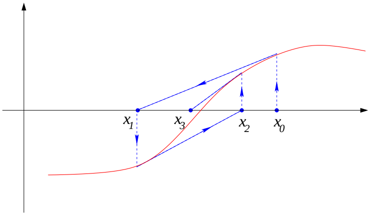
$$[F(z)]_k = x_{(k+1) \bmod n} - P(x_k) \quad \text{for } 0 \le k < n \ ,$$

  where $z = (x_0, \ldots, x_{n-1})$. $F(z) = 0$ if and only if x_0 is a fixed point of f^n.
- **Motivation:** The high-dimensional space in Σ^n compensates for the problems associated with the long integration times needed for P^n.

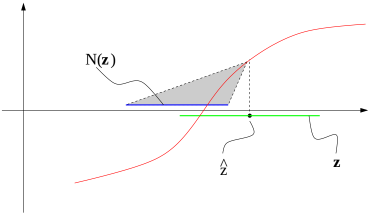
Interval operators for zero finding

- The interval Newton operator: $\mathrm{N}(\mathbf{z}) = \hat{z} - (F'(\mathbf{z}))^{-1}F(\hat{z})$, where $\hat{z} \in \mathbf{z}$. One usually chooses $\hat{z} = \mathrm{mid}(\mathbf{z})$.

Standard Newton



Interval Newton



- Existence and uniqueness of zeros:
  If $\mathrm{N}(\mathbf{z}) \cap \mathbf{z} = \emptyset$, then $\mathbf{z}$ contains no roots of F.
  If $\mathrm{N}(\mathbf{z}) \subseteq \mathrm{int}\mathbf{z}$, then $\mathbf{z}$ contains exactly one root of F.

Interval operators for zero finding

The Krawczyk operator avoids inverting an interval matrix:

$$\mathrm{K}(\mathbf{z}) = \hat{z} - CF(\hat{z}) - (CF'(\mathbf{z}) - I)(\mathbf{z} - \hat{z}),$$

We will use $\hat{z} = \mathrm{mid}(\mathbf{z})$, $C = (F'(\hat{z}))^{-1}$.
In order to evaluate K, we must be able to compute enclosures of the return map P and its partial derivatives P' over sets (rectangles in Σ).

Finding all short cycles

- Find a trapping region,
- generate a graph representation,
- find all cycles in the graph of length n,
- for each cycle generate an interval vector $\mathbf{z}$, evaluate the interval operator $\mathrm{K}(\mathbf{z})$ and check the existence condition,
- decreasing the number of cycles to be verified: a refinement technique
  - after generation of cycles the size of boxes is increased,
  - an example (period–8 orbits for the Roessler system)
    - the number of cycles of length 8 for boxes of size $(0.0125, 0.000025)$ is 114106,
    - for $\varepsilon = (0.0015625, 0.000003125)$ there are 21655 cycles of length 8, after increasing the box size 8 times, $\varepsilon = (0.0125, 0.000025)$ there are 27 cycles,

The Roessler system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_1 + ax_2 \\ b + x_3(x_1 - c) \end{pmatrix},$$

- parameter values: $a = 0.2$, $b = 0.2$, $c = 5.7$,
- the Poincaré map P, $\Sigma = \{x \in \mathbb{R}^3 \colon x_1 = 0, \dot{x}_1 > 0\}$,
- the graph representation (the number of boxes b, the number of nonforbidden transitions c),

box size	b	c
$(0.1, 0.0002) \times 2^{-2}$	793	4391
$(0.1, 0.0002) \times 2^{-4}$	2808	12323
$(0.1, 0.0002) \times 2^{-6}$	10477	43681
$(0.1, 0.0002) \times 2^{-8}$	52167	241556

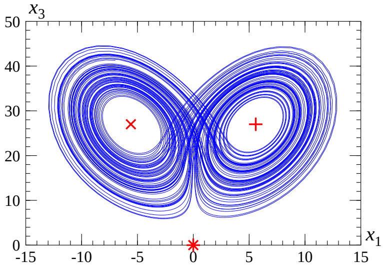
The Roessler system: periodic orbits with period $n \leq 20$

n	Q_n	P_n	H_n	C_n	D_n
1	1	1	0	1	1
2	1	3	0.54931	7	2
3	2	7	0.64864	17	3
4	1	7	0.48648	30	3
5	2	11	0.47958	43	4
6	3	27	0.54931	287	8
7	4	29	0.48104	220	5
8	7	63	0.51789	1972	10
9	10	97	0.50830	3475	18
10	15	163	0.50938	12842	28
11	24	265	0.50725	29896	34
12	36	463	0.51148	116437	65
13	58	755	0.50975	469835	111
14	88	1263	0.51009	1041859	164
15	138	2087	0.50957	2678303	297
16	216	3519	0.51037	9290861	542
17	340	5781	0.50955	24156174	780
18	531	9675	0.50985	75523431	1442
19	848	16113	0.50986	222765071	4584
20	1330	26767	0.50975	649120739	3911

The Lorenz equations

Introduced in 1963:

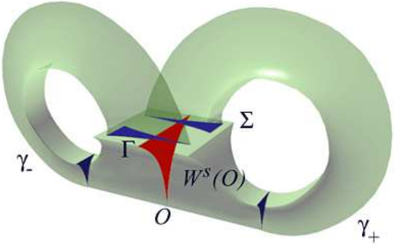
$$\dot{x}_1 = -\sigma x_1 + \sigma x_2$$
$$\dot{x}_2 = \varrho x_1 - x_2 - x_1 x_3$$
$$\dot{x}_3 = -\beta x_3 + x_1 x_2,$$

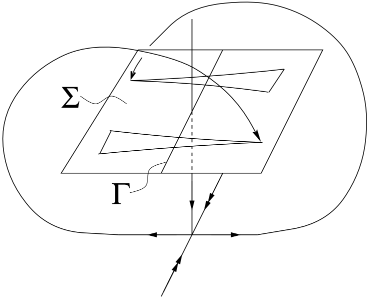


Classical parameters: $\sigma = 10$, $\beta = 8/3$, $\varrho = 28$.
Symmetry: $S(x_1, x_2, x_3) = (-x_1, -x_2, x_3)$.

The geometric model

Introduced by Guckenheimer and Williams (1979).
Assumes certain properties of the Poincaré map $P\colon \Sigma \to \Sigma$.

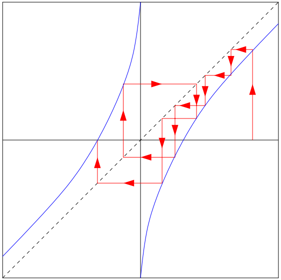




The geometric model

Projecting along stable leaves gives a 1-d function $f\colon [-1,1] \to [-1,1]$ satisfying:

- $f(-x) = -f(x)$,
- $\lim_{x\to 0} f'(x) = +\infty$,
- $f''(x) < 0$ on $(0,1]$,
- $f'(x) > \sqrt{2}$.

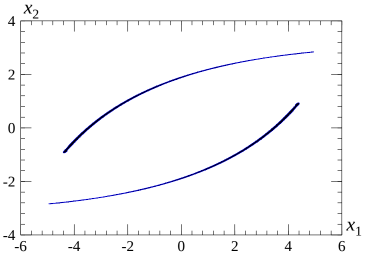


It follows that f is topologically transitive on $[-1,1]$.

Theorem [Tucker-98]: For the classical parameter values, the Lorenz equations support a robust strange attractor — the Lorenz attractor.

The graph representation

- Consider the Poincaré map P defined by the section $\Sigma = \{x = (x_1, x_2, x_3) \colon x_3 = r - 1, \dot{x}_3 < 0\}$,
- there is a trapping region $N \subset \Sigma$ for the map P. N is composed of 14518 boxes of size $1/2^7 \times 1/2^7$,



- the dynamics of P is represented in the form of a directed graph (N, E), where boxes N_i are the vertices of the graph, and non-forbidden connections are graph edges:

$$E = \{(i, j) \colon P(N_i) \cap N_j \neq \emptyset\},$$

$\#E = 514126$.

Average return time

- Using information on return times for individual boxes, we can find lower bounds t_n of the return time for P^n.
- the flow–time of any periodic orbit corresponding to a period–n cycle of P is larger than $n \cdot 0.6397$.

n	t_n	t_n/n
1	0.537044	0.537044
2	1.147434	0.573717
5	2.981325	0.596265
10	6.047894	0.6047894
100	63.721019	0.63721019
10000	6397.362682	0.6397362682

Finding all short periodic orbits using graph representation

- All period–n cycles in the graph are found.
- For each period–n cycle we evaluate the Krawczyk operator K, and check for existence/uniqueness: $\mathrm{K}(\mathbf{z}) \subset \mathbf{z}$ or $\mathrm{K}(\mathbf{z}) \cap \mathbf{z} = \emptyset$.
- Application of the standard method:
  - no fixed points,
  - one self–symmetric period–2 orbit,
  - for $p > 2$ the method failed,
- An additional section:
  - a pair of symmetric period–3 orbits,
  - three period–4 orbits: a pair of symmetric orbits, and one self–symmetric orbit.
  - For $p > 4$ the method failed: more sections needed, computation time increases very fast with p.

Using dynamical information for finding all short orbits

- γ — the first intersection of the (two-dimensional) stable manifold of the origin with the return plane Σ,
- labelling trajectories: if the trajectory intersects Σ to the left (right) of γ, then the intersection point is labelled with L (R).
- Fact: A periodic symbol sequence corresponds to *at most* one periodic orbit.
- Consequence: We know how many period-n orbits to expect.
- Strategy: Locate all period–n orbits by non-rigorous, heuristic methods. Verify the existence with the Krawczyk operator.
- Potential: There are 111011 periodic symbol sequences of length ≤ 20 [Viswanath, 2003].

Short periodic orbits for the Lorenz system

- Finding all short periodic orbits:
  - a trajectory of P composed of 1000000 points was generated,
  - all symbol sequences of length $p \leq 14$ were generated, for each of them an approximate position of the corresponding periodic orbit is guessed using the symbol information from the data set, a Newton iteration is used to improve the approximation and finally the existence of a nearby true periodic orbit is proved using the Krawczyk operator,
  - we have shown that for each symbol sequence s of length $p = 2, 3, \ldots, 14$ with the minimum period p there exists one periodic point of P with the symbol sequence s.

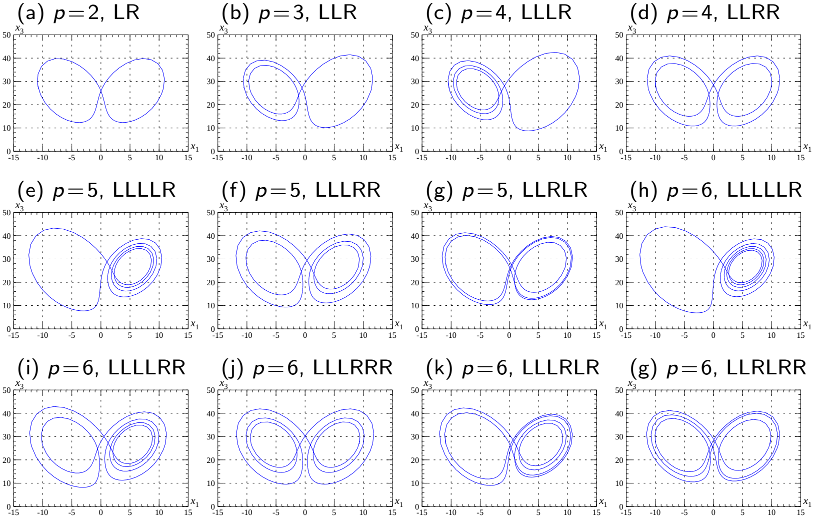
Results

- There are 2536 periodic orbits of P with period $p \leq 14$.

p	n_p	flow times
2	1	$[1.5586, 1.5587]$
3	2	$[2.3059, 2.3060]$
4	3	$[3.0235, 3.0843]$
5	6	$[3.7256, 3.8696]$
6	9	$[4.4177, 4.6372]$
7	18	$[5.1030, 5.4292]$
8	30	$[5.7834, 6.1947]$
9	56	$[6.4602, 6.9880]$
10	99	$[7.1346, 7.7531]$
11	186	$[7.8073, 8.5467]$
12	335	$[8.4792, 9.3117]$
13	630	$[9.1509, 10.1054]$
14	1161	$[9.8231, 10.8703]$

- The flow-time of any other periodic orbit of P is larger than $15 \cdot 0.6397 = 9.5955$

Periodic orbits with period $p \leq 6$



Longer periodic orbits

- The method can be used to prove the existence of longer periodic orbits. Examples: