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Abstract

A *packing* of a graph G is a set $\{G_1, G_2\}$ such that $G_1 \cong G$, $G_2 \cong G$, and G_1 and G_2 are edge disjoint subgraphs of K_n. Let $\mathcal{F}$ be a family of graphs. A *near packing admitting* $\mathcal{F}$ of a graph G is a generalization of a packing. In a near packing admitting $\mathcal{F}$, the two copies of G may overlap so the subgraph defined by the edges common to both copies is a member of $\mathcal{F}$. In the paper we study three families of graphs (1) $\mathcal{E}_k$ – the family of all graphs with with at most k edges, (2) $\mathcal{D}_k$ – the family of all graphs with maximum degree at most k, and (3) $\mathcal{C}_k$ – the family of all graphs that do not contain a subgraph of connectivity greater than or equal to $k+1$. By $m(n, \mathcal{F})$ we denote the maximum number m such that each graph of order n and size less than or equal to m has a near-packing admitting $\mathcal{F}$. It is well known that $m(n, \mathcal{C}_0) = m(n, \mathcal{D}_0) = m(n, \mathcal{E}_0) = n - 2$ because a near packing admitting $\mathcal{C}_0$, $\mathcal{D}_0$ or $\mathcal{E}_0$ is just a packing. We prove some generalization of this result, namely we prove that $m(n, \mathcal{C}_k) \approx (k+1)n$, $m(n, \mathcal{D}_1) \approx \frac{3}{2}n$, $m(n, \mathcal{D}_2) \approx 2n$. We also present bounds on $m(n, \mathcal{E}_k)$. Finally, we prove that each graph of girth at least five has a near packing admitting $\mathcal{C}_1$ (i.e. a near packing admitting the family of acyclic graphs).

1 Introduction

In this paper we use the term *graph* to refer to simple graphs without loops or multiple edges. The vertex and edge set of a graph G is denoted by $V(G)$ and $E(G)$, respectively. The maximum degree of G is denoted by $\Delta(G)$. A graph is called k-connected if any two of its vertices can be joined by k internally vertex disjoint paths. A complete graph K_1 is 0-connected.

Definition 1 *Let G_1 and G_2 be two graphs such that $|V(G_1)| = |V(G_2)| = n$. A* packing *of G_1 and G_2 is a pair of edge-disjoint subgraphs $\{H_1, H_2\}$ of K_n such that $H_1 \cong G_1$ and $H_2 \cong G_2$.*

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Definition 2 Let $\mathcal{F}$ be any family of graphs and let G_1, G_2 be two graphs such that $|V(G_1)| = |V(G_2)| = n$. A *near packing admitting* $\mathcal{F}$ of G_1 and G_2 is a pair of edge-disjoint subgraphs $\{H_1, H_2\}$ of K_n such that $H_1 \cong G_1$ and $H_2 \cong G_2$, and the subgraph having edges $E(H_1) \cap E(H_2)$ is a member of $\mathcal{F}$.

Given a graph G and a permutation σ of $V(G)$, by $\sigma(G)$ we denote the graph with $V(\sigma(G)) = V(G)$ and such that $\sigma(u)\sigma(v) \in E(\sigma(G))$ if and only if $uv \in E(G)$ for any $u, v \in V(G)$. The spanning subgraph of G having edges $E(G) \cap E(\sigma(G))$ is denoted by G^*_σ (abbreviated to G^* if no confusion arises). Thus, in case when $G_1 \cong G_2 \cong G$ the problem of finding a near packing admitting $\mathcal{F}$ of G_1 and G_2 is equivalent to the problem of finding a permutation σ of $V(G)$ such that $G^*_\sigma \in \mathcal{F}$. Such a permutation σ of $V(G)$ is called *a near packing of* G *admitting* $\mathcal{F}$.

We consider three families of graphs : (1) $\mathcal{E}_k$ being the family of all graphs with with at most k edges, (2) $\mathcal{D}_k$ being the family of all graphs with maximum degree at most k, and (3) $\mathcal{C}_k$ being the family of all graphs that do not contain a subgraph of connectivity greater than or equal to $k+1$. Notice that $\mathcal{D}_0 = \mathcal{C}_0 = \mathcal{E}_0$ is a family of edgeless graphs. Furthermore $\mathcal{C}_1$ is a family of acyclic graphs and $\mathcal{C}_1 \cap \mathcal{D}_2$ is a family of linear forests (i.e. disjoint unions of paths).

Let $\mathcal{F}$ be any family of graphs. By $m(n, \mathcal{F})$ we denote the maximum number m such that each graph of order n and size less than or equal to m has a near-packing admitting $\mathcal{F}$. A classic result in this area, obtained independently in [1, 2, 8], states that

Theorem 3 ([1, 2, 8]) $m(n, \mathcal{C}_0) = m(n, \mathcal{D}_0) = m(n, \mathcal{E}_0) = n - 2$,

because a near packing admitting $\mathcal{C}_0$, $\mathcal{D}_0$ or $\mathcal{E}_0$ is just a packing. Our aim is to prove some generalizations of Theorem 3. For every $k \geq 1$, we determine $m(n, \mathcal{C}_k)$ up to a constant depending only on k. We find the problem concerning near packings admitting $\mathcal{D}_k$ considerably harder. We determine only $m(n, \mathcal{D}_1)$ up to a constant, while $m(n, \mathcal{D}_2)$ is determined assymptotically. We also give bounds on $m(n, \mathcal{E}_k)$.

The notion of a near packing was introduced by Eaton [3] in order to obtain some investigations concerning the following conjecture of Bollobás and Eldridge:

Conjecture 4 ([1]) *If* $|V(G_1)| = |V(G_2)| = n$ *and* $(\Delta(G_1)+1)\cdot(\Delta(G_2)+1) \leq n+1$, *then there is a packing of* G_1 *and* G_2.

The following theorem is a special case of a more general result proved by Eaton.

Theorem 5 ([3]) *If* $|V(G_1)| = |V(G_2)| = n$ *and* $(\Delta(G_1) + 1) \cdot (\Delta(G_2) + 1) \leq n + 1$, *then there is a near packing admitting* $\mathcal{D}_1$ *of* G_1 *and* G_2.

We also investigate another conjecture of graph packing by Faudree, Rousseau, Schelp and Schuster [4]:

Conjecture 6 *For every non-star graph* G *of girth at least 5, there is a packing of two copies of* G.

In particular, Conjecture 6 is true for sufficiently large planar graphs [7]. On the other hand, the statement from the above conjecture is true if G is a non-star graph of girth at least six [5]. In this paper we prove that the statement is true if the term 'packing' is replaced by the term 'near packing admitting $\mathcal{C}_1$'. This result is in some sense best possible, since for every permutation σ of $V(K_{n,n})$ with $n \geq 3$, $K^*_{n,n}$ contains a cycle C_4.

2 Lemmas

Lemma 7 *Let G be a graph and $k, l, q \geq 0$ integers. Suppose that G contains an independent set $U \subset V(G)$ that satisfies the following conditions:*

1. *$d_G(u) \leq k$ for each $u \in U$,*
2. *$|N_G(u) \cap N_G(v)| \leq q$ for every $u, v \in U$.*

If $|U| \geq \frac{2(k-q)}{l-q+1}$, then for every permutation σ' of $V(G) \setminus U$ there exists a permutation σ of $V(G)$ such that $\sigma|_{G-U} = \sigma'$ and $d_{G^*_\sigma}(u) \leq l$ for each $u \in U$.

Proof. Let $G' := G - U$ and σ' be any permutation of $V(G')$. Below we show that we can extend σ' to a permutation σ as required of G.
For any $v \in V(G')$ let us define $\sigma(v) := \sigma'(v)$. Then let us consider a bipartite graph B with partition sets $X := U \times \{0\}$ and $Y := U \times \{1\}$. For $u, v \in U$ the vertices $(u, 0)$, $(v, 1)$ are joined by an edge in B if and only if $|\sigma'(N_G(u)) \cap N_G(v)| \leq l$. So, if $(u, 0)$, $(v, 1)$ are joined by an edge in B we can put $\sigma(u) = v$. In other words, if $(u, 0)$, $(v, 1)$ are not neighbors in B, then $|\sigma'(N(u)) \cap N(v)| \geq l + 1$. Therefore, since $|N_G(u) \cap N_G(v)| \leq q$ and $d_G(u) \leq k$ for $u \in U$, we have $d_B((u, 0)) \geq |U| - \frac{k-q}{l-q+1} \geq \frac{k-q}{l-q+1}$, by the assumption on $|U|$. Similarly, $d_B((v, 1)) \geq \frac{k-q}{l-q+1}$.

Let $S \subset X$. If $|S| \leq |U| - \frac{k-q}{l-q+1}$ then obviously $|N_B(S)| \geq |S|$. Notice that if $|S| > |U| - \frac{k-q}{l-q+1}$ then $N_B(S) = Y$. Indeed, othervise let $(v, 1) \in Y$ be a vertex which has no neighbour in S. Thus,

$$d_B((v, 1)) \leq |A| - |S| = |U| - |S| < |U| - (|U| - \frac{k-q}{l-q+1}) = \frac{k-q}{l-q+1},$$

a contradiction. Hence, in any case $|S| \leq |N(S)|$. Thus, by the Hall's theorem there is a matching M in G. Therefore we can define $\sigma(u) = v$ for $u, v \in U$ such that $(u, 0)$, $(v, 1)$ are incident with the same edge in M. □

Proposition 8 *Let G be a graph of order n and size m with $m \leq an - f(n)$, where a is a real number and $f(n)$ is a non-decreasing function. If $U \subset V(G)$ and vertices from U cover at least $a|U|$ edges, then*

$$m' \leq an' - f(n'),$$

where n' and m' are respectively the order and the size of $G - U$.

Proof.

$$
\begin{aligned}
m' &\leq an - f(n) - a|U| = a(n - |U|) - f(n) \\
&\leq a(n - |U|) - f(n - |U|) = an' - f(n'),
\end{aligned}
$$

because $f(n) \geq f(n - |U|)$. □

A *starry tree* is a graph H such that (1) $V(H)$ can be partitioned into three sets V_1, V_2 and $\{x\}$ that each induce a tree, (2) there is at least one edge incident to x, and (3) all edges not belonging to the trees induced by V_1 and V_2 are incident to x. A vertex x we call *a middle vertex* of H. Note that a starry tree need not be connected.

Lemma 9 *Let H be a starry tree. Then there is a near-packing of H admitting $\mathcal{C}_1 \cap \mathcal{D}_2$ such that the middle vertex of H is the image of its neighbor.*

Proof. The proof is by induction on $|V_1| + |V_2|$. If $|V_1| + |V_2| = 2$, then the existence of a near-packing as required is obvious. Assume that $|V_1| + |V_2| \geq 3$. Without loss of generality we may assume that $|V_1| \geq 2$. Let l be a leaf in T_1 and let l' be the neighbor of l other than x. We distinguish two cases:

Case 1. The middle vertex x is not joined with l.

Case 2. The middle vertex x is joined with l.

In Case 1, by the induction hypothesis, there exists a near packing σ' of $H' = H - \{l\}$ (admitting $\mathcal{C}_1 \cap \mathcal{D}_2$) such that x is the image of its neighbor. If l' is a fixed point of σ', then σ_1 such that $\sigma_1(l) = l'$, $\sigma_1(l') = l$ and $\sigma_1(u) = \sigma'(u)$ for any $U \in V(H) \setminus \{l, l'\}$ is a near packing as required of H. Otherwise, σ_2 such that $\sigma_2(l) = l$ and $\sigma_2(u) = \sigma'(u)$ for any $U \in V(H) \setminus \{l\}$ is a near packing as required of H.

Consider Case 2. By Theorem 3, there is a packing σ'' of two copies of $H - \{x, l\}$. Then σ_3 such that $\sigma_3(x) = l$, $\sigma_3(l) = x$ and $\sigma(u) = \sigma'(u)$ for any $U \in V(H) \setminus \{x, l\}$ is a near packing as required of H. □

3 Near packings admitting $\mathcal{E}_k$

The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ together with all the edges joining V_1 and V_2.

Lemma 10 *If $n \geq 2k + 2$ then $m(n, \mathcal{E}_{2k}) \leq \left\lceil \frac{(k+2)(n-1)}{2} \right\rceil - 1$.*

Proof. Let H be a k-regular graph of order $n-1$ provided that k is even or $n-1$ is even. Otherwise, let H be a graph with all but one vertices having degree k and one vertex having degree $k+1$. Let $G = K_1 + H$ and $V(K_1) = \{u\}$. It is easily seen that for any permutation σ of $V(G)$, the vertex u as well as $\sigma(u)$ has degree at least $k+1$ in G^*_σ. Thus, if $u \neq \sigma(u)$ then G^*_σ has at least $2k+1$ edges. If $u = \sigma(u)$ then u has degree $n-1$ in G^*_σ. Since $n \geq 2k+2$, G^*_σ has at least $2k+1$ edges. Therefore, G does not have a near packing admitting $\mathcal{E}_{2k}$. Furthermore, $E(G) = \frac{(k+1)(n-1)+n-1}{2} = \frac{(k+2)(n-1)}{2}$ if k is even or $n-1$ is even, or $E(G) = \frac{(k+1)(n-2)+(k+2)+(n-1)}{2} = \frac{(k+2)(n-1)+1}{2}$ otherwise. $\square$

Theorem 11 $m(n, \mathcal{E}_k) \geq \sqrt{\frac{k}{2}n(n-1)}$.

Proof. Let G be a graph of order n and size m. We will prove that if $m \leq \sqrt{\frac{k}{2}n(n-1)}$ then there is a near-packing of G admitting $\mathcal{E}_k$. Consider the probability space whose $n!$ points are the permutations of $V(G)$. For any two edges $e, f \in E(G)$ let X_{ef} denote the indicator random variable with value 1 if f is an image of e. Then

$$E(X_{ef}) = Prob(X_{ef} = 1) = \frac{2(n-2)!}{n!} = \binom{n}{2}^{-1}.$$

Let $X = \sum_{e,f \in E(G)} X_{ef}$. Thus, by the linearity of expectation, we have

$$E(X) = \sum_{e,f \in E(G)} E(X_{ef}) \leq m^2 \binom{n}{2}^{-1} \leq k.$$

This implies that there exists a permutation σ of $V(G)$ such that G^*_σ has at most k edges. Thus, σ is a near packing of G admitting $\mathcal{E}_k$. $\square$

4 Near packings admitting $\mathcal{C}_k$

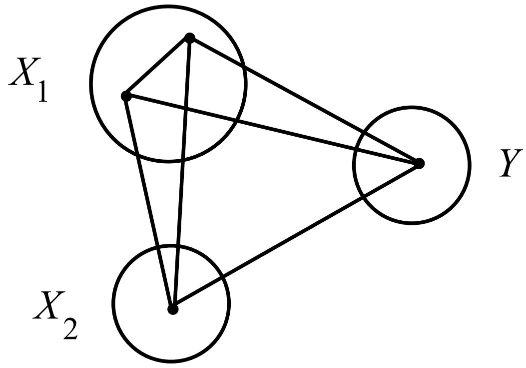
Recall that $m(n, \mathcal{C}_0) = n - 2$. We start with the following construction. Let $K^+_{s,k-s,k-s}$ denote a graph with vertex set $V(K^+_{s,k-s,k-s}) = X_1 \cup X_2 \cup Y$ such that X_1, X_2, Y are pairwise disjoint and $|X_1| = s$, $|X_2| = |Y| = k - s$. Furthermore, $E(K^+_{s,k-s,k-s}) = E_1 \cup E_2$, where $E_1 = \{xy : x \in X_1 \cup X_2, y \in Y\}$ and $E_2 = \{xz : x \in X_1, z \in X_1 \cup X_2\}$. In other words, $K^+_{s,k-s,k-s}$ arises from a tripartite graph by adding all possible edges having two endpoints in X_1, see Figure 1. It is easily seen that any two vertices of $K^+_{s,k-s,k-s}$ are joined by at least k internally vertex disjoint paths, so $K^+_{s,k-s,k-s}$ is k connected. In what follows $\bar{G}$ denotes the complement of a graph G, i.e. a graph on the same vertex set as G and with the property that $e \in E(\bar{G})$ if and only if $e \notin E(G)$.

Lemma 12 $m(n, \mathcal{C}_k) \leq (k+1)n - (k+1)\frac{k+2}{2} - 1$

Proof. Let $G = \overline{K_{k+1}} + K_{n-k-1}$. Clearly, $|E(G)| = (k+1)n - (k+1)\frac{k+2}{2}$. We will show that G does not have a near packing admitting $\mathcal{C}_k$. Consider an arbitrary permutation σ of $V(G)$. Let $S \subset V(K_{k+1})$ be a maximal set of vertices with the property that $\sigma(S) \subset V(K_{k+1})$. Let $|S| = s$. Then, G^*_σ contains a $K^*_{s,k+1-s,k+1-s}$ with $X_1 = S$, $Y = V(K_{k+1}) \setminus S$ and $X_2 \subset K_{n-k-1}$. $\square$

Theorem 13 $m(n, \mathcal{C}_k) \geq (k+1)n - 4k(k+1)^2 - 2$

Figure 1: $K^+_{2,1,1}$



Proof. For $k = 0$ the result follows from Theorem 3. Fix $k \geq 1$ and let $c_k = 4k(k+1)^2 + 2$. We will prove that each graph of order n and size at most $(k+1)n - c_k$ has a near packing admitting $\mathcal{C}_k$.

Suppose that G is a counterexample with minimum order n. Let m denote the size of G, so $m \leq (k+1)n - c_k$. Note that if $n \leq 4(k+1)^2$, then

$$m \leq (k+1)n - c_k = kn - c_k + n \leq k(4(k+1)^2) - (4k(k+1)^2 + 2) + n = n - 2.$$

Hence G has a near packing admitting $\mathcal{C}_k$, by Theorem 3, which contradicts our assumption on G. Thus, we may assume that $n \geq 4(k+1)^2 + 1$. Furthermore, if $\Delta(G) \leq 2(k+1) - 1$ then $(\Delta(G) + 1)^2 \leq 4(k+1)^2 < n + 1$. Hence, G has a near packing admitting $\mathcal{C}_k$ by Theorem 5 (because $\mathcal{D}_1 \subset \mathcal{C}_k$), a contradiction again. Therefore, we may assume that $\Delta(G) \geq 2(k+1)$. Let $w \in V(G)$ with $d_G(w) \geq 2(k+1)$.

Suppose first that G contains a vertex u with $d_G(u) \leq k$. By Proposition 8 and by the minimality assumption, $G' := G - \{u, w\}$ has a near packing σ' admitting $\mathcal{C}_k$. We claim that $\sigma := (u, w)\sigma'$ is a near packing of G admitting $\mathcal{C}_k$. Indeed, since $d_G(u) \leq k$ then $d_{G^*}(u) \leq k$ as well as $d_{G^*}(w) \leq k$. Hence, neither u nor w can be in a subgraph of G^* of connectivity $k+1$ or more. Moreover, since $\sigma|_{G'}$ is a near packing of G' admitting $\mathcal{C}_k$, then $G^* - \{u, w\}$ does not contain a subgraph of connectivity $k+1$ or more, neither. Therefore, σ is a near packing of G admitting $\mathcal{C}_k$.

Thus, we may assume that $d_G(u) \geq k+1$ for every $u \in V(G)$. Let S be a maximum set of vertices of G such that S is independent, $k+1 \leq d_G(u) \leq 2k+1$ for each $u \in S$, and $|N_G(u) \cap N_G(w)| \leq k$ for every $u, w \in S$. Since S is independent, by Proposition 8 and by the minimality assumption, $G - S$ has a near packing σ'' admitting $\mathcal{C}_k$. By Lemma 7 (with k, l, q replaced by $2k+1, k, k$, respectively), if $|S| \geq 2k+2$ then there is a permutation σ of G, such that $\sigma|_{G-S} = \sigma''$ and $d_{G^*}(u) \leq k$ for every $u \in S$. Simirarly as before, we can see that σ is a near packing of G admitting $\mathcal{C}_k$, a contradiction.

Therefore $|S| \le 2k+1$ and so $|N_G(S)| \le (2k+1)^2$. Let $V_j = \{v \in V(G) \setminus N_G(S) : d_G(v) = j\}$. Note that by the definition of S, we have $|N_G(S) \cap N_G(u)| \ge k+1$ for every $u \in V_{k+1} \cup \dots \cup V_{2k+1}$. Hence, vertices from $N_G(S)$ are incident (in common) to at least $(k+1)(|V_{k+1}| + \dots + |V_{2k+1}|)$ edges. Thus,

$$\begin{aligned}
&(2k+2)n - 8k(k+1)^2 - 4 \ge 2m \\
&= \sum_{u \in N_G(S)} d_G(u) + \sum_{v \in V(G) \setminus N_G(S)} d_G(v) \\
&\ge (k+1)(|V_{k+1}| + \dots |V_{2k+1}|) + (k+1)|V_{k+1}| + \dots + (2k+1)|V_{2k+1}| \\
&+ (2k+2)(n - |V_{k+1}| + \dots |V_{2k+1}| - |N_G(S)|) \\
&\ge (2k+2)n - (2k+2)(2k+1)^2,
\end{aligned}$$

a contradiction. Hence, we deduce no counterexample to Theorem 13 exists. □

Theorem 14 *Every graph of girth at least 5 has a near packing admitting $\mathcal{C}_1$.*

Proof. Let G be a minimum counterexample to Theorem 14. Let $u \in V(G)$ with $d_G(u) = \Delta(G)$. Let $G' = G - u$ and $U = N_G(u)$. By the girth assumption, U is an independent set in G' (as well as in G), and $N_{G'}(x) \cap N_{G'}(y) = \emptyset$ for every $x, y \in U$. By the minimality assumption $G'' := G' - U$ has a near packing σ'' admitting $\mathcal{C}_1$. Moreover, $|U| = \Delta(G)$ and $d_{G'}(u) \le \Delta(G) - 1$. Hence, by Lemma 7 (with $k = \Delta(G) - 1, l = 1, q = 0$), G' has a near packing σ' such that $\sigma'|_{G''} = \sigma''$ and $d_{G'^*}(u) \le 1$ for each $u \in U$. Thus, since G''^* is acyclic, G'^* is also acyclic. Let u be any vertex from U. It is easy to see that the permutation σ such that $\sigma(u) = x$, $\sigma(x) = u$ and $\sigma(y) = \sigma'(y)$ for every $y \in V(G) \setminus \{u, x\}$ is a near packing of G admitting $\mathcal{C}_1$, a contradiction. □

5 Near packings admitting $\mathcal{D}_k$

Recall that $m(n, \mathcal{D}_0) = n - 2$.

Lemma 15 $m(n, \mathcal{D}_k) \le \left\lceil \frac{(k+2)(n-1)}{2} \right\rceil - 1$.

Proof. Let H be a k-regular graph of order $n-1$ provided that k is even or $n-1$ is even. Otherwise, let H be a graph with all but one vertices having degree k and one vertex having degree $k+1$. Let $G = K_1 + H$ and $V(K_1) = \{u\}$. It is easily seen that for any permutation σ of $V(G)$, the vertex u (as well as its image) has degree at least $k+1$ in G^*_σ. Thus, G does not have a near packing admitting $\mathcal{D}_k$. Furthermore, $E(G) = \frac{(k+1)(n-1)+n-1}{2} = \frac{(k+2)(n-1)}{2}$ if k is even or $n-1$ is even, or $E(G) = \frac{(k+1)(n-2)+(k+2)+(n-1)}{2} = \frac{(k+2)(n-1)+1}{2}$ otherwise.□

Theorem 16 $m(n, \mathcal{D}_1) \ge \frac{3}{2}n - 10$

Proof. Let G be a counterexample of minimum order n. Without loss of generality we assume that $m := |E(G)| = \frac{3}{2}n - 10$. Note that if $n \le 16$ then

$\frac{3}{2}n - 10 \leq n - 2$. Thus, by Theorem 3, G has a packing which contradicts our assumption on G. Hence, we may assume that $n \geq 17$. Furthermore, if $\Delta(G) \leq 3$, then $(\Delta(G)+1)^2 \leq 16 < n+1$, so G has a near packing admitting $\mathcal{D}_1$ by Theorem 5. Thus, we may assume that $\Delta(G) \geq 4$. Let $w \in V(G)$ with $d_G(w) \geq 4$.

Suppose first that G has a vertex u with $d_G(u) = 0$. Then, by Proposition 8 and by the minimality assumption, $G_1 := G - \{u, w\}$ has a a near packing σ_1 admitting $\mathcal{D}_1$. Clearly, $(u, w)\sigma_1$ is a near packing of G admitting $\mathcal{D}_1$.

So we may assume that G has no isolated vertex. Suppose now that G has a vertex u with $d_G(u) = 1$ and let v be the neighbor of u. If $d_G(v) \geq 3$ then, by Proposition 8 and by the minimality assumption, $G_2 := G - \{u, v\}$ has a near packing σ_2 admitting $\mathcal{D}_1$. Clearly, $(u, v)\sigma_2$ is a near packing admitting $\mathcal{D}_1$ of G. Similarly, if $d_G(v) = 1$ then $(u)(w, v)\sigma_3$ is a near packing admitting $\mathcal{D}_1$ of G where σ_3 is a near packing admitting $\mathcal{D}_1$ of $G - \{u, v, w\}$ (σ_3 exists by the minimality assumption). Thus we may assume that $d_G(v) = 2$. Let x be the neighbor of v different from u. If $x \neq w$ then $(u)(v, w, x)\sigma_4$ is a near packing admitting $\mathcal{D}_1$ of G where σ_4 is a near packing admitting $\mathcal{D}_1$ of $G - \{u, v, w, x\}$ (σ_4 exists by the minimality assumption). Finally, if $x = w$ then $(u)(v, w)\sigma_5$ is a near packing admitting $\mathcal{D}_1$ of G where σ_5 is a near packing admitting $\mathcal{D}_1$ of $G - \{u, v, w\}$ (σ_5 exists by the minimality assumption).

Therefore, we may assume that $d_G(u) \geq 2$ for each $u \in V(G)$. Let $S \subset V(G)$ be a maximal set such that S is independent in G, $d_G(v) = 2$ for every $v \in S$, and $N_G(u) \cap N_G(v) = \emptyset$ for every $u, v \in S$. Note that $S \neq \emptyset$. By Proposition 8 and by the minimality assumption, $G - S$ has a near packing σ' admitting $\mathcal{D}_1$. Note that if $|S| \geq 4$, then by Lemma 7 (with $k = 2$, $q = 0$ and $l = 0$), there exists a near packing of G admitting $\mathcal{D}_1$, a contradiction with the assumption on G. Thus, $|S| \leq 3$ and so $|N_G(S)| \leq 6$. Let $V_j = \{v \in V(G) \setminus N_G(S) : d_G(v) = j\}$. Note that by the definition of S, we have $|N_G(S) \cap N_G(u)| \geq 1$ for every $u \in V_2$. Therefore,

$$\begin{aligned} 3n - 20 = 2m &= \sum_{u \in N_G(S)} d_G(u) + \sum_{v \in V(G) \setminus N_G(S)} d_G(v) \\ &\geq |V_2| + 2|V_2| + 3(n - |V_2| - |N_G(S)|) \geq 3n - 18, \end{aligned}$$

a contradiction. Hence, we deduce no counterexample to Theorem 16 exists. □

If σ is a packing of G and $\sigma(u) \neq u$ for every $u \in V(G)$ then we say that G is fixed-point-free packable.

In the proof of our next result we apply the idea that was first used in [6]. We will need also the following strengthening of Theorem 3:

Theorem 17 ([9]) *Every graph of order n and size at most $n-2$ is fixed-point-free packable.*

Theorem 18 $m(n, \mathcal{D}_2 \cap \mathcal{C}_1) \geq 2n - 10n^{2/3} - 7$.

Proof. Let $t = \lfloor n^{1/3} \rfloor$. We will prove that each graph of order n and size at most $\max\{2n - 10n^{2/3} - 7, 0\}$ has a near packing admitting $\mathcal{D}_2 \cap \mathcal{C}_1$. Clearly, we may focus only on the case when the maximum is equal to $2n - 10n^{2/3} - 7$ which implies that $n \geq 125$. Suppose that G is a counterexample with minimum order n, $n \geq 125$. Hence, by Theorem 5, we may assume that $\Delta(G) > \sqrt{n+1} > 11$. Let $u \in V(G)$ with $d_G(u) = \Delta \geq 12$. First we prove the following claim.

Claim 19 *G has no isolated vertices and G has at most 7 vertices of degree 1.*

Proof of Claim. Suppose first that there is a vertex $v \in V(G)$ with $d_G(v) = 0$. By Proposition 8 and by the minimality assumption on G, there is a near packing σ_1 of $G - \{u, v\}$ admitting $\mathcal{D}_2 \cap \mathcal{C}_1$. Hence, σ such that $\sigma_(u) = v$, $\sigma(v) = u$ and $\sigma(x) = \sigma'(x)$ for every $x \in V(G) \setminus \{v, u\}$ is a near-packing as required of G, a contradiction.

Suppose now that G has at least 8 vertices of degree 1. Let $W = \{v_1, ..., v_8\} \subset V(G)$ be a set of vertices such that $d_G(v_1) = ... = d_G(v_8) = 1$. Let y_i be the neighbor of v_i. We distinguish two cases:

Case 1. There is $i \in \{1, ..., 8\}$ such that $y_i \in W$. Then by the minimality assuption, there is a near packing as required of $G_1 := G - \{v_i, y_i, u\}$. Let σ_1 be a near packing of G_1 admitting $\mathcal{D}_2 \cap \mathcal{C}_1$. Then σ such that $\sigma(v_i) = y_i$, $\sigma(y_i) = u$, $\sigma(u) = v_i$ and $\sigma(x) = \sigma_1(x)$ for every $x \in V(G) \setminus \{v_i, y_i, u\}$ is a near-packing as required of G.

Case 2. W is a set of independent vertices. If there is $i \in \{1, ..., 8\}$ such that $d(y_i) \geq 4$, then by Proposition 8 and the minimality assumption there is a near packing σ_2 of $G_2 := G - \{v_i, y_i\}$ admitting $\mathcal{D}_2 \cap \mathcal{C}_1$. Thus, σ such that $\sigma(v_i) = y_i$, $\sigma(y_i) = v_i$ and $\sigma(x) = \sigma_2(x)$ for every $x \in V(G) \setminus \{v_i, y_i\}$ is a near-packing as required of G.

Assume that $d(y_i) \leq 3$ for each $i = 1, ..., 8$. In particular, u is not a neighbor of any v_i. Suppose next that a vertex y is a common neighbor of two vertices $v, v' \in W$. Then, by Proposition 8 and the minimality assumption, there exists a near packing σ_3 of $G_3 = G - \{v, v', y, u\}$ admitting $\mathcal{D}_2 \cap \mathcal{C}_1$. Thus, σ such that $\sigma(v) = u$, $\sigma(u) = v$, $\sigma(v') = y$, $\sigma(y) = v'$, and $\sigma(x) = \sigma_3(x)$ for every $x \in V(G) \setminus \{v, v', y, u\}$ is a near-packing as required of G.

Hence, we may assume that $y_i \neq y_j$ if $i \neq j$. Let $w \in W$ and let z be the neighbor of w. Recall that $d_G(z) \leq 3$. Therefore, there are at least 5 vertices $v^1, ..., v^5 \in W$ such that $N(v^i) \cap N(z) = \emptyset$. Consider now graphs $G_4 = G - \{v^1, ..., v^5, w, z, u\}$ and $G_5 = D - \{v^1, ..., v^5, z\}$. By Proposition 8 and by the minimality assumption, G_4 has a near packing σ_4 admitting $\mathcal{D}_2 \cap \mathcal{C}_1$. Then σ such that $\sigma(u) = w$, $\sigma(w) = u$ and $\sigma(x) = \sigma_4(x)$ for every $x \in V(G_4) \setminus \{u, w\}$ is a near-packing of G_4 admitting $\mathcal{D}_2 \cap \mathcal{C}_1$. Furthermore, vertices $v^1, ..., v^5, z$ form an independent set in G, have degrees less than or equal to 3 and have mutually disjoint sets of neighbours. Thus, by Lemma 7 (with $l = q = 0$), G has a near packing admitting $\mathcal{D}_2 \cap \mathcal{C}_1$. □

Hence, we may assume that $d_G(v) \geq 2$ for each vertex v in G except at most seven of degree one. We choose a maximal set S of $V(G)$ such that (1) S is independent in G, (2) for each $u \in S$ we have $2 \leq d_G(u) \leq t$ and (3) vertices of

S have pairwise disjoint neighborhoods. Observe that the number q of vertices of degree greater than t does not exceed $2n^{2/3}$. Indeed

$$4n - 20n^{2/3} - 14 \geq 2||G|| = \sum_{v \in V(G)} d_G(v) \geq 7 + 2(n - 7 - q) + qn^{1/3},$$

hence

$$q \leq \frac{2n - 20n^{2/3} - 7}{n^{1/3} - 2} < 2n^{2/3}.$$

In particular $S \neq \emptyset$. Thus, by Proposition 8 and by the minimality assumption, $G - S$ has a near packing admitting $\mathcal{D}_2 \cap \mathcal{C}_1$. Hence, by Lemma 7 (with $q = l = 0$), there is a near packing of G admitting $\mathcal{D}_2 \cap \mathcal{C}_1$, if $|S| \geq 2t$. Hence $|S| < 2t$, and so $|N(S)| < 2t^2 \leq 2n^{2/3}$. Let $V_j := \{v \in V(G) \setminus N(S) : d_G(v) = j\}$. By the definition of S, every vertex from $V_2 \cup ... \cup V_t$ has a neighbor in $N(S)$. Therefore

$$|N(N(S))| \geq |V_2 \cup ... \cup V_t| \geq n - 7 - m - |N(S)| > n - 7 - 4n^{2/3}. \quad (1)$$

Thus, vertices from $N(S)$ cover at least $n - 7 - 4n^{2/3}$ edges.

Consider now the graph $G - N(S)$. Let T_1,...,T_p denote connected components of $G - N(S)$ which are trees such that each vertex of T_i is incident with at most one vertex in $N(S)$. We call these components *minimal components* of $G - N(S)$. Let $R := G - N(S) - V(T_1) - ... - V(T_p)$. Let r denote the sum of the size of R and the number of all vertices in R which are joined (in G) with $N(S)$ by at least two edges. Since R does not contain minimal components, every component of R which is a tree contains a vertex joined with $N(S)$ by at least two edges. On the other hand, every component of R which is not a tree has at least as many edges as vertices. Hence, $r \geq |R|$. Moreover, r counts all edges in R and some edges between R and $N(S)$ which are not counted in inequality (1), because this inequality counts only the number of vertices in $N(N(S))$ and ignores the number of connections.

Note that there are exactly $n - |N(S)| - |R| - p$ edges in $\bigcup_{i=1}^{p} T_i$. Below we show that p is greater than or equal to $2|N(S)|$. By the assumption and by inequality (1), the size of G satisfies:

$$\begin{aligned} 2n - 10n^{2/3} - 7 \geq ||G|| &\geq n - 7 - 4n^{2/3} + (n - |N(S)| - p - |R|) + r \\ &> 2n - 6n^{2/3} - 7 - p - |R| + r. \end{aligned}$$

Thus

$$p \geq 4n^{2/3} - |R| + r \geq 2|N(S)| - |R| + r \geq 2|N(S)|. \quad (2)$$

Let $G' := G[N(S) \cup V(T_1) \cup ... \cup V(T_{2|N(S)|})]$ and $G'' := G - G'$. Below we show that there exists a near packing of G' admitting $\mathcal{D}_2 \cap \mathcal{C}_1$ such that the image of every vertex in $N(S)$ is not in $N(S)$. Let L be a set of maximum cardinality l of vertex-disjoint starry trees, such that each starry tree is formed of two of the trees T_i, $1 \leq i \leq 2|N(S)|$, and one vertex (the middle vertex) from $N(S)$. Let $H_1, ..., H_l$, $l \leq |N(S)|$, denote the starry trees. Suppose first that $l = |N(S)|$.

By Lemma 9, there is a near packing of H_i admitting $\mathcal{D}_2 \cap \mathcal{C}_1$. Let σ_i be the near packing as required of H_i. We claim that the product $\sigma = \sigma_1 \dots \sigma_{|N(S)|}$ is a near-packing of G' admitting $\mathcal{D}_2 \cap \mathcal{C}_1$. Since σ_i is a near packing as required of H_i, only edges between different starry trees may spoil the near packing of G'. Furthermore, every middle vertex is mapped on a non-middle vertex. Since there are no edges between T_i and T_j for $i \neq j$, the edges between different T_i as well as the edges between middle vertices do not appear in G'^*_σ. It remains to check the edges of the form xy where x is the middle vertex of some starry tree and y is a non-middle vertex of another starry tree. However, since the middle vertex of each starry tree is the image of one of its neighbors in the same starry tree and this neighbor has no other neighbors outside its minimal component, these edges also do not appear in G'^*_σ. Suppose now, that $l < |N(S)|$. Again, we pack every starry tree in such a way that the middle vertex is the image of one of its neighbors. Moreover, since L is maximal, each remaining vertex of $N(S)$ has no neighbors in each of the remaining minimal components (otherwise, we would have an extra starry tree). Hence, by Theorem 17, each of the remaining vertices from $N(S)$ together with two non-trivial minimal components (not involved in any starry tree) can be (properly) packed without fixed points. We claim that the product σ of near packings σ_i and the above mentioned proper packings of the rest of G' is a near packing of G' admitting $\mathcal{D}_2 \cap \mathcal{C}_1$. It suffices to prove that the edges between different starry trees do not appear in G'^*_σ. Suppose for a contradiction that the image of such an edge e in G' coincides with some other edge e' in G'. Using the previous argument, e' must join a vertex $z \in N(S)$ which is not in any starry tree from L with a non-middle vertex of some starry tree H. Moreover, e must join the middle vertex of H with some minimal component which is not in any starry tree from L. By replacing the middle vertices incident to e and e' we obtain more than l starry trees and we get a contradiction. Hence there is a near packing of G' admitting $\mathcal{D}_2 \cap \mathcal{C}_1$.

Recall that $r \geq ||R||$. Furthermore, by (2) we have

$$\begin{aligned} ||G''|| &= ||R \cup T_{2|N(S)|+1} \cup \dots \cup T_p|| \\ &= ||R|| + |T_{2|N(S)|+1}| + \dots + |T_p| - (p - 2|N(S)|) \\ &< ||R|| + |T_{2|N(S)|+1}| + \dots + |T_p| - (r - |R|) - 1 \\ &\leq |R| + |T_{2|N(S)|+1}| + \dots + |T_p| - 1 \\ &= |R \cup T_{2|N(S)|+1} \cup \dots \cup T_p| - 1 = |G''| - 1. \end{aligned}$$

Thus, by Theorem 3, G'' is packable.

Let σ', σ'' denote near packings of G' and G'' admitting $\mathcal{D}_2 \cap \mathcal{C}_1$, respectively. Then $\sigma = \sigma'\sigma''$ is a near packing as required of G. To prove this it suffices to show that edges between $V(G')$ and $V(G'')$ do not appear in G^*_σ. Suppose for a contradiction that the image of an edge xy in G, where $x \in V(G')$ and $y \in V(G'')$, coincides with some other edge $\sigma(x)\sigma(y)$ in G. Then $x, \sigma(x) \in V(G')$ and $y, \sigma(y) \in V(G'')$. Since there are no edges between T_i and R in G, both x and $\sigma(x)$ belong to $N(S)$. Then we get a contradiction, since the image of every vertex in $N(S)$ is not in $N(S)$. The near-packing σ contradicts the assumption that G did not have a near packing admitting $\mathcal{C}_1 \cap \mathcal{D}_2$, so we deduce no counterexample to Theorem 18 exists. □

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