MATEMATYKA
DYSKRETNA
www.ii.uj.edu.pl/preMD/

Rafał KALINOWSKI and Monika PILŚNIAK

Distinguishing graphs by edge-colourings

Preprint Nr MD 067
(otrzymany dnia 11.06.2013)

**Kraków
2013**

Distinguishing graphs by edge-colourings

Rafał Kalinowski*

AGH University of Science and Technology,

al. Mickiewicza 30, 30-059 Krakow, Poland

kalinows@agh.edu.pl

Monika Pilśniak†

AGH University of Science and Technology,

al. Mickiewicza 30, 30-059 Krakow, Poland

pilsniak@agh.edu.pl

Abstract

We investigate the *distinguishing index* $D'(G)$ of a graph G as the least number d such that G has an edge-colouring with d colours that is only preserved by the trivial automorphism. This is an analog to the notion of the distinguishing number $D(G)$ of a graph G, which is defined for colourings of vertices. We obtain a general upper bound $D'(G) \leq \Delta(G)$ unless G is a small cycle C_3, C_4 or C_5.

We also investigate the *distinguishing chromatic index* $\chi'_D(G)$ defined for proper edge-colourings of a graph G. We prove that $\chi'_D(G) \leq \Delta(G) + 1$ except for four graphs C_4, K_4, C_6 and $K_{3,3}$. It follows that, surprisingly, each connected Class 2 graph G admits a minimal proper edge-colouring, i.e., with $\Delta(G) + 1$ colours, preserved only by the trivial automorphism.

Keywords: distinguishing index; distinguishing chromatic index; automorphism; symmetry breaking in graphs

Mathematics Subject Classifications: 05C25, 05C80, 03E10

*The research was partially supported by the Polish Ministry of Science and Higher Education.

†The research was partially supported by the Polish Ministry of Science and Higher Education.

1 Introduction and definitions

Albertson and Collins [1] introduced the *distinguishing number* $D(G)$ of a graph G as the least number d such that G admits a vertex-colouring with d colours that is only preserved by the trivial automorphism. Ten years later Collins and Trenk [2] defined the *distinguishing chromatic number* $\chi_D(G)$ of a graph G for proper colourings, so $\chi_D(G)$ is the least number d such that G has a proper colouring with d colours that is only preserved by the trivial automorphism. This concept has spawned numerous papers on finite graphs and infinite graphs. For endomorphisms instead of automorphisms this concept was investigated in [6].

Note that $D(G) = 1$ for all asymmetric graphs. This means that almost all finite graphs have distinguishing number 1, because almost all graphs are asymmetric (see Erdős and Rényi [4]). Clearly, $D(G) \geq 2$ for all other graphs. Again, it is conjectured that almost all of them have distinguishing number 2. This is supported by some observations of Conder and Tucker [3].

On the other hand, for a complete graph K_n, and a complete bipartite graph $K_{n,n}$ we have $D(K_n) = n$, and $D(K_{n,n}) = n + 1$. Furthermore, the distinguishing number of a cycles C_3, C_4, C_5 is 3, while cycles C_n of length $n \geq 6$ have distinguishing number 2.

This compares with a more general result of Collins and Trenk [2], and of Klavžar, Wong and Zhu [10].

Theorem 1 *[2],[10] If G is a connected graph with maximum degree Δ, then $D(G) \leq \Delta + 1$. Furthermore, equality holds if and only if G is a K_n, $K_{n,n}$ or C_5.*

In the same paper, Collins and Trenk obtained a general result for the distinguishing chromatic number.

Theorem 2 *[2] If G is a connected graph with maximum degree Δ, then $\chi_D(G) \leq 2\Delta$. Furthermore, equality is achieved if and only if G is a $K_{n,n}$ or C_6.*

The aim of this paper is a presentation of fundamental results for colourings of edges instead of vertices. We obtain general bounds, and an interesting relationship between the distinguishing chromatic index and the vertex distinguishing index by walks (introduced in [9]).

Definition 3 *The distinguishing index $D'(G)$ of a graph G is the least number d such that G has an edge-colouring with d colours that is preserved only by the identity automorphism of G.*

Definition 4 *The distinguishing chromatic index $\chi'_D(G)$ of a graph G is the least number d such that the G has a proper edge-colouring with d colours that is preserved only by the identity automorphism of G.*

One may use the term *labeling* instead of colouring. Obviously, none of these two invariants is defined for graphs having K_2 as a connected component.

Given an edge-colouring c, *the palette at a vertex* x is the set of colours of the edges incident to x. Clearly, if different vertices have different palettes, then the only automorphism preserving c is the identity.

Sometimes $D'(G) = D(G)$. Clearly it holds for all graphs with a trivial automorphism group. And also for paths and cycles.

Proposition 5 $D'(P_n) = D'(C_p) = 2$, *for any* $n \geq 3$ *and* $p \geq 6$, *while* $D'(C_3) = D'(C_4) = D'(C_5) = 3$.

Proof. The distinguishing number for paths and cycles equals $D(P_n) = 2$ for $n \geq 2$, $D(C_p) = 2$ for $p \geq 6$ and $D(C_3) = D(C_4) = D(C_5) = 3$. Our observation follows immediately from the fact that the $L(P_n) \cong P_{n-1}$ and $L(C_p) \cong C_p$, where L denotes the line graph. □

However, quite frequently $D'(G) < D(G)$. We shall show that $D'(K_n) = D'(K_{n,n}) = 3$ while $D(K_{n-1}) = D(K_{n,n}) = n + 1$ (cf. Theorem 1).

Proposition 6 $D'(K_n) = D'(K_{p,p}) = 3$, *for any* $n \geq 3$ *and* $p \geq 2$.

Proof. First, we show how to colour a complete bipartite graph $K_{p,p}$ with three colours. Let $x_1, ..., x_p$ be vertices of one independent set X and $y_1, ..., y_p$ be vertices of the other independent set Y. For each i, we colour with red all edges $x_i y_j$, for $j \in \{1, ..., i\}$. Next we colour two edges $x_1 y_p$ and $x_2 y_p$ with green, and all the remaining edges we colour with blue. In this way, every vertex from X has a different number of incident red edges. The same is true for vertices from Y. Furthermore, every automorphism preserving this colouring fixes y_p since it is the only vertex with two green edges. Consequently, it fixes every vertex in Y, and thus, also in X. Hence, the unique automorphism preserving this colouring is the identity.

It is easy to see that two colours are not enough. We have to distinguish at least $p+1$ vertices (all from one independent set and at least one from another one). If we use two colours, say red and blue, then in every vertex, we obtain a pair of numbers of red and blue edges summing up to p. Observe that we cannot have at once a vertex with all blue edges and another one with all red edges. So we have only p possibilities for such pairs.

Now, we colour a complete graph with three colours. Let $V(K_n) = \{x_1, ..., x_n\}$. For each $i \in \{1, ..., \lfloor \frac{n}{2} \rfloor\}$ we colour with red all edges $x_i x_j$, for $j \in \{i+1, ..., n-i+1\}$. Next we colour with green two edges $x_n x_{\lceil \frac{n}{2} \rceil}$ and $x_{n-1} x_{\lceil \frac{n}{2} \rceil}$ and we colour all the remaining edges blue. In this way, every vertex has a different number of incident red edges except for two vertices $x_{\lceil \frac{n}{2} \rceil}$ and $x_{\lceil \frac{n}{2} \rceil + 1}$ which have $\lfloor \frac{n}{2} \rfloor$ red edges. But none automorphism preserving this colouring permutes these two vertices because of the green edges. Hence, the unique automorphism preserving this colouring is the identity. It is easy to observe that two colours are not enough, because, as above, we need n different pairs of numbers summing up to n, but if a pair $(0, n)$ appears, then $(n, 0)$ is excluded.. □

In the next section we first present some special classes of trees with the distinguishing index greater than the distinguishing number. Next, we show that $D'(G) \leq D(G)$ for all remaining graphs (see Theorem 8). It follows that $D'(G)$ is not greater then $\Delta(G) + 1$ for any connected graph. And we show independently that$D'(G) \leq \Delta(G)$ (Theorem 12).

In the last section we investigate proper colourings of edges of G. By Vizing's Theorem every graph has a colouring with $\Delta(G)$ or $\Delta(G)+1$ colours. We show that $\Delta(G) + 1$ colours suffice to find a proper colouring preserved only by the trivial automorphism unless G is C_4, K_4, C_6 or $K_{3,3}$.

2 General bounds for trees and connected graphs

2.1 Trees

Recall that every finite tree T has either a *central vertex* v_c or a *central edge* e_c, which is fixed by every automorphism of T. We say that a tree T is *bisymmetric* if it has a central edge e_c, all leaves are at the same distance from e_c and every vertex that is not a leaf has the same degree.

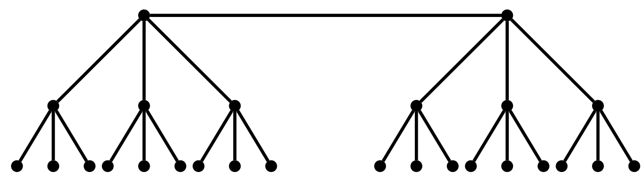

Figure 1: An example of a bisymmetric tree with maximum degree four.

If a tree has a central vertex, all leaves are at the same distance from a vertex v_c and every vertex that is not a leaf has the same degree, then such a tree is called *symmetric*. Observe that a path is either a symmetric or a bisymmetric tree depending of the parity of its length.

Collins and Trenk in [2], obtained a general bound for the distinguishing number of trees. We cite it improving a small mistake in the original paper.

Theorem 7 *[2] If T is a tree of order $n \geq 3$, then $D(T) \leq \Delta(T)$. Furthermore, equality is achieved if and only if T is a symmetric tree or a path of odd length.*

We show that the distinguishing index of a tree is not greater than its distinguishing number unless it is a bisymmetric tree different from a path.

Theorem 8 *If T is a tree of order $n \geq 3$, then*

$$D'(T) \leq D(T) + 1$$

with equality holding if and only if T is a bisymmetric tree with maximum degree at least three.

Proof. Let $\hat{c} : V \to \{1, 2, \ldots, D(T)\}$ be a vertex-colouring preserved only by the identity.

Case 1. A tree T has a central vertex v_c. If xy is an edge of T such that $d(x, v_c) = d(y, v_c) + 1$, then we colour it as $c(xy) := \hat{c}(x)$. Suppose ϕ is a nontrivial automorphism of T preserving the colouring c. As ϕ fixes the central vertex v_c, it also fixes the distance from any vertex x to v_c. Hence, $\hat{c}(\phi(x)) = c(\phi(x)\phi(y)) = c(xy) = \hat{c}(x)$, that is, ϕ preserves the vertex colouring $\hat{c}$, a contradiction.

Case 2. A tree T has a central edge e_c. The central edge e_c is fixed by every automorphism. We colour every edge xy such that the distance from x to the central edge e_c is greater by one than the distance from y to e_c

with $c(xy) := \hat{c}(x)$. We colour e_c arbitratrily. Suppose that ϕ is a nontrivial automorphism of T preserving the colouring c. So there exist two edges x_1y_1 and x_2y_2 with the same colour such that $\phi(x_1y_1) = x_2y_2$. As ϕ fixes the edge e_c, the distances from e_c to x_1 and x_2 are equal and $\hat{c}(x_1) = \hat{c}(x_2)$. Hence, either ϕ is a nontrivial automorphism preserving the colouring $\hat{c}$ or T is a bisymmetric tree and $\hat{c}$ distinguishes the end vertices of e_c. In the latter case, we need one additional colour for one edge adjacent to e_c. In consequence, the unique automorphism of T preserving this colouring c is trivial. □

Theorems 7 and 8 immediately imply the following result.

Corollary 9 *If T is a tree of order $n \geq 3$, then $D'(T) \leq \Delta(T)$. Moreover, equality is achieved if and only if T is either a symmetric or a bisymmetric tree.* □

2.2 Connected graphs

Theorem 10 *If G is a connected graph of order $n \geq 3$, then $D'(G) \leq D(G) + 1$.*

Proof. If G is a tree then the claim is true by Theorem 9. Suppose that G is a cyclic graph. If G is just a cycle, then the claim follows from Proposition 5.

Let $\hat{c} : V \to \{1, 2, \ldots, D(G)\}$ be a colouring preserved only by the identity. Obviously, if $D(G) = 1$, then G is asymmetric and $D'(G) = 1$. So, let $\hat{c}$ use at least two colours.

We will define an edge-colouring c with $D(G)+1$ colours. Denote by 0 an additional colour not used by $\hat{c}$. Let C be a cycle of G. We first colour the edges of C according to its length p in such a way that the only automorphism of C preserving this colouring is the identity. By Proposition 5, we can do this with colours $0, 1, 2$ if $p \leq 5$, and with colours $0, 1$ if $p \geq 6$. Then we retire 0, so C will be fixed by every automorphism of G. Next we colour every edge xy such that the distance from x to the cycle C is one more than the distance from y to C. Here we put $c(xy) = \hat{c}(x)$. Finally, we colour every edge xy such that x and y are of the same distance to the cycle C arbitrarily, say with colour 1.

Suppose that ϕ is a nontrivial automorphism of G preserving a colouring c. Then ϕ fixes the vertices of the cycle C, hence it preserves the distances from vertices of G to C. Therefore there exist edges xy such that the distance from x to C is greater by one than that of y, and $c(\phi(xy)) = c(xy)$. For each such edge, we have $\hat{c}(\phi(x)) = c(\phi(xy)) = c(xy) = \hat{c}(x)$. As ϕ fixes C, it follows that the colouring $\hat{c}$ of vertices is preserved by ϕ, a contradiction. □

Theorem 10 is interesting in view of the conjecture, mentioned in Section 1, that almost every non-asymmetric graph has the distinguishing number two.

Theorems 1 and 10 immediately imply the following.

Corollary 11 *If G is a connected graph of order $n \geq 3$, then $D'(G) \leq \Delta(G) + 1$.* □

We can strenghten the above corollary as follows.

Theorem 12 *If G is a connected graph of order $n \geq 3$, then $D'(G) \leq \Delta(G)$ unless G is C_3, C_4 or C_5.*

Proof. Denote $\Delta = \Delta(G)$. Due to Proposition 5, we may assume that $\Delta \geq 3$. Denote by $N_r(\xi)$ the set of all vertices of distance r from ξ, where ξ is either a vertex or an edge.

Consider first an irregular graph $G = (V, E)$. Let xy be an edge of G such that $\deg(x) = \delta(G)$. We colour xy with 1, then all other edges incident to x with a colour from $\{2, \ldots, \Delta(G)\}$, and all other edges incident with y from $\{\Delta - \deg(y) + 2, \ldots, \Delta\}$. As $\delta(G) \neq \Delta(G)$, the sets of colours of edges incident to x or to y are different. We will not use colour 1 any more, so vertices x and y are fixed by any automorphism of G. Moreover, all vertices from $N_1(xy)$ are fixed by any automorphism. Now, for $r \geq 1$ let u be a vertex from $N_r(xy)$. We colour all edges uv, for $v \in N_{r+1}(xy)$, with colours from $\{2, \ldots, \Delta\}$. Therefore, all vertices of G are fixed by any automorphism of G. Observe that colours of edges between two vertices of the same $N_i(x)$ could be arbitrary.

Now, let G be a regular graph. Due to Proposition 5, we may assume $\Delta \geq 3$. Fix any vertex x of G and colour all edges incident to it with 1. In our edge-colouring c of the graph G, the vertex x will be the unique vertex with the monochromatic palette $\{1\}$, hence it will be fixed by every automorphism ϕ preserving c. The neighbourhood $N_1(x)$ can be partitioned into subsets M_k, for $k = 0, 1, \ldots, \Delta - 1$, defined as

$$M_k = \{v \in V \ : \ |N_1(v) \cap N_2(x)| = k\}.$$

Denote $M_k = \{v_1, \ldots, v_{l_k}\}$, $k = 0, 1, \ldots, \Delta - 1$. Thus, $l_0 + l_1 + \ldots + l_{\Delta-1} = \Delta$. If $l_0 = \Delta$, then G is a complete graph $K_{\Delta+1}$, and we done by Proposition 6. Otherwise, if $l_0 \geq 1$, we can colour the edges incident to the vertices of M_0 with two colours 2 and 3 such that the palette of v_i contains exactly $l_0 + 1 - i$ edges coloured with 2. Thus, each vertex of M_0 is fixed.

Let $k \geq 1$. For every $i = 1, \ldots, l_k$, we colour the edges $v_i u$, where $u \in N_2(x)$, with a distinct colour from $\{1, \ldots, k+1\}$ in such a way that the colour i is missing in the palette of v_i. Then we colour all the remaining edges incident to v_i with 2. Clearly, each vertex of $N_1(x) \cup N_2(x)$ is fixed by every automorphism preserving the colouring c.

For $v_j \in Nj(x)$, $j \geq 2$, we colour all edges $v_j u$, $u \in N_{j+1}(x)$ with distinct colours from $\{2, \ldots, \Delta\}$ and the remaining edges incident to v_j arbitrarily.

Then we recursively colour the edges incident to consecutive spheres $N_j(x)$ in such a way that distinct vertices of $N_j(x)$ have distinct palettes. It is easily seen that it is always possible. Hence, all vertices of G are fixed by any automorphism ϕ preserving our colouring c. □

3 Distinguishing chromatic index

3.1 General bound

Let c be a proper edge-coloring of a connected graph G of order $n \geq 3$. For every vertex x, each walk starting at x defines a sequence of colours (α_i), called a *colour walk*. Denote by $W_c(x)$ the set of all colour walks starting at x. A new invariant $\mu(G)$, called the *distinguishing index by colour walks* of a graph G, was introduced in [9] as the minimum number of colours required in an edge-colouring of G such that $W_c(x) \neq W_c(y)$ for every two distinct vertices x, y.

Theorem 13 *[9] Let G be a connected graph of order $n \geq 3$. Then*

$$\mu(G) \leq \Delta(G) + 1$$

except for four graphs of small order: C_4, K_4, C_6, $K_{3,3}$.

The next lemma exhibits a relationship between $\mu(G)$ and $\chi'_D(G)$.

Lemma 14 *Every connected graph G of order $n \geq 3$ fulfils the inequality*

$$\chi'_D(G) \leq \mu(G).$$

Proof. Let c be an edge-colouring distinguishing vertices by colour walks, i.e., $W_c(x) \neq W_c(y)$ if $x \neq y$. Suppose ϕ is a nontrivial automorphism of G preserving c. Then there exists a vertex x such that $x \neq \phi(x)$. An automorphism ϕ preserves the colouring, so every sequence $(\alpha_i) \in W_c(x)$ belongs also to $W_c(\phi(x))$. And every sequence (β_i) starting at $\phi(x)$, starts also at $\phi^{-1}(\phi(x)) = x$. Hence, x and $\phi(x)$ are not distinguished by colour walks in this colouring. □

In consequence, we obtain a sharp upper bound for the distinguishing chromatic index of connected graphs.

Theorem 15 *If G is a connected graph of order $n \geq 3$, then*

$$\chi'_D(G) \leq \Delta(G) + 2$$

and equality is achieved if and only if $G \in \{C_4, K_4, C_6, K_{3,3}\}$. □

This theorem immediately implies an interesting result. An edge-colouring of G with $\chi'(G)$ colours is called *minimal*.

Theorem 16 *Every connected Class 2 graph admits a minimal edge-colouring that is not preserved by any nontrivial automorphism.* □

Let us now modify the notion of distinguishing graphs by colour walks by considering colour paths with palettes. More precisely, given an edge-colouring, let $P_c(x)$ denote the set of sequences of colours of paths starting at a vertex x. We say that two vertices x and y are *similar* if $P_c(x) = P_c(y)$, and moreover, the paths P_x, P_y starting at x, y respectively, with the same sequence of colours terminate in vertices with equal palettes.

If there are no two similar vertices, then we say that the vertices of a graph G are *distinguished by colour paths with palettes*, and the least number of colours in such a colouring we denote by $\mu_p(G)$. It is easily seen that $\mu_p(G) \leq \mu(G)$. In the next subsection, we shall show that the inequality cannot be replaced by the equality.

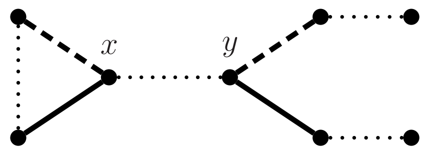

Figure 2: The vertices x and y are not similar because the paths (dashed, dotted) starting at x and y terminate in vertices with distinct palettes.

Proposition 17 *Let c be an proper edge-colouring of a connected graph G. If c is preserved only by the trivial automorphism, then all vertices of a graph G are distinguished by colour paths with palettes. In other words,*

$$\mu_p(G) \leq \chi'_D(G).$$

Proof. Suppose that x and y are two similar vertices. Define a nontrivial automorphism ϕ of a graph G as follows. First put $\phi(x) = y$. Next, for every vertex $v \in V(G) \setminus \{x\}$, take any path P_x from x to v and a path P_y with the same sequence of colours which terminates at u, and put $\phi(v) = u$.

We have to prove that ϕ is a well-defined automorphism preserving the colouring c. It is not difficult to see that to do this, it suffices to show that the definition of $\phi(v)$ does not depend on the choice of a path P_x from x to v. Assume P'_x is another path from x to v. Let $P_y, P'_y \in P_c(y)$, the sequences of colours of P_x and P_y are equal, as well as those of P'_x and P'_y, and let P_y, P'_y terminate at u, u', respectively. If $u = u'$, we are done. Then suppose $u \neq u'$. Let $P_x = xw_1 \dots w_k$ and $P'_x = xw'_1 \dots w'_l$ with $w'_l = w_k$. Consider a path $Q = xw_1 \dots w_k w'_{l-1} \dots w'_1$. Since x and y are similar, there exists a path $Q' \in P_c(y)$ with the same colour sequence as Q. The end vertex of Q' has the same palette as w'_1, therefore Q' can be prolonged by an edge coloured with $c(w'_1 v)$ to a path $Q'' \in P_c(y)$, because $u \neq u'$. But $P_c(x)$ does not contain a path with the colour sequence of Q'' since adding an edge $w'_1 v$ to Q yields a cycle. Thus $P(x) \neq P(y)$, a contradiction. □

In consequence, we obtain the equality of two indices.

Corollary 18 *Let G be a connected graph of order $n \geq 3$. Then every proper edge-colouring c of G is preserved only by the trivial automorphism if and only if c distinguishes the vertices of G by colour paths with palettes, i.e.,*

$$\mu_p(G) = \chi'_D(G).$$

3.2 Some Class 1 graphs

As it follows from the previous subsection, $\chi'_D(G) = \mu(G) = \mu_p(G) = \chi'(G)$ for every connected Class 2 graph. For Class 1 graphs, some of the first two equalities may not hold.

We shall first show that there are graphs for which $\chi'_D < \mu$. By Proposition 17, this means that distinguishing vertices by colour walks is not equivalent with distinguishing by colour paths with palettes.

Every regular graph G of Class 1 satisfies $\mu(G) = \Delta(G) + 1$. Indeed, for every minimal edge-colouring of G, the palette of each vertex is the same. Hence, for any vertex x, the set $W_c(x)$ is the same, as it comprises all sequences of colours of c. By Theorem 13, one additional colour is enough to distinguish all vertices by colour walks.

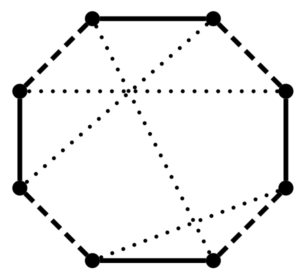

Figure 3: A graph G such that $\chi'_D(G) < \mu(G)$.

Consider the cubic graph G drawn in Figure 3. The edges of a cycle C_8 are properly coloured with two colours, and the remaining edges, creating a perfect matching, have a third colour. Let ϕ be an automorphism preserving this colouring. The unique triangle of G has to be mapped onto itself. Regarding the colours of its edges, it has to be fixed by ϕ. Hence, the cycle C_8 also is fixed. Thus $\chi' - D(G) = 3$ while $\mu(G) = 4$.

This example can easily be generalized for higher orders and degrees by taking a longer even cycle with a perfect matching creating only one triangle, and then introducing more arbitrary perfect matchings.

For Class 1 graphs, we sometimes need one colour more than χ' for χ'_D, and in four cases, two additional colours. Also for paths of odd length we $\chi'_D(P_{2k}) = \chi'(P_{2k}) + 1$ colours. If we have a proper colouring of P_{2k}, then it is enough to recolour a hanging edge with a new additional colour. For paths of even length, any proper colouring is preserved only by the identity. This observation can be extended to trees in general.

Proposition 19 *If T is a tree of order $n \geq 3$, then*

$$\chi'_D(T) = \Delta(T) + 1$$

if and only if T is a bisymmetric tree.

Proof. Consider any proper edge-colouring of T with $\Delta(T)$ colours.

Case 1. T has a central vertex v_c fixed by every automorphism. A colouring is proper, so every edge incident to v_c has a distinct colour. Hence, all vertices adjacent to v_c are fixed by every automorphism of T. By induction on the distance from v_c, all vertices of T are fixed.

Case 2. T has a central edge e_c fixed by every automorphism. Let T_1 and T_2 be subtrees created by deleting an edge e_c. If these subtrees are not isomorphic, then the end vertices of e_c are fixed by every automorphism, and we are done by similar arguments as in Case 1. Let then T_1 and T_2 be isomorphic. Suppose there exist vertices $x_1 \in V(T_1)$ and $x_2 \in V(T_2)$ that are not leaves with degree in T smaller than $\Delta(T)$. If the sets of colours of edges incident to x_1 and x_2 are different, the unique automorphism preserving this colouring of T is the identity. If not, let 0 be a colour which is not in a set of colours of edges incident to x_2. We recolour one edge incident to x_2 with 0, and, if necessary, we recolour a Kempe path in T_2 induced by 0 and the colour of the edge incident with x_2 before recolouring.

If T_1 and T_2 are isomorphic and all vertices in T that are not leaves have degree $\Delta(T)$, then T is a bisymmetric tree. We need an additional colour to distinguish end vertices of e_c, which are then fixed by every automorphism, and we can use the same arguments as in Case 1 to finish the proof. □

References

[1] M. O. Albertson and K. L. Collins, Symmetry breaking in graphs, Electron. J. Combin. 3 (1996) R18.

[2] K. L. Collins and A. N. Trenk, The distinguishing chromatic number, Electron. J. Combin. 13 (2006) R16.

[3] M. Conder and T. Tucker, The Motion Lemma and Distinguishability Two, Ars Mathematica Contemporanea, 4 (2011), 63-72.

[4] P. Erdős and A. Rényi, Asymmetric graphs, Acta Math. Acad. Sci. Hungar. 14 (1963) 295–315.

[5] W. Imrich, J. Jerebic and S. Klavžar, The distinguishing number of Cartesian products of complete graphs, European J. Combin. 29 (2008) 922–929.

[6] W. Imrich, R. Kalinowski, F. Lehner, M. Pilśniak, Endomorphism Breaking in Graphs, preprint 2012, http://www.geometrie.tugraz.at/lehner/krakauWRFM-ejc.pdf.

[7] W. Imrich, S. Klavžar and V. Trofimov, Distinguishing infinite graphs, Electron. J. Combin. 14 (2007) R36

[8] W. Imrich, S. M. Smith, T. Tucker and M. E. Watkins, Infinite motion and the distinguishing number of graphs, manuscript 2012.

[9] R. Kalinowski, M. Pilśniak, J. Przybyło, M. Woźniak, How to personalize the vertices of a graph?, preprint 2013, http://www.ii.uj.edu.pl/index.php..

[10] S. Klavžar, T.-L. Wong and X. Zhu, Distinguishing labelings of group action on vector spaces and graphs, J. Algebra 303 (2006) 626–641.

[11] A. Russell and R. Sundaram, A note on the asymptotics and computational complexity of graph distinguishability, Electron. J. Combin. 5 (1998), R23.

[12] S. M. Smith, T. Tucker and M. E. Watkins, Distinguishability of Infinite Groups and Graphs, Electron. J. Combin. 19 (2012) P27.

[13] T. Tucker, Distinguishing Maps, Electron. J. Combin. 18 (2011) R50.

[14] M. E. Watkins and X. Zhou, Distinguishability of Locally Finite Trees, Electron. J. Combin. 14 (2007) R29.