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Abstract

Let $H = (V, E)$ be a hypergraph of order n. A distance magic labeling of H is a bijection $l\colon V \to \{1, 2, \dots, n\}$ for that there exists a positive integer k such that $\sum_{x \in N(v)} l(x) = k$ for all $v \in V$, where $N(v)$ is the neighborhood of v. In this paper we deal with (r, t)-hypercycles. It was proved that $(1, 2)$-hipercycle of order n is a distance magic graph if and only if $n = 4$ ([7]). In this paper we solve the similar problem for $t = 3, 4$.
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1 Introduction

A hypergraph H is a pair $H = (V, E)$ where V is a set of vertices and E is a set of non-empty subsets of V called hyperedges. The order of a hypergraph H is denoted by $|H|$ and the size is denoted by $\|H\|$. If all edges have the same cardinality t, the hypergraph is said to be t-uniform. Hence a graph is 2-uniform hypergraph. Two vertices in a hypergraph are *adjacent* if there is an edge containing both of them. The *neighborhood* $N_H(x)$ of a vertex

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$x \in V(H)$ is the set of vertices adjacent to x.
The (r,t)-hypercycle, $1 \leq r \leq t-1$, is defined as t-uniform hypergraphs whose vertices can be ordered cyclically in such a way that the edges are segments of that cyclic order and every two consecutive edges share exactly r vertices [6].

Distance magic labeling (also called *sigma labeling*) of a hypergraph $H = (V,E)$ of order n is a bijection $l\colon V(H) \to \{1,2,\ldots,n\}$ with the property that there is a positive integer k (called *magic constance*) such that $\sum_{y\in N_H(x)} l(y) = k$ for every $x \in V(H)$. If a hypergraph H admits a distance magic labeling, then we say that H is *distance magic hypergraph.*

The idea of distance magic labelling of a graph has been motivated by the construction of magic squares. Finding an r-regular distance magic labeling turns out equivalent to finding equalized incomplete tournament EIT(n,r) [2]. A *fair incomplete tournament* of n teams with k rounds is a tournament in which every team plays exactly k other teams and the total strength of the opponents that each team misses during the tournament is the same for all teams. For a survey, we refer the reader to [1].

The following observations were independently proved:

Observation 1 ([5], [7], [8], [9]) *Let G be a r-regular distance magic graph on n vertices. Then $k = \frac{r(n+1)}{2}$.*

Observation 2 ([5], [7], [8], [9]) *No r-regular graph with r-odd can be a distance magic graph.*

It was proved in [7]:

Theorem 3 ([7]) *The cycle C_n of length n is a distance magic graph if and only if $n = 4$.*

In this paper we consider the corresponding problem for (r,t)-hypercycles. We show that if $r \leq \frac{t}{2}$ then the (r,t)-hypercycle is not distance magic. We will give also some results for $(t-1,t)$-hypercycles. In particular we complete solve the case for $t \in \{3,4\}$.

The paper is organized as follows. In the next section we show correspondence between distance magic labeling (r,t)-hypercycles of order n and distance magic labeling of some graphs. Some preliminary lemmas will be proved in the third section. In forth section we characterize whenever C_n^p for $p = 2,3$ is distance magic graph. The main result and open problems are stayed in the last section.

2 Equivalent problem

For a hypergraph H of order n we define a graph G_H as follows $V(G_H) = V(H)$ and $x_ix_j \in E(G_H)$ in and only if there exists an edge $e \in E(H)$ such that $x_i, x_j \in e$. Let $l'\colon V(H) \to \{1, 2, \dots, n\}$ be any bijection. Define $l\colon V(G_H) \to \{1, 2, \dots, n\}$ such that $l(x_i) = l'(x_i)$. Notice that $\sum_{y \in N_H(x)} l'(y) = \sum_{v \in N_{G_H}(x)} l(v)$. Hence H is distance magic hypergraph if and only if G_H is distance magic graph.

The pth power of a graph G is a graph G^p with the same set of vertices as G and an edge between two vertices if and only if there is a path of length at most p between them. In this paper we will consider the pth power of a cycle C_n. Notice that C_n^p is $2p$-regular graph.

Notice that if H is $(t-1, t)$-hypercycle then $G_H \cong C_n^{t-1}$.

3 Lemmas

In this section we present several useful lemmas.
Let $w(x) = \sum_{y \in N_H(x)} l(y)$ for every $x \in V(H)$. We start with the observations:

Observation 4 *Let C_n^p be distance magic graph with magic constance k, then for any $\gamma \in \mathbb{N}$:*

$$\begin{aligned}
&l(x_0) + l(x_{p+1}) = l(x_p) + l(x_{2p+1}) = \dots = l(x_{\gamma p}) + l(x_{(\gamma+1)p+1}) = k_1,\\
&l(x_1) + l(x_{p+2}) = l(x_{p+1}) + l(x_{2p+2}) = \dots = l(x_{\gamma p+1}) + l(x_{(\gamma+1)p+2}) = k_2,\\
&\vdots\\
&l(x_{p-1}) + l(x_{2p}) = l(x_{2p-1}) + l(x_{3p}) = \dots = l(x_{(\gamma+1)p-1}) + l(x_{(\gamma+2)p}) = k_p.
\end{aligned}$$

and $k_1 + k_2 + \dots + k_p = k$.

Proof. Since C_n^p is distance magic we obtain that $w(x_0) - w(x_1) = w(x_1) - w(x_2) = \ldots = w(x_{n-1}) - w(x_0) = 0$. Hence $w(x_i) - w(x_{i+1}) = l(x_{i-p}) + l(x_{i+1}) - (l(x_i) + l(x_{i+1+p})) = 0$ for $i \in \{0, 1, \dots, n\}$. ■

Observation 5 *Let C_n^p be distance magic graph, then for any $i \in \{0, 1, \dots, n-1\}$:*

$$l(x_i) + l(x_{i+1}) + \ldots + l(x_{i+p-1}) = l(x_{i+2p+2}) + l(x_{i+2p+3}) + \ldots + l(x_{i+3p+1}).$$

Proof. Since C_n^p is distance magic we obtain that $w(x_{i+p}) - w(x_{i+2p+1}) = 0$. ■

We show now some families of graphs C_n^p which are **not** distance magic.

Lemma 6 *If* $\gcd(2p+2, n) = 1$ *and* $n > 2p+1$*, then* C_n^p *is not distance magic graph.*

Proof. Since $\gcd(2p+2, n) = 1$ then by Bézout's lemma there exist coefficients α, β such that $\alpha(2p+2) + \beta n = 1$. It follows that

$$\begin{aligned}
&0 + \alpha(2p+2) \equiv 1 (\bmod n)\\
&1 + \alpha(2p+2) \equiv 2 (\bmod n)\\
&\vdots\\
&p - 1 + \alpha(2p+2) \equiv p (\bmod n)
\end{aligned} \tag{1}$$

Suppose that C_n^p is distance magic, by (1) and Observation 5 we obtain that

$$l(x_0) + l(x_1) + \cdots + l(x_{p-1}) = l(x_1) + l(x_2) + \cdots + l(x_p).$$

Furthermore $l(x_0) = l(x_p)$, a contradiction. ■

Lemma 7 *If* $\gcd(2p+2, n) = p+1$ *and* $n > 2p+1$*, then* C_n^p *is not distance magic graph.*

Proof. Since $\gcd(2p+2, n) = p+1$ then there exist α, β such that $\alpha(2p+2) + \beta n = p+1$. It follows that

$$\begin{aligned}
&0 + \alpha(2p+2) \equiv p + 1 (\bmod n)\\
&1 + \alpha(2p+2) \equiv p + 2 (\bmod n)\\
&\vdots\\
&p - 1 + \alpha(2p+2) \equiv p (\bmod n)
\end{aligned} \tag{2}$$

Let C_n^p be distance magic graph with magic constance k, then by (2) and Observation 5 we obtain that

$$l(x_0) + l(x_1) + \cdots + l(x_{p-1}) = l(x_{p+1}) + l(x_{p+2}) + \cdots + l(x_{2p}) = \frac{k}{2}.$$

Analogously by (2) and Observation 5 we obtain $l(x_1) + l(x_2) + \cdots + l(x_p) = l(x_{p+2}) + l(x_{p+3}) + \cdots + l(x_{2p+1}) = \frac{k}{2}$. It follows that $l(x_1) = l(x_p)$, a contradiction. ■

Lemma 8 *If* $\gcd(p,n)=1$ *and* $n\neq 2p+2$*, then* C_n^p *is not distance magic graph.*

Proof. Since $\gcd(p,n)=1$ then there exist coefficients α, β such that $\alpha n+\beta p=1$. Suppose that C_n^p is distance magic. By Observation 4 we obtain that

$$l(x_0)+l(x_{p+1})=l(x_p)+l(x_{2p+1})=\cdots=l(x_{-(\beta+1)p})+l(x_{-\beta p+1})=k_1$$

Applying $-\beta p+1\equiv 0(\bmod n)$ we have $l(x_{p+1})=l(x_{n-p-1})$. Since $n\neq 2p+2$, a contradiction. ■

Lemma 9 *If* p *is odd and* $n>2p(p+1)$*, then* C_n^p *is not distance magic graph.*

Proof. Suppose that C_n^p is distance magic. Let k be a magic constance for C_n^p. Then by Observation 4

$$\begin{aligned}
&l(x_0)+l(x_{p+1})=l(x_p)+l(x_{2p+1})=\cdots=l(x_{\gamma p})+l(x_{(\gamma+1)p+1})=k_1,\\
&l(x_1)+l(x_{p+2})=l(x_{p+1})+l(x_{2p+2})=\cdots=l(x_{\gamma p+1})+l(x_{(\gamma+1)p+2})=k_2,\\
&\vdots\\
&l(x_{p-1})+l(x_{2p})=l(x_{2p-1})+l(x_{3p})=\cdots=l(x_{(\gamma+1)p-1})+l(x_{(\gamma+2)p})=k_p,
\end{aligned}$$

and $k_1+k_2+\cdots+k_p=k$.

Let $l(x_0)=k_0$, then:

$$l(x_{j(p+1)})=\sum_{i=0}^{j}(-1)^{j-i}k_i$$

for $j=1,2,\ldots,p$. If p is odd then $l(x_{p(p+1)})=k_p-k_{p-1}+k_{p-2}-\cdots+k_1-k_0$. It follows that

$$\begin{aligned}
&l(x_{(p+1)(p+1)})=-k_p+k_{p-1}-k_{p-2}+\cdots+k_2+k_0\\
&l(x_{(p+2)(p+1)})=k_p-k_{p-1}+k_{p-2}-\cdots+k_3-k_0\\
&\vdots\\
&l(x_{2p(p+1)})=k_0
\end{aligned}$$

It follows that $l(x_0)=l(x_{2p(p+1)})=k_0$, a contradiction. ■

The following lemma shows that there exist infinitely many p's such that C_{2p+2}^p admits distance magic labeling.

Lemma 10 *If $n = 2p + 2$, then C_n^p is distance magic graph.*

Proof. Let

$$\begin{array}{lllll} l(x_0) = 1, & l(x_1) = 2, & l(x_2) = 3, & \dots & l(x_p) = p + 1, \\ l(x_{p+1}) = n, & l(x_{p+2}) = n - 1, & l(x_{p+3}) = n - 2, & \dots & l(x_{2p+2}) = p + 2. \end{array}$$

Notice that $k = p(n + 1) = 2p(p + 1)$. Observe that $\sum_{y \in N(x_i)} l(y) = \frac{(n+1)n}{2} - l(x_i) - l(x_{(i+p+1)(\operatorname{mod} n)}) = \frac{(n+1)n}{2} - (n + 1) = 2p(p + 1)$ for every $x_i \in V(C_{2p+2}^p)$. ■

4 Distance magic labeling for C_n^2 and C_n^3

Observe that if $n \leq 2p + 1$ then $C_n^p \cong K_n$ that is not distance magic. From now on we will assume that $n > 2p + 1$.

Theorem 11 *A graph C_n^2 is not distance magic graph unless $n = 6$.*

Proof. There exists distance magic labeling of C_6^2 by Lemma 10.

Let now $n > 6$. By Lemma 8 we can also assume that n is even. Assume that C_n^2 is distance magic. If k is a magic constance for C_n^2, then $k = 2(n+1)$. We will consider few cases on congruency on n modulo 6.

Case 1: $n \equiv 0 (\operatorname{mod} 6)$
Let $n = \alpha 6$ and $\alpha > 1$. By Observation 4 we obtain:

$$l(x_0) + l(x_3) = l(x_2) + l(x_5) = \dots = l(x_{\alpha 6-2}) + l(x_1) = k_1$$
$$l(x_1) + l(x_4) = l(x_3) + l(x_6) = \dots = l(x_{\alpha 6-3}) + l(x_0) = l(x_{\alpha 6-1}) + l(x_2) = k_2$$

Putting $l(x_0) = k_0$, we have:

$$l(x_{6i}) = ik_2 - ik_1 + k_0$$
$$l(x_{6i+3}) = -ik_2 + (i + 1)k_1 - k_0$$

for $j = 1, 2, \dots, \alpha - 1$.
Hence $l(x_{\alpha 6-3}) = -(\alpha-1)k_2 + \alpha k_1 - k_0$. Furthermore because $k_2 = l(x_{\alpha 6-3}) + l(x_0) = -(\alpha-1)k_2 + \alpha k_1$ we obtain that $k_1 = k_2$. It implies that $l(x_0) = l(x_6)$,

a contradiction.

Case 2: $n \equiv 2 (\mathrm{mod}\, 6)$ or $n \equiv 4 (\mathrm{mod}\, 6)$
By equation (5) we obtain:

$$l(x_0) + l(x_1) = l(x_2) + l(x_3) = l(x_4) + l(x_5) = \cdots = l(x_{n-2}) + l(x_{n-1}) = k_1$$

and

$$l(x_1) + l(x_2) = l(x_3) + l(x_4) = l(x_5) + l(x_6) = \cdots = l(x_{n-1}) + l(x_0) = k_2$$

Since $l(x_0) + l(x_1) + l(x_3) + l(x_4) = k$, $k_2 + k_2 = k$. Let $l(x_0) = k_0$, then:

$$\begin{array}{l} l(x_{2i}) = ik - 2ik_1 + k_0 \\ l(x_{2i+1}) = (2i+1)k_1 - ik - k_0 \end{array}$$

for $i = 0, 1, \ldots, \frac{n-2}{2}$. Hence $l(x_{n-1}) = (n-1)k_1 - (n-2)k - k_0$. Recall that $l(x_{n-1}) + l(x_0) = k - k_1$. It implies that $k_1 = k_2 = \frac{k}{2}$ and moreover $l(x_0) = l(x_2)$, a contradiction. ■

Theorem 12 *A graph C_n^3 is not distance magic graph unless $n = 8$ or $n = 24$.*

Proof. By Lemma 9 we can assume that $n \leq 24$.

Suppose first that $n = 24$, then let

$$\begin{array}{llllll} l(x_0) = 2, & l(x_1) = 7, & l(x_2) = 15, & l(x_3) = 5, & l(x_4) = 22, & l(x_5) = 18, \\ l(x_6) = 11, & l(x_7) = 19 & l(x_8) = 3, & l(x_9) = 8, & l(x_{10}) = 13, & l(x_{11}) = 6, \\ l(x_{12}) = 23, & l(x_{13}) = 16, & l(x_{14}) = 12, & l(x_{15}) = 20, & l(x_{16}) = 1, & l(x_{17}) = 9, \\ l(x_{18}) = 14, & l(x_{19}) = 4, & l(x_{20}) = 24, & l(x_{21}) = 17, & l(x_{22}) = 10, & l(x_{23}) = 21. \end{array}$$

It is easy to check that function l defined above is a distance magic labeling for C_{24}^3.

Let now $n < 24$. For $n = 8$ by Lemma 10 there exists distance magic labeling of C_8^3. By Lemmas 6, 7 and 8 we need to consider only case when $n = 18$. Assume that C_{18}^3 is distance magic. Let k be a magic constance for C_{18}^3.

By Observation 4 we obtain:

$$\begin{array}{l} l(x_0) + l(x_4) = \ldots = l(x_9) + l(x_{13}) = l(x_{12}) + l(x_{16}) = l(x_{15}) + l(x_1) = k_1 \\ l(x_1) + l(x_5) = l(x_4) + l(x_8) = l(x_7) + l(x_{11}) = l(x_{10}) + l(x_{14}) = l(x_{13}) + l(x_{17}) = k_2 \\ l(x_2) + l(x_6) = l(x_5) + l(x_9) = l(x_8) + l(x_{12}) = l(x_{11}) + l(x_{15}) = l(x_{14}) + l(x_0) = k_3 \end{array}$$

Putting $l(x_0) = k_0$ and $l(x_2) = k_0'$, we have:

$$\begin{array}{lll} l(x_4) = k_1 - k_0, & l(x_8) = k_2 - k_1 + k_0, & l(x_{12}) = k_3 - k_2 + k_1 - k_0, \\ l(x_6) = k_3 - k_0', & l(x_{10}) = k_1 - k_3 + k_0', & l(x_{14}) = k_2 - k_1 + k_3 - k_0'. \end{array}$$

Since $l(x_{14}) + l(x_0) = k_3$ we obtain $k_0' = k_2 - k_1 + k_0$ what implies that $l(x_6) = k_3 - k_2 + k_1 - k_0 = l(x_{12})$, a contradiction. ■

5 Distance magic labeling for (r,t)-hypercycles

We will start this section with few observations:

Observation 13 *If $t > 2$ and $r \leq \frac{t}{2}$ then (r,t)-hypercycle is not distance magic.*

Proof. Let H be a (r,t)-hypercycle of order n and size m. It easy to check that if $t = 3$ and $m = 2$, then H is not distance magic hypergraph. Let $m > 2$ or $t > 3$ and construct a graph G_H as in Section 2.

It follows that there exist $x, y \in V(G_H)$ such that they are adjacent and $N_{G_H}(x) = (N_{G_H}(y) \setminus \{x\}) \cup \{y\}$. Suppose that G_H is distance magic graph, then in particular the magic constance $k = \sum_{v \in N_{G_H}(x)} l(v) = \sum_{w \in N_{G_H}(y)} l(w)$. Hence $l(x) = l(y)$, a contradiction. ■

Observation 14 *If t is even then $(t-2,t)$-hypercycle is not distance magic.*

Proof. Let H be a (r,t)-hypercycle of order n and size m. Let construct a graph G_H as in Section 2. Observe that if t is even the graph G_H is $(2t-3)$-regular graph. By Observation 2 G_H is not distance magic. ■

Now we will prove our main theorem:

Theorem 15 *If $t \in \{3, 4\}$, then (r,t)-hypercycle of order n is distance magic if and only if $r = t - 1$ and one of the following condition holds:*

- *$r = 2$ and $n = 6$,*
- *$r = 3$ and $n = 8$ or $n = 24$.*

Proof. Let H be a (r,t)-hypercycle of order n and size m. Let construct a graph G_H as in Section 2. Recall that if $r = t-1$ then $G_H \cong C_n^{t-1}$.

Suppose first that $t = 3$ by Observation 13 and Theorem 11 H is distance magic if and only if $r = 2$ and $n = 6$.

Let $t = 4$, then by Observations 13, 14 and Theorem 12 H is not distance magic unless $r = 3$ and $n = 8$ or $n = 24$. ■

Since for H to be $(t-1,t)$-hypercycle of order n we have $G_H \cong C_n^{t-1}$ it is worthy also to notice the facts that follows immediately by Lemmas 6, 8, 9 and 10:

Corollary 16 *Let H be $(t-1,t)$-hypercycle of order n then:*

- *If $\gcd(2t,n) = 1$ then H is not distance magic hypergraph.*
- *If $\gcd(t-1,n) = 1$ and $n \neq 2t$ then H is not distance magic hypergraph.*
- *If t is even and $n > 2t(t-1)$ then H is not distance magic hypergraph.*
- *If $n = 2t$ then H is distance magic hypergraph.*

At the end of the section we will put the following open problems:

Problem 17 *Decide if (r,t)-hypercycle of is distance magic hypergraph for $\frac{t}{2} < n \leq t-2$.*

Problem 18 *Decide if $(t-1,t)$-hypercycle of order n is distance magic hypergraph for t even and $2t < n \leq 2t(t-1)$.*

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