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On the adjacent-vertex-distinguishing index by sums in total proper colorings *

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Abstract

We consider a proper coloring c of edges and vertices in a simple graph and the sum $f(v)$ of the color of a vertex v and colors of all the edges incident to v. We say that a coloring c distinguishes adjacent vertices by sums, if every two adjacent vertices have different values of f. We conjecture that $\Delta + 3$ colors suffice to distinguish adjacent vertices in any simple graph. We show that this holds for complete graphs, cycles, bipartite graphs, cubic graphs and graphs with maximum degree at most three.

Keywords: simple graph, total proper coloring, adjacent vertex distinguishing index, neighbor sum distinguishing coloring.
2000 Mathematics Subject Classification: 05C15.

1 Introduction

We use Bondy and Murty's book [1] for terminology and notation not defined here. Let $G = (V, E)$ be a simple graph with the maximum degree $\Delta = \Delta(G)$.

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Without loss of generality, we may assume that G is connected. Suppose that $c : V \cup E \to \{1, 2, ..., k\}$ is a proper total coloring of G. For a vertex v, let $f(v)$ denote the total sum of colors of the edges incident to v and the color of v. We try to answer the question, how large k must be to guarantee that there is a proper coloring of vertices and edges with k colors, so that the function f distinguishes neighbor vertices of G. The smallest such k is called adjacent-vertex-distinguishing index, denoted by $\mathrm{tndi}_{\sum}(G)$. Complete graphs, cycles, bipartite graphs, cubic graphs and graphs with maximum degree at most three show that $k \leq \Delta + 3$ in these cases, but can we always do it with $\Delta + 3$ colors?

Conjecture 1 *For every graph $G = (V, E)$, the adjacent-vertex-distinguishing index by sums $\mathrm{tndi}_{\sum}(G)$ in proper total colorings satisfies the inequality*

$$\mathrm{tndi}_{\sum}(G) \leq \Delta + 3.$$

Proper total colorings of simple graphs were considered first by Rosenfeld in [9]. He showed that $\Delta + 2$ colors are enough for cliques, for complete bipartite and tripartite graphs, for balanced k-partite graphs and for graphs with maximum degree at most three. Next Kostochka showed the same bound for graphs with maximum degree at most four and five (see [4] and [5]). In the general case the conjecture is still open.

About ten years ago, a new trend originated in the topic of graph colorings. Many mathematicians considered colorings (proper, general, total or from lists) such that vertices (all or adjacent) are distinguished by either by sets or multisets or sums. In this paper we investigate distinguishing adjacent vertices by sums. At first, Karoński, Łuczak and Thomasson in [7] considered general colorings of edges, and they conjectured that three colors are enough to distinguish adjacent vertices by sums. This conjecture, and a similar one that two colors are enough by general total coloring (see [8]), are almost proved - first one for five colors, the other one for three (see [6]). Recently, some other authors in [3] considered a proper coloring of edges distinguishing adjacent vertices by sums.

In [10], the authors investigated also a proper total coloring of G, but for every vertex v they assigned a set $S(v)$ of colors of the edges incident to v and the color of v. By $\mathrm{tndi}(G)$ they denoted the smallest number k of colors so that there exists a proper total coloring c for G and a function sets S distinguishing neighbor vertices ($S(u)$ is different from $S(v)$ for every pair of adjacent vertices u, v). They considered the cases of cliques, paths, cycles, fans, wheels, stars, complete graphs, bipartite complete graphs and trees. They showed (giving exact bounds for tndi) that $\Delta + 3$ colors are enough in these cases and made the following conjecture:

Conjecture 2 *For every graph $G = (V, E)$, the adjacent-vertex-distinguishing index by sets* $\mathrm{tndi}(G)$ *in proper total colorings satisfies the inequality*

$$\mathrm{tndi}(G) \le \Delta + 3.$$

Next in [2] this conjecture was proved for bipartite graphs and for graphs with maximum degree at most three.

It is easy to observe, that if vertices are distinguished by sums then they are also distinguished by sets, but not necessarily conversly. In this sense our results are stronger. Namely, in this paper we prove all (accept the case of bipartite graphs) values and bounds of $\mathrm{tndi}_{\sum}(G)$ which were shown earlier for $\mathrm{tndi}(G)$.

2 Results for several classes of graphs

Observe that a vertex with maximum degree in G needs exactly $\Delta + 1$ colors to color it, and all edges incident to it, so in general, a lower bound of the invariant $\mathrm{tndi}_{\sum}(G)$ is $\Delta + 1$. This is the same as in a proper total coloring of G. However, if there exist two neighbor vertices x and y with maximum degree in G, then we have to use an additional color to obtain different sums $f(x)$ and $f(y)$. This is a very useful observation in proofs in this section.

Observation 3 *If a graph G contains two adjacent vertices x, y such that* $\deg(x) = \deg(y) = \Delta$*, then* $\mathrm{tndi}_{\sum}(G) \ge \Delta + 2$.

2.1 Paths, cycles, stars and complete graphs

Proposition 4 *Let P_n be a path of order $n \ge 3$. Then* $\mathrm{tndi}_{\sum}(P_n) = \Delta + 2$ *for $n \ge 4$, and* $\mathrm{tndi}_{\sum}(P_3) = \Delta + 1$.

Proof. Color the consecutive vertices of P_n for $n \ge 4$, with $1, 2, 1, 2, \ldots$ and the consecutive edges with $3, 4, 3, 4, 3, 4, \ldots$. Clearly, this is a proper total coloring distinguishing adjacent vertices by sums and no such proper total coloring with less colors exists by Observation 3. □

Proposition 5 *Let C_n be a cycle of order n. Then* $\text{tndi}_{\sum}(C_n) = \Delta + 2$ *for $n \geq 4$, and* $\text{tndi}_{\sum}(C_3) = \Delta + 3$.

Proof. If n is even the coloring of C_n looks like that of P_n. Let n be odd. Color the consecutive vertices of C_n with $1, 2, 1, 2, \ldots, 1, 2, 3$, and the corresponding consecutive edges with $4, 3, 4, 3, \ldots, 4, 1, 2$. Clearly, this is a proper total coloring distinguishing adjacent vertices by sums, and no proper total coloring with less colors is neighbor sum distinguishing by Observation 3. □

Proposition 6 *Let S_n be a star of order $n+1 \geq 3$. Then* $\text{tndi}_{\sum}(S_n) = \Delta + 1$.

Proof. First we color the edges of S_n with colors $1, 2, \ldots, \Delta$ and the vertex of the maximum degree with the color $\Delta + 1$. Next we use color 2 for a vertex incident to the edge with color 1 and all remaining vertices we can color with 1. Clearly, this is a proper total coloring distinguishing adjacent vertices by sums, and no such proper total coloring with less colors is possible. □

Proposition 7 *Let K_n be a complete graph of order $n > 1$. Then*

$$\text{tndi}_{\sum}(K_n) = \begin{cases} \Delta + 2, & n \quad \text{even}, \\ \Delta + 3, & n \quad \text{odd}. \end{cases}$$

Proof. Clearly, no proper total coloring with less colors then $\Delta + 2 = n + 1$ is neighbor sum distinguishing, by Observation 3.

First, let n be even. We know from [10] that $\text{tndi}(K_{2p}) = \Delta + 2$. So there exists a proper total coloring distinguishing adjacent vertices by sets such that for any vertex, exactly one number is missing in its set. Obviously, in every set is missing another color and sums of colors in all vertices are different.

Next, let n be odd. If there existed a total coloring of K_n distinguishing adjacent vertices by sums with $n+1 = \Delta+2$ colors, then it would be missing exactly one number (different every once) in every sum. On the other hand, every vertex must have another color, hence we use n colors for vertices. Let x be a vertex with color 1 and let y be a vertex without 1 in its sum. And if we consider edges in color 1, then we see that a matching in color 1 uses $n - 3$ vertices ($n - 3$ is even), so that there exists a vertex $z \neq y$ in which sum is missing color 1. Therefore, we must use at least $\Delta + 3$ colors in a proper total coloring distinguishing adjacent vertices by sums, if n is odd.

Now, we show that $\mathrm{tndi}_{\sum}(K_n) = \Delta+3$, if n is odd. We view K_n such that its vertices $v_1, \ldots, v_n$ are situated equidistantly on a circle. In first step we color all edges incident with v_1 such that $c(v_1 v_i) = i$, for $i = 2, \ldots, n = \Delta+1$ and a vertex v_1 with color 1. Next we consider v_2: one edge is colored with 2, so we put $c(v_2) = 3$ and $c(v_2 v_i) = i + 1$, for $i = 3, \ldots, n = \Delta + 1$. During the j-th step, if $j \leq \Delta+1$, we have colored edges with colors $j, j+1, \ldots, 2j-2$ (for a color greater than $\Delta+3$ we consider mod $(\Delta+3)$). So we put $c(v_j) = 2j-1$, as before mod $(\Delta+3)$ and $c(v_j v_i) = j+i-1 \mod (\Delta+3)$, for $i = j+1, \ldots, n = \Delta + 1$. This way we obtain a proper total coloring distinguishing all vertices by sums, because the numbers $i-2$ and $i-1$ (where 0 is a color $\Delta+3$ and -1 is a color $\Delta+2$) are missing in the sum for a vertex v_i. □

2.2 Bipartite graphs

Proposition 8 *Let $K_{p,q}$ be a complete bipartite graph of order $p+q$. Then*

$$\mathrm{tndi}_{\sum}(K_{p,q}) = \begin{cases} \Delta+1, & p < q, \\ \Delta+2, & p = q. \end{cases}$$

Proof. We can color the edges of $K_{p,q}$ with colors $1, 2, \ldots, \Delta$ using König's theorem. If $p = q$, then all vertices are of maximum degree and by Observation 3, we have to use two additional colors: $\Delta+1$ for the first independent set and $\Delta+2$ for the other one. Clearly, this is a proper total coloring distinguishing adjacent vertices by sums. If $p < q$, we color with $\Delta+1$ all vertices with maximum degree. Now, let v be a vertex with degree $p < \Delta$. We can choose at least one color for v from the set $\{1, 2, \ldots, \Delta\}$, which wasn't used for any edge incident to v. It can be checked that this is a proper total coloring distinguishing adjacent vertices by sums. □

Quite a similar proof to the case $p = q$ can be done for k-regular bipartite graph.

Proposition 9 *Let k be an integer and G be a k-regular bipartite graph. Then* $\mathrm{tndi}_{\sum}(G) = \Delta + 2$.

Theorem 10 *Let T be a tree of order n. Then* $\mathrm{tndi}_{\sum}(T) = \Delta + 2$, *if there exist two neighbor vertices with maximum degree in T, and* $\mathrm{tndi}_{\sum}(T) = \Delta+1$ *otherwise.*

Proof. We proceed by induction on n. Observe that the theorem is trivial for $n = 2$, since $T = K_2$.

Let v be a leaf of T and $e = uv$ be the incident edge. Call $T' = T - v$. Recall that $S(u)$ is the set of colors of all the edges incident to u and a color of a vertex u. We can color v and e using the following rules:

– if $c(u) = 1$, then $c(e) = \min\{\{1, 2, \ldots, \Delta + 2\} \setminus S(u)\}$ and $c(v) = 3$, if $c(e) = 2$, or $c(v) = 2$ otherwise;

– if $c(u) = 2$ or 3, then $c(v) = 1$ and $c(e) = \min\{\{2, 3, \ldots, \Delta + 2\} \setminus S(u)\}$.

It is not difficult to check that we don't use color $\Delta + 2$, if there does not exist two neighbor vertices with maximum degree in T. And we obtain a proper total coloring distinguishing adjacent vertices by sums. □

Theorem 11 *If $G = (X, Y; E)$ is a bipartite graph, then* $\text{tndi}_{\sum}(G) \leq \Delta + 3$.

Proof. We color the edges of G with colors $1, 3, 4, 5, \ldots, \Delta + 1$ using König's theorem. And we will show that we can choose one free color from the set $\{1, 2, 3, \ldots, \Delta + 1\}$ for every vertex from X such that its sum will be odd, if $\Delta \equiv 0$ or 1 (mod 4), and even, if $\Delta \equiv 2$ or 3 (mod 4). Next we color the vertices of Y with color $\Delta + 2$ or $\Delta + 3$ such that its sums will be even, if $\Delta \equiv 0$ or 1 (mod 4), and odd, if $\Delta \equiv 2$ or 3 (mod 4). Clearly, this is a proper total coloring distinguishing adjacent vertices by sums.

So, let $\Delta = 4k$. We consider a vertex x from X such that $\deg(x) = \Delta$. Note that

$$\sum_{e:\, x \in e} c(e) = 1 + \frac{3 + 4k + 1}{2}(4k - 1) = (4k - 1)(2k + 2) + 1$$

is an odd number. So if we put $c(x) = 2$, we obtain an odd sum $f(x)$. Moreover, observe that there are $2k + 1$ odd numbers in the set $\{1, 3, 4, 5, \ldots, 4k + 1\}$, hence we can always find a right color for any vertex from X in such a way that its sum will be odd.

Now let $\Delta = 4k + 2$. Like above we consider a vertex x from X such that $\deg(x) = \Delta$. Now, the sum

$$\sum_{e:\, x \in e} c(e) = 1 + \frac{3 + 4k + 3}{2}(4k + 1) = (4k + 1)(2k + 3) + 1 = (4k + 1)2k'$$

is even. So if we put $c(x) = 2$ we obtain an even sum $f(x)$. Moreover, observe that there are $2k + 2$ odd numbers in the set $\{1, 3, 4, \ldots, 4k + 3\}$, hence we can always find a right color for any vertex from X in such a way that its sum will be even.

The proofs of the remaining two cases are similar to the described above and we would give under reader's consideration. $\square$

2.3 Graphs with maximum degree three

Theorem 12 *If G is a cubic graph, then* $\mathrm{tndi}_{\sum}(G) \leq \Delta + 3$.

Proof. The claim holds for K_4 by Proposition 7, so G is biparite or tripartite by Brook's theorem ($\chi(G) \leq \Delta$). Hence, by Theorem 11, we can assume that G is a tripartite graph with independent sets A, B, C. First we color all edges using colors $2, 3, 4, 5$ - we can do this by Vizing's theorem. Next we can put colors on vertices as follows: $c(a) = 1$, $c(b) = 6$ and $c(c)$ is one free color from set $\{2, 3, 4, 5\}$. Now we count the sums: $f(a) \in \{10, 11, 12, 13\}$, $f(b) \in \{15, 16, 17, 18\}$ and $f(c) = 14$. So this is a proper total coloring distinguishing adjacent vertices by sums. $\square$

Theorem 13 *If G is a graph with $\Delta \leq 3$, then* $\mathrm{tndi}_{\sum}(G) \leq \Delta + 3$.

Proof. We proceed by induction on $n := V(G)$. Observe that the theorem is trivial for $n = 2$, since $G = K_2$. If all vertices are of degree three in G, then a theorem holds by Theorem12. So we assume that there exists vertices of degree one and two. We consider three cases.

Case 1. There exists a vertex x of degree one in G.

We consider a graph $G' = G - \{x\}$ and it has a good coloring c' with a function of sums f' by induction hypothesis. Denote a adjacent vertex to x in G by y, and two adjacent vertices to y (if there exists only one, then the proof is the same) by u_1 and u_2. Now we define a coloring c of G such that $c(v) = c'(v)$, if $v \in V(G')$, $c(e) = c'(e)$, if $e \in E(G')$. Observe that we have at lest one color free for an edge xy: $c(xy)$ have to be different from $c(y)$, $c(yu_1)$, $c(yu_2)$, $f'(u_1) - f'(y)$, $f'(u_2) - f'(y)$. And now we can choose a color for x such that it will be different from $c(y)$, $c(xy)$, f'_y.

Case 2. There exists vertices of degree only two and three in G and every vertex of degree two is adjacent to an other vertex of degree two.

If all vertices are of degree two, then G is cycle and a good coloring exists by Proposition 5.

Let y be a vertex of degree three and $x_1, \ldots, x_k$ be vertices of degree two such that all x_i and y create a cycle C (see Fig.). Moreover, let y be adjacent to a vertex u. We consider a graph $G' = G - \{x_1, \ldots, x_k\}$ and it has a good coloring c' with a function of sums f' by induction hypothesis. Now we define a coloring c of G such that $c(v) = c'(v)$, if $v \in V(G')$, $c(e) = c'(e)$, if $e \in E(G')$. Observe that we can choose two colors c_1 and c_2 for edges of the cycle C – they have to be different from $c(y)$, $c(yu)$ and $c_1 + c_2 \neq f'(u) - f'(y)$. Next we choose two colors c_3 and c_4 for the vertices $x_1, \ldots, x_k$ such that it will be different from $c(y)$, c_1, c_2, $f'(y)$. We color the consecutive vertices of C with c_3, c_4 and the consecutive egdes of C with c_1, c_2 with one exception: if k is odd, then $c(x_{\frac{k+1}{2}}) = c(y)$, and if k is even, then $c(x_{\frac{k}{2}} x_{\frac{k}{2}+1}) = c(y)$.

Now, let y_1 and y_2 be a vertex of degree three and $x_1, \ldots, x_k$ be vertices of degree two such that the vertices $y_1, x_1, \ldots, x_k, y_2$ create a path P (see Fig.). We consider a graph $G' = G - \{x_1, \ldots, x_k\}$ and it has a good coloring c' with a function of sums f' by induction hypothesis. Now we define a coloring c of G such that $c(v) = c'(v)$, if $v \in V(G')$, $c(e) = c'(e)$, if $e \in E(G')$. Observe that we can choose a color c_1 (and respectively c_2) for an edge $x_1 y_1$ (for $x_n y_2$) in the same way like in case 1. We would like to have $c_1 \neq c_2$. If $c_1 = c_2$, then $c_1 := c(x_n y_2)$ and c_2 has to be different from c_1, $f'(y_1) - c_1$, $f'(y_2) - c_1$. So we color consecutive edges by c_1 and c_2. If k is even, then $c(x_{\frac{k}{2}} x_{\frac{k}{2}+1}) = c_5$, where we define below color c_5. Next we choose two colors c_3 for the vertices x_{2i-1} and c_4 for the vertices x_{2i} such that c_3 will be different from c_1, c_2, $f'(y_1) - c_2$ and c_4 will be different from c_1, c_2, $f'(y_2) - c_1$, c_3, $c_1 + c_3 - c_2$. If k is odd, then $c(x_{\frac{k+1}{2}}) = c_5$. And let c_5 be a color different from c_1, c_2, c_3, c_4.

Case 3. There exists vertices of degree only two and three in G and every vertex of degree two is adjacent to a vertex of degree three.

Let x be a vertex of degree two and y_1, y_2 be two vertices of degree three adjacent to x. We consider a graph H, which is created from two copies G_1 and G_2 isomorphic to G, where every two responsible vertices $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$ of degree two are adjacent in H. So graph H is cubic and it has a good coloring c_H by Theorem 12. Now we explain how we can modify c_H such that it will be "good" coloring on G_1 (a proper total coloring distinguishing adjacent vertices by sums).

First we consider a bipartite graph H with independent sets A and B. We can color all edges using colors $2, 3, 4, 5$ by Vizing's theorem and $c(a) = 1$ for $a \in A$ and $c(b) = 6$ for $b \in B$. Then the sums are different in both sets, because $f(a) \in \{10, 11, 12, 13\}$ and $f(b) \in \{15, 16, 17, 18\}$. Observe that if $x_1 \in A$, then we can delete an edge $x_1 x_2$ and we will still have a proper total coloring distinguishing adjacent vertices by sums. Now let

$x_1 \in B$. If we delate an edge x_1x_2, then we can obtain a conflict: $f(x_1) = f(y_1)$ or $f(x_1) = f(y_2)$. Then we can choose a good color for x_1 from set $\{2,3,4,5\} \setminus \{c(x_1y_1), c(x_1y_2)\}$.

Next we suppose that H is tripartite and c_H is define by proof of Theorem 12. Recall that $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$ and $y_1, y_2 \in V(G_1)$ are incident to x_1. Observe that if $c_H(x_1) = 1$, then $f_H(x_1) \in \{10, 11, 12, 13\}$ and the adjacent vertices have different color from 1 (it is a proper coloring) and sums in a set $\{14, \ldots, 18\}$. So we can delete an edge x_1x_2 (its color is from set $\{2,3,4,5\}$) and we will still have a proper total coloring distinguishing adjacent vertices by sums.

Let $c_H(x_1) = 2$, then $f_H(x_1) = 14$ and the adjacent vertices have different color from $2,3,4,5$ (for all of them the sum is 14) and sums in a set $\{10, \ldots, 13, 15, \ldots, 18\}$. If $c_H(x_1x_2) = 5$, then we can delete an edge x_1x_2 (then $f_{G_1}(x_1) = 9$). If $c_H(x_1x_2) = 4$, then we can delete an edge x_1x_2 and if we obtain conflict, that means that $f_H(y_1) = 10$, so we can put other $c_{G_1}(x_1) = 4$ (if $f_H(y_2) \neq 12$) or $c_{G_1}(x_1) = 6$ (if $f_H(y_2) = 12$, that means $c_H(y_2) = 1$). If $c_H(x_1x_2) = 3$, then we can delete an edge x_1x_2 and if we obtain conflict, that means that $f_H(y_1) = 11$, so we can put other $c_{G_1}(x_1) = 3$ (if $f_H(y_2) \neq 12$) or $c_{G_1}(x_1) = 6$ (if $f_H(y_2) = 12$, that means $c_H(y_2) = 1$).

The proofs of the remaining three cases if $c_H(x_1) = 3$, $c_H(x_1) = 4$, $c_H(x_1) = 5$ are similar and we left them to the reader.

Let $c_H(x_1) = 6$. Then $f_H(x_1) \in \{15, \ldots, 18\}$, the adjacent vertices have different color from 6 (it is a proper coloring) and sums in a set $\{10, \ldots, 14\}$. Let $f_H(x_1) = 18$, so $c_H(x_1x_2) = 3$ or $c_H(x_1x_2) = 4$ or $c_H(x_1x_2) = 5$. If $c_H(x_1x_2) = 3$, then $f_{G_1}(x_1) = 15$ and all is right. If $c_H(x_1x_2) = 4$, then :

– if $f_H(y_1) = f_H(y_2) = 14$ then $c_{G_1}(x_1) = 1$;

– if $c_H(y_1) = c_H(y_2) = 1$ then $c_{G_1}(x_1) = 6$;

– if $f_H(y_1) = 14$, $c_H(y_2) = 1$ then $c_{G_1}(x_1) = 4$ (if $f_H(y_2) \neq 12$) or $c_{G_1}(x_1) = 2$ (if $f_H(y_2) = 12$ and $c_H(y_1) \neq 2$) or $c_{G_1}(x_1) = 6$ and $c_{G_1}(y_1) = 3$, $c_{G_1}(x_1y_1) = 2$ (if $f_H(y_2) = 12$ and $c_H(y_1) = 2$).

If $c_H(x_1x_2) = 5$, then:

– if $f_H(y_1) = f_H(y_2) = 14$ then all is right;

– if $f_H(y_1) = 14$, $c_H(y_2) = 1$ then $c_{G_1}(x_1) = 5$ or $c_{G_1}(x_1) = 6$;

– if $c_H(y_1) = c_H(y_2) = 1$ then $c_{G_1}(x_1) = 2$.

The proofs of the remaining three cases if $f_H(x_1) = 15$, $f_H(x_1) = 16$, $f_H(x_1) = 17$ are similar and we left them to the reader. □

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