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Universal third parts of any complete 2-graph and none of DK_5

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Abstract

It is shown that there is no digraph F which could decompose the complete digraph on 5 vertices minus any 2-arc remainder into three parts isomorphic to F for each choice of the remainder. On the other hand, for each $n \geq 3$ there exists an analogous 3-decomposition of the complete 2-graph on n vertices minus any 2-edge remainder if necessary, i.e., necessary if $n \bmod 3 = 2$.

1 Introduction

By a 2-graph we mean a multigraph with edge multiplicity at most two. The problem we deal with is a specification ($t = 3$) of the edge (arc) t-decomposition of the complete 2-graph (complete digraph) into t isomorphic parts with an edge (arc) *t-remainder* R, under the restriction that the size of the remainder is as small as possible. If those parts are isomorphic to an F then the isomorphism class of F is called a tth *part*, with remainder R if $|R| \neq 0$. The symbol $\langle R\rangle$ stands for the 2-graph (digraph) induced by R. Moreover, the isomorphism class of the $\langle R\rangle$ is called a *shape* of R. In case t-packings of a tth part F of ${}^2K_n$ realize all possible shapes of t-remainder R, then F is called a *universal* tth *part*. A decomposition (packing) with parts

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isomorphic to F is called an *F-decomposition* (*F-packing*). We use notation and terminology of graph theory as in [1,3,12].

If $t = 2$, the second parts of the complete digraph are known as self-complementary digraphs and were described by Read [9]. If $|R| = 0$, the existence of tth parts was proved by Harary, Robinson and Wormald in [4]. Related results on complete multigraphs and digraphs, involving remainders, can be found in [5,6,7,8,10]. The following conjecture is a motivation of our study.

Conjecture 1 (Skupień [11]) *A universal tth part of the complete graph exists.*

We state the following conjecture.

Conjecture 2 *A universal tth part of any complete 2-graph exists.*

Our main results follow.

Theorem 1 *A universal third part of any complete 2-graph exists.*

Theorem 2 *There is no universal third part of the complete digraph on 5 vertices.*

2 Proof of Theorem 1

2.1 Recursive step in the proof

Assume that for $G = {}^2K_n$ with $n \geq 2$, there is an F-decomposition F_1, F_2, F_3 with 3-remainder R. Note that $|R| = \|{}^2K_n\| \bmod 3 = 2$ if $n \bmod 3 = 2$ and $|R| = 0$ otherwise. Consider $\tilde{G} = {}^2K_{n+3}$ which includes G and three new vertices x_1, x_2, x_3. Then $\{\tilde{F}_1, \tilde{F}_2, \tilde{F}_3\}$ is a required 3-decomposition of $\tilde{G}$ if we assume that each $\tilde{F}_j$ includes F_j, all double edges joining x_j to G and the double edge which joins together the two remaining new vertices, $j = 1, 2, 3$. Another possibility is that $\tilde{F}_j$ includes both F_j and only single edges joining all of G to two new vertices different from x_j together with single edges joining those two vertices to x_j.

Therefore in order to prove Theorem 1 it is enough to find a universal third part of ${}^2K_n$ for n equal to 3,4,5.

A 3-decomposition of the multigraph ${}^2K_n$ into three parts isomorphic to an F, with a nonempty 3-remainder R if $n \bmod 3 = 2$, is represented by an $n \times n$ matrix, in which the entry k in row i and column j (with $i \neq j$) means that the edge ij belongs to R if $k = 0$ and to part k otherwise,

$k = 1, 2, 3$. This matrix with zeroes on the main diagonal replaced by dots is a modification of the adjacency matrix of ${}^2K_n$ and is called (a 3- or an F-) *decomposition/packing matrix*.

2.2 On the complete 2-graphs on 3 or 4 vertices

For $n = 3, 4$, the 3-remainder is empty and an F-decomposition of ${}^2K_n$ into three parts exists. Namely, $F = P_3$ or $F = {}^2K_2$ if $n = 3$. If $n = 4$, F can be any of seven 2-graphs with 4 edges on 3 or 4 vertices under the assumption that F is different from the 2-graph obtained from the path P_4 by doubling the middle edge. For corresponding decomposition matrices, see Table 1.

Table 1. 3-decomposition matrices for $n = 4$:

. 1 2 3	. 1 1 1	. 1 1 3	. 1 1 2	. 1 1 2	. 1 2 1	. 1 1 1
1 . 3 2	2 . 2 2	1 . 2 2	1 . 2 3	1 . 1 3	3 . 1 2	2 . 1 2
2 3 . 1	3 3 . 3	1 2 . 3	3 3 . 1	2 3 . 2	3 2 . 1	3 2 . 2
3 2 1 .	1 2 3 .	3 2 3 .	2 3 2 .	2 3 3 .	2 3 3 .	3 3 3 .

2.3 On the complete 2-graph on 5 vertices

Let R be a 3-remainder in ${}^2K_5$. Since R includes two edges, there are three shapes of R. Let $A = A(R)$ be the degree sequences of ${}^2K_5 - R$. Then

$$A = \begin{cases} (8,8,8,6,6) & \text{for } \langle R\rangle = C_2, \\ (8,8,7,7,6) & \text{for } \langle R\rangle = P_3, \\ (8,7,7,7,7) & \text{for } \langle R\rangle = 2K_2. \end{cases}$$

We are going to find all degree sequences of would-be third parts of ${}^2K_5$. The order and the size of those parts are 5 and 6 respectively, multiplicity of edges being at most 2. Therefore we find all partitions of 12 into 5 or less parts, each of which is at most 6. There are 29 of such partitions. We note that if $\Delta = 6$ then remaining parts are to be 2 or 1, and if $\Delta = 5$, at most 3. This observation eliminates 12 of the partitions without any 2-graphic realization. The remaining 17 partitions can be proved to be 2-graphic. A few mutually equivalent characterization of r-graphic partitions are presented by Chungphaisan [2]. One of those characterizations, which is a generalized Erdős-Gallai theorem, is as follows.

Theorem 3 (Chungphaisan [2]) *A nonincreasing sequence $d = (d_1, \ldots, d_n)$ of nonnegative integers is r-graphic if and only if $\sum_{i=1}^{n} d_i$ is even, and for* *every positive integer $k \le n$,*

$$\sum_{i=1}^{k} d_i \le rk(k-1) + \sum_{i=k+1}^{n} \min\{rk, d_i\}.$$

Let F be a third part of ${}^2K_5 - R$. Then F is of size 6. Therefore if $\alpha = (a_1, \ldots, a_5)$ is a (nonincreasing) degree sequence of F and $A(R) = (A_1, \ldots, A_5)$ then the following condition is satisfied.

(i) There exist three permutations $\sigma_1, \sigma_2, \sigma_3$ such that $a_i + a_{\sigma_1(i)} + a_{\sigma_2(i)} = A_{\sigma_3(i)}$, for $i = 1, \ldots, 5$.

An F-decomposition of the multigraph ${}^2K_5 - R$ into three parts is represented by a 3×5 matrix, called a *degree-decomposition matrix*, in which the first row is a degree sequence of F and the remaining two are permutations of it. Moreover, column sums make up a permutation of $A(R)$. Two degree-decomposition matrices are called *equivalent matrices* if interchanging columns and/or rows in one of the matrices gives the other. A degree-decomposition matrix M is called a *standard degree-decomposition matrix* if the concatenation of the consecutive columns of M is a sequence which is a lexicographical maximum among all matrices equivalent to M.

The following Table 2 summarizes results of computer calculations. The symbol + therein means that the condition (i) is satisfied for the corresponding A.

Table 2. Partitions of 12 which are 2-graphic:

$\alpha \backslash A$	88866	88776	87777
62220	+	-	-
62211	+	-	-
53310	-	+	-
53220	+	+	+
53211	-	+	+
52221	+	-	-
44400	-	-	-
44310	+	+	-
44220	+	-	-
44211	-	-	-
43320	+	+	+
43311	-	+	-
43221	+	+	+
42222	+	-	-
33330	-	-	-
33321	-	+	+
33222	+	+	+

Each partition α listed in Table 2 is a degree sequences of a 2-graph F. Every α which is accompanied by three symbols + therein is called to be an *acceptable F-sequence*. Thus only the following four partitions are acceptable F-sequences:

$$(5,3,2,2,0),\ (4,3,3,2,0),\ (4,3,2,2,1),\ (3,3,2,2,2).$$

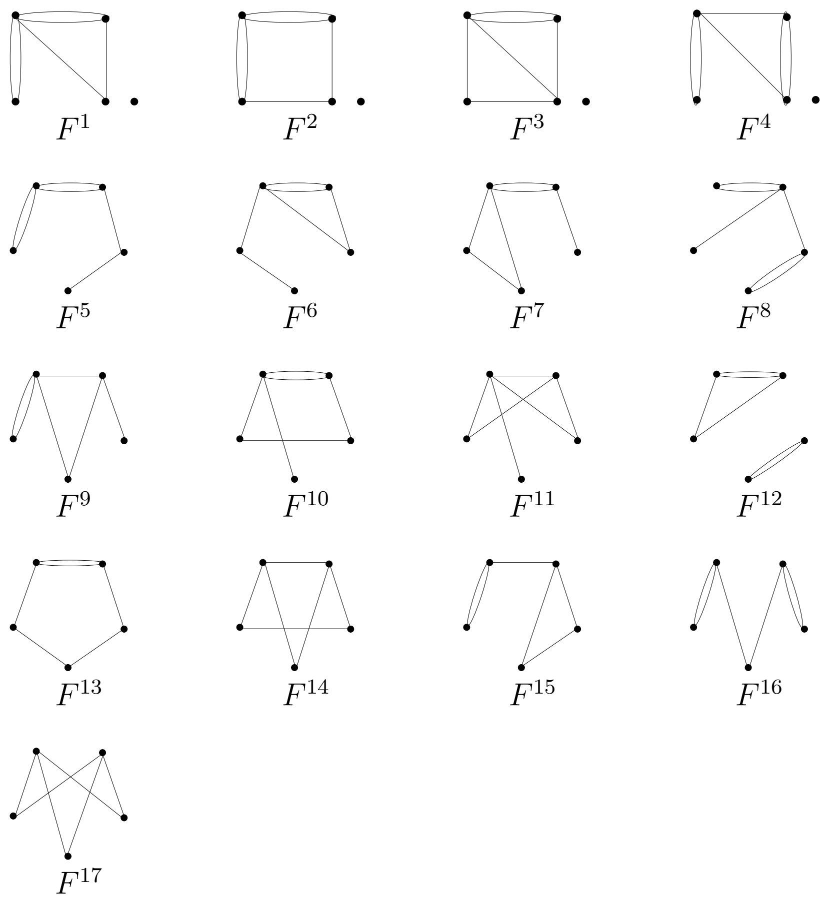

Fig. 1. All 2-graphic realizations of acceptable F-sequences

All 2-graphic realizations of those sequences are presented in Fig. 1 and are a result of our exhaustive search for drawings. Note that F^1, F^2–F^4, F^5–F^{11}, and F^{12}–F^{17} make up the corresponding four lists of realizations.

For each of the four acceptable F-sequences, all standard degree-decomposition matrices M^j have been generated by the above-mentioned computer program. Matrices are listed in Tables 3, 4 and 5. The superscript j increases if we move from left to right along any row of the list as well as if we go down to a new row of matrices. The name M^j is put at a matrix only in case the matrix is referred to later on.

Table 3. Standard degree-decomposition matrices M^j for the remainder $\langle R \rangle = P_3$, $j = 1, 2, \ldots, 29$:

53220 53220
32205 32205
03252 02253

43320 43320 43320 43320 43320 43320
43023 42033 42033 32043 33204 33024
02433 03423 03324 03324 02343 02343

43221 43221 43221 43221 43221 43221 43221 43221 43221 43221
32421 23421 23421 22431 32412 32412 23412 23412 23412 31422
12234 21234 22134 22134 12243 12234 22134 21243 22143 13224

43221 43221 43221 43221 43221 43221 43221 43221 43221
31422 32214 32214 32214 23214 23214 22314 31224 31224
12234 12432 13242 12243 21432 22341 21342 13422 12432

33222 33222
33222 32322
22332 23232

Table 4. Standard degree-decomposition matrices M^j for the remainder $\langle R \rangle = C_2$, $j = 30, \ldots, 38$:

53220
30225
05223

43320 43320
43203 43023 (M^{31}, M^{32})
02343 02343

43221 43221 43221 43221 43221
32421 31422 31422 23214 31224
13224 14223 12243 22431 14223

33222
32322 (M^{38})
23322

Table 5. Standard degree-decomposition matrices M^j for the remainder $\langle R\rangle = 2K_2$, $j = 39, 40, \ldots, 46$:

53220 53220
32205 23205 (M^{39}, M^{40})
02352 02352

43320
33024 (M^{41})
02433

43221 43221 43221 43221
22413 32214 23214 23214 $(M^{42}, \ldots, M^{45})$
22143 12342 12342 21342

33222
32322 (M^{46})
22233

Let $\left\lfloor \frac{^2K_n}{t} \right\rfloor$ denote the set of universal tth parts of the complete 2-graph ^2K_n. Note that the following result completes the proof of Theorem 1.

Theorem 4 $\left\lfloor \frac{^2K_5}{3} \right\rfloor = \{F^1, F^6, F^9, F^{11}, F^{14}, F^{15}\}$.

Proof. We first prove that each of six listed multigraphs is a universal part for $n = 5$. To this end, the 3-packing matrices are presented in Table 6.

Table 6. Decomposition matrices for $n = 5$:

F^1	F^6	F^9	F^{11}	F^{14}	F^{15}
$\langle R\rangle = 2K_2$					
. 1 1 1 2	. 1 1 1 2	. 1 1 1 2	. 1 1 1 1	. 1 1 2 1	. 1 2 2 1
1 . 1 3 2	1 . 2 1 2	2 . 1 2 1	2 . 1 1 2	3 . 1 1 3	2 . 1 1 3
2 0 . 3 2	0 3 . 3 1	3 3 . 3 2	2 3 . 3 2	2 2 . 3 2	3 0 . 1 2
1 3 3 . 3	2 3 3 . 3	1 3 0 . 3	3 3 0 . 2	3 2 0 . 1	3 2 3 . 3
2 2 3 0 .	3 2 2 0 .	0 2 2 3 .	2 0 3 3 .	2 0 3 3 .	1 3 2 0 .

$\langle R \rangle = C_2$

. 1 1 1 2	. 1 1 1 2	. 1 1 1 2	. 1 1 1 1	. 1 1 2 1	. 1 1 1 2
1 . 3 1 3	1 . 1 2 3	2 . 1 3 1	3 . 1 1 2	3 . 2 1 1	3 . 2 2 1
1 3 . 0 2	2 3 . 0 2	2 3 . 2 2	2 3 . 0 2	2 3 . 1 2	2 3 . 1 3
2 3 0 . 2	3 3 0 . 1	1 3 2 . 0	2 3 0 . 2	3 2 3 . 0	3 3 2 . 0
2 3 2 3 .	2 3 3 2 .	3 3 3 0 .	2 3 3 3 .	2 3 3 0 .	2 1 3 0 .

$\langle R \rangle = P_3$

. 1 1 1 2	. 1 1 1 2	. 1 1 1 3	. 1 1 1 1	. 1 2 1 1	. 1 3 1 2
1 . 3 1 3	1 . 2 1 3	2 . 1 2 1	2 . 1 1 3	2 . 1 2 1	2 . 1 2 1
1 3 . 3 2	2 3 . 2 1	2 2 . 3 2	2 2 . 2 2	3 2 . 2 1	3 2 . 2 1
2 3 0 . 2	3 2 2 . 3	1 3 0 . 3	2 3 3 . 3	2 3 3 . 3	1 3 3 . 3
2 3 2 0 .	3 0 0 3 .	3 3 2 0 .	3 0 3 0 .	3 3 0 0 .	2 3 0 0 .

We next show that if F^* is any of remaining eleven multigraphs in Fig. 1, then a 3-remainder exists which is not realized by 3-packings of F^* in ${}^2K_5$, see Lemma 5. □

Table 7. Obstructions for being universal:

F^i	j in M^j	edges		j in M^j	edges	
F^2	41	4-5	lack			
F^3	31	4-5	excess	32	3-5	excess
F^4	41	1-2	excess			
F^5	42	3-4	lack	43	1-2 or 1-5	excess
	44	2-4	excess	45	1-2 or 2-5	excess
F^7, F^{10}	42	1-2	excess	43	3-4	excess
	44	2-5	excess	45	4-5	excess
F^8	42	1-5	lack or exc.	43	1-2	lack or exc.
	44	1-2	lack or exc.	45	1-2	lack or exc.
F^{12}	46	2-4 or 3-5	excess			
F^{13}	46	1-2 or 1-3	excess			
F^{16}	46	4-5	lack			
F^{17}	46	4-5	lack			

Lemma 5 *The multigraph F^3 is not a third part of ${}^2K_5$ with remainder C_2. Moreover, none of remaining ten multigraphs in Fig.* 1 *is a third part with remainder $2K_2$.*

Proof. There are 11 cases to deal with. For example, relatively hard is case of the multigraph F^5 and the matrix M^{44}. We first note that degree-3 and degree-4 vertices in each row of M^{44} are mutually doubly adjacent, see F^5 in Fig. 1. Therefore degree-1 vertex in row 3 is adjacent to column 5 and that in row 2 to column 1. Consequently, degree-4 vertex in column 1 is doubly adjacent to column 3 in row 1. Thus a third edge 2-4 in row 1 is required and this makes a 2-4 excess as stated in Table 7. In a similar way we deal with the multigraph F^{12} and matrix M^{46}. Remaining cases are rather simple and detailed proofs can be derived from Table 7. $\square$

Corollary 6 *The following three sequences* $(5, 3, 2, 2, 0)$, $(4, 3, 2, 2, 1)$ *and* $(3, 3, 2, 2, 2)$ *are the only degree sequences among universal third parts of the complete 2-graph* ${}^2K_5$. $\square$

This implies the following observation to be used in what follows.

Corollary 7 *Standard degree-decomposition matrices of the universal third parts for the remainder* C_2*, see Table* 4*, are among matrices* M^{30} *and* M^{33}*–*M^{38}. $\square$

3 Proof of Theorem 2

Lemma 8 *There are exactly two half-degree sequences, namely,*

$$s^1 := (3, 2, 1, 0, 0), \quad s^2 := (2, 2, 1, 1, 0),$$

among universal third parts of DK_5.

Proof. Because $|R| = 2$, there are 5 shapes of t-remainder R in DK_5. The digraph $DK_5 - R$ has one of the following five sequences of degree pairs (outdegree, indegree),

$$\begin{array}{ll}
((4,4),(4,4),(4,4),(3,3),(3,3)) & \text{for } \langle R\rangle = \vec{C}_2, \\
((4,4),(4,4),(4,3),(3,4),(3,3)) & \text{for } \langle R\rangle = \vec{P}_3, \\
((4,4),(4,4),(4,3),(4,3),(2,4)) & \text{for } \langle R\rangle = \vec{P}_{3out}, \\
((4,4),(4,4),(4,2),(3,4),(3,4)) & \text{for } \langle R\rangle = \vec{P}_{3in}, \\
((4,4),(4,3),(4,3),(3,4),(3,4)) & \text{for } \langle R\rangle = 2\vec{P}_2.
\end{array} .$$

Therefore any corresponding half-degree sequence (indegree or outdegree alike) of $DK_5 - R$ is one of the two sequences $B = (4, 4, 4, 4, 2)$ or $B = (4, 4, 4, 3, 3)$. Suppose that F is a universal third part of DK_5. Then F is of size 6. Fix either $B =: (B_1, \ldots, B_5)$ and assume that $\beta = (b_1, \ldots, b_5)$ is a half-degree sequence of F. Then the corresponding half-degree-decomposition matrices for F impose the following condition.

(*ii*) There exist three permutations $\sigma_1, \sigma_2, \sigma_3$ such that $b_i + b_{\sigma_1(i)} + b_{\sigma_2(i)} = B_{\sigma_3(i)}$ for $i = 1, \ldots, 5$ and for any choice of the pair B and β.

There are 8 partitions of 6 into at most five summands of which the largest is at most 4. Each of the partitions gives rise to a half-degree sequence, say $\tilde{\beta}$, of a digraph on 5 vertices, see Table 8 wherein the symbol + indicates that the condition (*ii*) is satisfied. It is easy to see that Table 8 is correct. Hence it follows that β, if exists, is as stated. □

Table 8. Partitions of 6 satisfying the condition (*ii*):

$\tilde{\beta}\backslash B$	(4,4,4,3,3)	(4,4,4,4,2)
(4,2,0,0,0)	-	+
(4,1,1,0,0)	+	-
(3,3,0,0,0)	-	-
(3,2,1,0,0)	+	+
(3,1,1,1,0)	+	-
(2,2,2,0,0)	-	+
(2,2,1,1,0)	+	+
(2,1,1,1,1)	+	-

In what follows we use the abbreviation DP for *degree pair* in the names DP-*sequence* and DP-*decomposition matrix*, the counterparts of degree sequence and degree-decomposition matrix, respectively.

Using results Theorem 4 and Corollary 6 on all universal third parts of ${}^2K_5$ we show the following result, which completes the proof of Theorem 2.

Lemma 9 *No universal third part of* ${}^2K_5$ *has an orientation which could be a universal third part of* DK_5.

Proof. Assume that $((c_1, d_1), \ldots, (c_5, d_5))$ is a DP-sequence of a universal third part of DK_5. Then the following two conditions are satisfied.

(*iii*) Both $(c_1, ...c_5)$ and $(d_1, ...d_5)$ are permutations of either sequence s^1 or s^2, or both.

(*iv*) The sequence $(c_1{+}d_1, ..., c_5{+}d_5)$ is a permutation of one of the sequences $(5, 3, 2, 2, 0)$, $(4, 3, 2, 2, 1)$ and $(3, 3, 2, 2, 2)$.

The condition (*iv*) follows from Corollary 6.

For each of the three degree sequences in Corollary 6, we use three decision trees (namely, s^1-s^1 tree, s^2-s^2 tree and s^1-s^2 tree) in order to split the degree sequence into all possible DP-sequences $((c_1, d_1), \ldots, (c_5, d_5))$ such that both conditions (*iii*) and (*iv*) are satisfied. The converse s^2-s^1 splits are omitted wlog. Results of this procedure are shown in Table 9.

Table 9. DP-sequences satisfying the conditions (*iii*) and (*iv*):

(5,3,2,2,0)	s^1-s^1	((3,2),(2,0),(1,1),(0,3),(0,0))
		((3,0),(2,3),(1,1),(0,2),(0,0))
	s^1-s^2	((3,2),(2,1),(1,1),(0,2),(0,0))
(4,3,2,2,1)	s^1-s^1	((3,1),(2,0),(1,0),(0,3),(0,2))
		((3,0),(2,0),(1,3),(0,2),(0,1))
	s^1-s^2	((3,1),(2,1),(1,0),(0,2),(0,2))
		((3,1),(2,0),(1,2),(0,2),(0,1))
		((3,0),(2,2),(1,1),(0,2),(0,1))
	s^2-s^2	((2,2),(2,1),(1,1),(1,0),(0,2))
		((2,2),(2,0),(1,2),(1,1),(0,1))
(3,3,2,2,2)	s^1-s^2	((3,0),(2,1),(1,1),(0,2),(0,2))
M^{38}	s^1-s^1	((3,0),(2,0),(1,1),(0,2),(0,3))
F^{14}, F^{15}	s^2-s^2	((2,1),(2,0),(1,2),(1,1),(0,2))

In order to complete the proof, we show that in each of degree-decomposition matrices for the remainder C_2, namely, in matrices M^{30} and M^{33}–M^{38}, see Corollary 7, there is a pair of columns which cannot be split into two columns of degree pairs (taken from Table 9) which could represent an orientation in any of underlying multigraphs. This way we show that no orientation in question can realize the remainder $\vec{C}_2$. Those columns follow:

$$\begin{matrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{matrix}\ (M^{30}, M^{37}, M^{38}),\quad \begin{matrix} 2 & 2 \\ 4 & 2 \\ 2 & 2 \end{matrix}\ (M^{33}, M^{34}),\quad \begin{matrix} 2 & 2 \\ 4 & 2 \\ 2 & 4 \end{matrix}\ (M^{35}),\quad \begin{matrix} 4 & 2 \\ 2 & 2 \\ 2 & 4 \end{matrix}\ (M^{36}).$$

Consider the case of a pair of all-2 columns in an M^j, $j = 30,\ 37,\ 38$. Each corresponding DP column has the sum $(3,3)$ in order to assure the remainder $\vec{C}_2$. However, the degree pair $(1,1)$ appears at most once among DP-sequences in Table 9. Consequently, both all-2 columns in M^j split into columns with distinct degree pairs $(1,1),(2,0),(0,2)$. Such degree pairs, all without any repetition, are available in Table 9 in the last two DP-sequences only. These are to be DP-sequences in a required orientation of F^{14} and/or F^{15}. Moreover, it follows that rows in the two DP columns are to represent distinct pairs of degree-2 vertices. Since degree-2 vertices are not independent in those multigraphs, no required orientation exists in this case.

It remains to deal with next three pairs of columns listed above, with degree sequence $(4,3,2,2,1)$, and with corresponding DP-sequences in Table 9. Then each degree column with sum 8 comprises degree 4 and twice degree 2. Moreover, the column should split into DP column with sum $(4,4)$. On the other hand, 4 as a degree splits into DP $(3,1)$, $(1,3)$ or $(2,2)$, see Table 9.

It is easy to see that the required splitting comprising DP's $(2, 2)$ and twice $(1, 1)$ is the only possible. Then no splitting exists for the accompanying column because $(1, 1)$ appears only once in the related DP-sequences. $\square$

References

[1] J. Bosák, Decompositions of Graphs, Kluwer Acad. Publ. (1990).

[2] V. Chungphaisan, Conditions for sequences to be r-graphic, Discrete Math. **7** (1974) 31–39.

[3] R. Diestel, Graph Theory, Springer (2005).

[4] F. Harary, R.W. Robinson, N. Wormald, Isomorphic factorisations V: directed graphs, Mathematika **25** (1978) 279–285.

[5] M. Meszka, Generating third parts of nearly complete digraphs [Polish], Jagiellonian University, MSc thesis (1994).

[6] M. Meszka, Z. Skupień, Self-converse and oriented graphs among the third parts of nearly complete digraphs, Combinatorica **18** (1998) 413–424.

[7] M. Meszka, Z. Skupień, On some third parts of nearly complete digraphs, Discrete Math. **212** (2000) 129–139.

[8] M. Meszka, Z. Skupień, Decompositions of nearly complete digraphs into t isomorphic parts, Discuss. Math. Graph Theory (2009) 563–572.

[9] R.C. Read, On the number of self-complementary graphs and digraphs, J. London Math. Soc. **38** (1963) 99-104.

[10] Z. Skupień, The complete graph t-packing and t-coverings, Graphs Comb. **9** (1993) 353–363.

[11] Z. Skupień, Clique parts independent of remainders, Discuss. Math. Graph Theory **22** (2002) 361.

[12] D.B. West, Introduction to Graph Theory, Prentice Hall (2001).