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A note on uniquely embeddable cycles

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Abstract

Let C_n be a cycle of order n. It is well known that if $n \geq 5$ then there is an embedding of C_n into its complement $\overline{C_n}$. In this note we consider a problem concerning the uniqueness of such an embedding.

1 Introduction

We shall use standard graph theory notation. We consider only finite, undirected graphs of order $n = |V(G)|$ and size $e(G) = |E(G)|$. All graphs will be assumed to have neither loops nor multiple edges.

We shall need some additional definitions in order to formulate the results. If a graph G has order n and size m, we say that G is an (n, m) graph.

Assume now that G_1 and G_2 are two graphs with disjoint vertex sets. The *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If a graph is the union of n (≥ 2) disjoint copies of a graph H, then we write $G = nH$.

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For our next operation, the conditions are quite different. Let now G_1 and G_2 be graphs with $V(G_1) = V(G_2)$ and $E(G_1) \cap E(G_2) = \emptyset$. The *edge sum* $G_1 \oplus G_2$ has $V(G) = V(G_1) = V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

An *embedding* of G (in its complement $\overline{G}$) is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$.

In others words, an embedding is an (edge-disjoint) *placement* (or *packing*) of two copies of G into a complete graph K_n.

The following theorem was proved, independently, in [1], [2] and [5].

Theorem 1 *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \leq n - 2$ then G can be embedded in its complement $\overline{G}$.* ■

The example of the star $K_{1,n-1}$ shows that Theorem 1 cannot be improved by raising the size of G. However if a tree is not a star then it is embeddable. This fact was first observed by H.J.Straight [unpublished]. The version given below comes from [7].

Theorem 2 *Let T be a non-star tree. Then there exists a cyclic permutation on $V(T)$ being an embedding of T.* ■

Let us consider now the problem of the uniqueness. First, we have to precise what we mean by *distinct* embeddings.

Let σ be an embedding of the graph $G = (V, E)$. We denote by $\sigma(G)$ the graph with the vertex set V and the edge set $\sigma^*(E)$ where the map σ^* is induced by σ. Since, by definition of an embedding, the sets E and $\sigma^*(E)$ are disjoint we may form the graph $G \oplus \sigma(G)$.

Two embeddings σ_1, σ_1 of a graph G are said to be *distinct* if the graphs $G \oplus \sigma_1(G)$ and $G \oplus \sigma_2(G)$ are not isomorphic. A graph G is called *uniquely embeddable* if for all embeddings σ of G, all graphs $G \oplus \sigma(G)$ are isomorphic.

The next theorem, proved in [8], characterizes all $(n, n-2)$ graphs that are uniquely embeddable.

Theorem 3 *Let G be a graph of order n and size $e(G) = n - 2$. Then either G is not uniquely embeddable or G is isomorphic to one of the seven following graphs (see also Fig. 1): $K_2 \cup K_1$, $2K_2$, $K_3 \cup 2K_1$, $K_3 \cup K_2 \cup K_1$, $C_4 \cup 2K_1$, $K_3 \cup 2K_2$, $2K_3 \cup 2K_1$.* ■

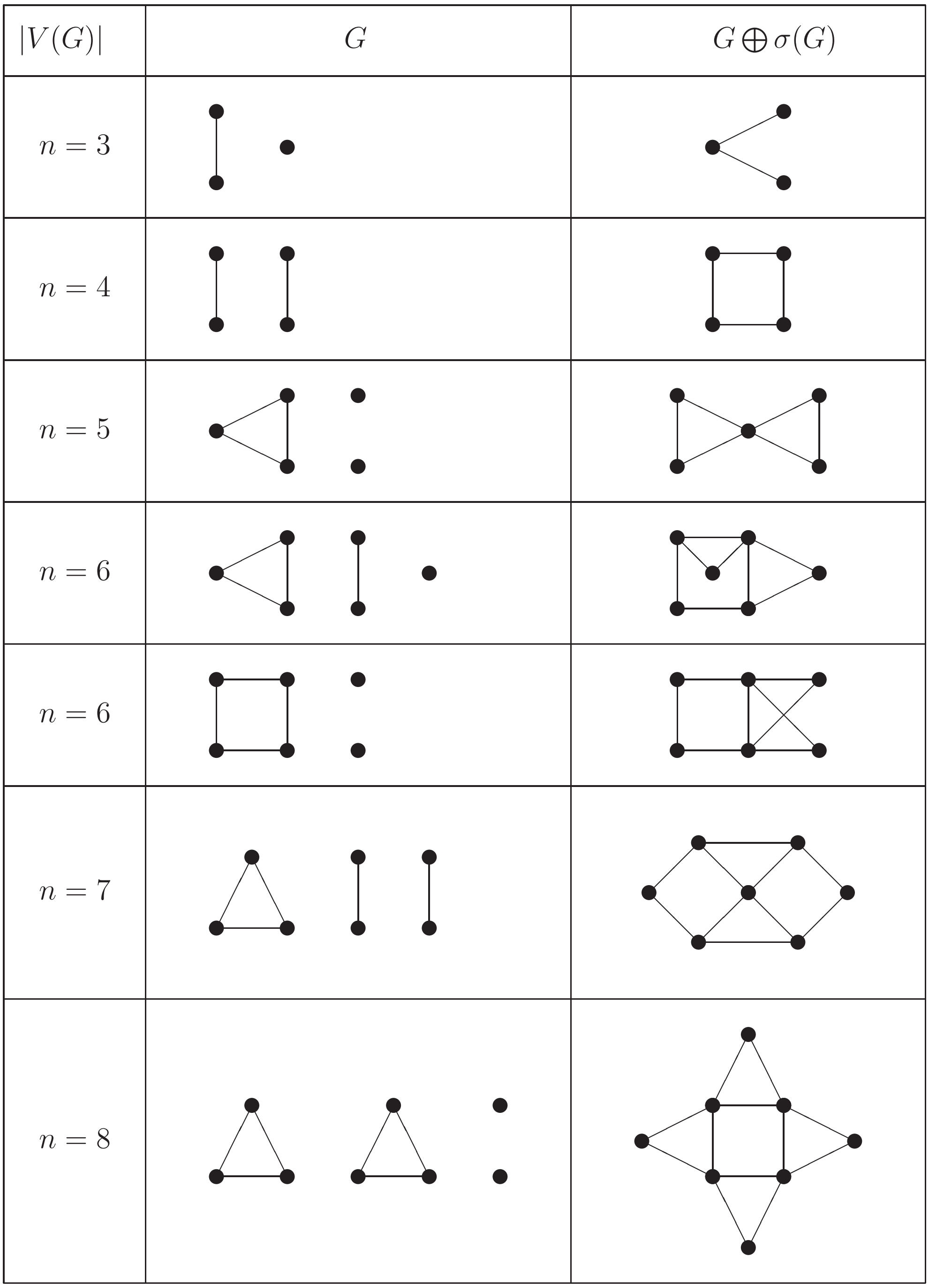

Figure 1: Uniquely embeddable $(n, n-2)$-graphs

The aim of this note is to consider the problem for cycles. We have the following

Theorem 4 *Let C_n be a cycle of order n. The cycles C_3 and C_4 are not embeddable. The cycles C_5 and C_5 are uniquely embeddable. For $n \geq 7$ there exist at least two distinct embeddins of C_n.*

The proof of Theorem 4 is given in the next section.

Remark. The main references of the paper and of other packing problems are the following survey papers [11], [9] or [10].

2 Proof of Theorem 5

It is easy to see that neither C_3 nor C_4 is embeddable.

The cycle C_5 is embeddable but for each embedding σ we have $C_5 \oplus \sigma(C_5) = K_5$. So, C_5 is uniquely embeddable.

The cycle C_6 is also embeddable. For each embedding σ the graph $C_6 \oplus \sigma(C_6)$ is a 4-regular subgraph of K_6. The complement of such a graph is a 1-factor in K_6. Thus, all these graphs are isomorphic. So, C_6 is uniquely embeddable.

Two distinct embeddings of C_7 are given in Figure 2. In the first one, the complement of the graph $C_n \oplus \sigma(C_n)$ is isomorphic to C_7 while in the second one, to $C_3 \cup C_4$.

For $n \geq 8$ we shall show that there are at least two distinct embeddings of C_n:

A) One such that the graph $C_n \oplus \sigma(C_n)$ contains a clique K_4 and

B) another one such that the graph $C_n \oplus \sigma(C_n)$ is K_4-free.

Case A.

Denote by x, a_1, a_2, a_3, a_4, y six consecutive vertices of C_n and by P the path joining x and y obtained from C_n by removing the vertices $\{a_1, a_2, a_3, a_4\}$. Since $n \geq 8$, P has at least four vertices. By Theorem 2, there is a cyclic permutation, say σ' being an embedding of P. Let $x' = \sigma'(x)$ and $y' = \sigma'(y)$. Figure 3 shows how to extend σ' to get an embedding of C_n. Let us observe that the vertices $\{a_1, a_2, a_3, a_4\}$ induce a clique K_4.

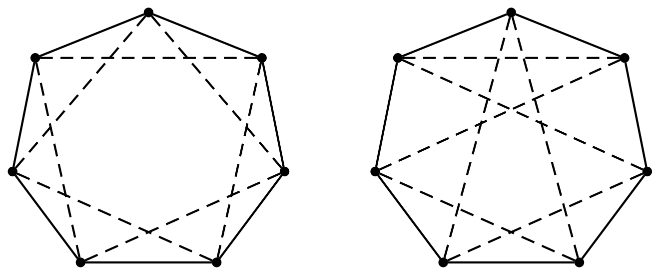

Figure 2: Two distinct embeddings of C_7

Case B. Denote by $v_1, v_2, v_3, \ldots, v_n$ consecutive vertices of C_n. We shall consider two cases.

Subcase B1. n is odd.

Then, the edges $v_i v_{i+2} \pmod n$ define a cycle of length n. This cycle can be considered as an image of C_n by a permutation, say σ. We shall show that the graph $H = C_n \oplus \sigma(C_n)$ is K_4-free. Suppose that H contains a clique on four vertices. It has six edges and it is easy to see that three of them should belong to the first copy of C_n and the remaining three to the second copy of C_n, each of these triples forming a path of length three in the corresponding copy. But a path of length three in C_n should be induced by four consecutive vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3} \pmod n$. The fact that v_i, v_{i+3} is not an edge of the second (dashed) copy of C_n finishes the proof of this case.

Subcase B2. n is even.

It is easy to see that the edges of the form $v_i v_{i+r} \pmod n$ define a cycle of length n if r and n are coprime. In order to prove the existence of such an integer $r$we can use, for instance, the well-known Chebyshev's theorem saying that for each integer $k \geq 4$ there is a prime number between k and $2k-2$. Denote by p such a number where $k = \frac{n}{2}$ and put $r = n - p$. Since a prime numer p and n are surely coprime, r and n are also coprime. Moreover, we have $3 \leq r \leq \frac{n}{2} - 1$. Similarly as above, it is easy to see that the graph formed by C_n and the edges of the form $v_i v_{i+r} \pmod n$ is K_4-free. This finishes the proof.

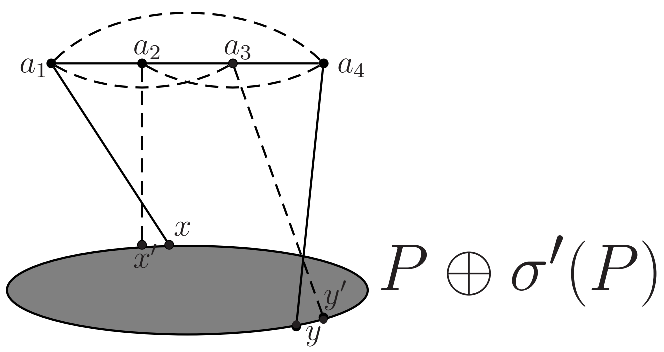

Figure 3: Case A

References

[1] B.Bollobás and S.E.Eldridge, Packings of graphs and applications to computational complexity, *J. Combin. Theory* **B 25** (1978), 105–124.

[2] D.Burns and S.Schuster, Every $(p, p-2)$ graph is contained in its complement, *J. Graph Theory* **1** (1977), 277–279.

[3] D.Burns and S.Schuster, Embedding $(n, n-1)$ graphs in their complements, *Israel J. Math.* **30** (1978), 313–320.

[4] B.Ganter, J.Pelikan and L.Teirlinck, Small sprawling systems of equicardinal sets , *Ars Combinatoria* **4** (1977), 133–142.

[5] N. Sauer and J. Spencer, Edge disjoint placement of graphs, *J. Combin. Theory* Ser. B **25** (1978), 295–302.

[6] M.Woźniak, Embedding graphs of small size, *Discrete Applied Math.* **51** (1994), 233–241.

[7] M.Woźniak, Packing three trees, *in Discrete Math.* **150** (1996), 393–402.

[8] M.Woźniak, A note on uniquely embeddable graphs, *Discussiones Mathematicae-Graph Theory*, **18** (1998), 15-21.

[9] M.Woźniak, Packing of graphs — some recent results and trends, *Studies, Math. Series* **16** (2003), 115–120.

[10] M.Woźniak, Packing of graphs and permutation — a survey, *Discrete Math.* **276** (2004), 379–391.

[11] H.P.Yap, Packing of graphs — a survey, *Discrete Math.* **72** (1988), 395–404.