COMPUTER ASSISTED PROOFS FOR NON-SYMMETRIC PLANAR CHOREOGRAPHIES AND FOR STABILITY OF THE EIGHT

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ABSTRACT. We present a general method to produce computer assisted proofs of the existence of choreographies in the N-body problem. This method allows to verify rigorously numerical data from computer simulations. As an example we use it to prove the existence of non-symmetric choreographies with 6 and 7 bodies. The method provides estimates for the initial conditions and for the monodromy matrix of the choreography. These data are used to show linear stability of the Eight solution restricted to the plane.

1. INTRODUCTION

The N-body problem with N equal unit masses is described by a differential equation

$$\ddot{q}_i = \sum_{i \neq j = 1}^{N} \frac{(q_j - q_i)}{||q_i - q_j||^3}, \tag{1.1}$$

where $q_i \in \mathbb{R}^n$, $i = 1, ..., N$. The gravitational constant is taken equal to 1.

A T-periodic *choreography* with N bodies is a solution of the N-body equation where all bodies move on the same curve with constant phase shift equal to $\frac{T}{N}$. The main goal of this paper is to introduce a general method to obtain computer assisted proofs of the existence of choreographies, which can be used, e.g., to verify solutions produced in a non-rigorous numerical way. No assumptions are done on the existence of symmetries of the curve. Before going into the details we here only briefly sketch the method.

When we are searching for an initial condition for a choreography of period T, we can look for a configuration of bodies (positions and velocities) such that after time $\frac{T}{N}$ the bodies, moving under the gravitational force, interchange themselves cyclically. Let q_i and p_i be the position and the velocity of the i-th body. Then we are searching for a point in the phase space $x = (q_1, p_1, q_2, p_2, \ldots, q_N, p_N)$ which is a fixed point of a function

$$G(x) := \sigma \cdot \varphi(x, T/N), \tag{1.2}$$

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where $\varphi(x,t)$ is the flow generated by the N-body equation (1.1) and $\sigma(q_1, p_1, q_2, p_2, \ldots, q_N, p_N) = (q_N, p_N, q_1, p_1, \ldots, q_{N-1}, p_{N-1})$ is a cyclical shift of the bodies. Instead of searching for a fixed point of G one can search for a zero of function

$$F(x) = G(x) - x. \tag{1.3}$$

To achieve this goal we want to use the Interval Krawczyk Method (see section 3), which can rigorously prove the existence of a locally unique zero of the function F. But we face difficulties: due to the existence of first integrals the solution of the N-body problem is never isolated. When we have one solution then another one can be obtained by rotation, translation, rescaling or phase shift. From this it follows that zeroes of F are never isolated. So there is no chance for Interval Krawczyk Method to succeed.

To get around this we reduce the phase space and we modify the function F. Then using interval arithmetic we rigorously verify the assumptions of Interval Krawczyk Method and, as a result, we prove the existence of a zero of the function F in the reduced space. But then we have an obvious question: can this guarantee the existence of a zero of F in the full phase space? In general the answer is negative. In this paper we will show that for a specific reduction and under some additional conditions we have also the existence of a zero of F in the full space and hence the existence of the desired choreography. In section 4 we describe a general method of reduction using the first integrals. In previous papers [KZ, Ka] the symmetries of the orbit have been used for this reduction. So only the existence of symmetric orbits could be proved. The new method is not assuming any symmetry. It is applied to prove the existence of planar (i.e. $\mathbb{R}^n = \mathbb{R}^2$) nonsymmetric choreographies with 6 and 7 bodies.

The outline of the paper is as follows: in Section 2 we introduce the notation of interval arithmetic and explain what we understand by a computer assisted proof. Section 3 recalls the definitions and properties of Krawczyk operator. Section 4 presents the main theorems which are the base of our method. In Section 5 we test the method reproving the case of the figure eight with 3 bodies and then it is applied to prove the existence of choreographies with 6 and 7 bodies. Section 6 shows one of the advantages of the present method: it gives not only existence results but also estimates for the initial conditions of the choreographies and the monodromy matrix. This information is used to show stability of the Eight solution for motions restricted to the plane and with fixed angular momentum equal to zero.

2. Computer assisted proofs

Due to representation errors, to be rigorous, instead of real numbers in computations we use representable intervals (intervals having representable numbers on both ends). For any set $X \in \mathbb{R}$ we denote by $[X]$ an interval hull, the smallest representable interval which contains X. Next we define interval arithmetic on representable intervals. Let X,Y be two representable intervals, $\diamond \in \{+, -, \times, /\}$ be any basic operation. Then we define

$$X \hat{\diamond} Y = [X \diamond Y].$$

For division we assume additionally that $0 \notin Y$. This can be easily extended to arithmetic on vectors and matrices with interval entries (we call them interval vectors and interval matrices or just interval sets). For any basic function f, using interval arithmetic, we define its interval version $\hat{f}$ in such a way that for any interval set X a value $\hat{f}(X)$ is some interval set containing $f(X)$. For any interval vector X we denote by $\operatorname{diam} X$ the vector of diameters of its components.

In computer assisted proofs we use computers in the following way. First of all we need some abstract theorem with assumptions of special kind (e.g. inequalities, inclusions). Next we use interval arithmetic to obtain rigorous upper bounds for the values of some parameters or functions and finally we rigorously check the assumptions of that theorem.

Below we present Krawczyk Theorem as an example of theorem whose assumptions can be verified using interval arithmetic. It is the basis for all other theorems in that paper.

3. The Interval Krawczyk Method

We cite the following result from Krawczyk [K].

Assume that:

- $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a C^1 function,
- $X \subset \mathbb{R}^n$ is an interval set,
- $\bar{x} \in X$,
- $C \in \mathbb{R}^{n\times n}$ is a linear isomorphism.

Then the Krawczyk operator is given by

$$K(\bar{x}, X, F) := \bar{x} - CF(\bar{x}) + (Id - C[DF(X)])(X - \bar{x}),$$

where $[DF(X)]$ denotes the interval hull of the derivative computed over the set X.

Theorem 3.1. *With the assumptions and notation introduced above, the following holds:*

(1) *If $x^* \in X$ and $F(x^*) = 0$, then $x^* \in K(\bar{x}, X, F)$.*
(2) *If $K(\bar{x}, X, F) \subset intX$, then there exists a unique $x^* \in X$ such that $F(x^*) = 0$.*
(3) *If $K(\bar{x}, X, F) \cap X = \emptyset$, then $F(x) \neq 0$ for all $x \in X$.*

Remark. To compute Krawczyk operator we need the value of F in the point $\bar{x}$ and the interval hull of its derivative $[DF]$ computed over the whole set X. Although C can be any non-degenerate matrix, the best choice is to take an approximation of the inverse of $DF(\bar{x})$.

4. Reduction method using first integrals

As it was mentioned in section 1, to isolate a fixed point of the function G given by equation (1.2), we need to reduce the dimension of the phase space. In a first step we reduce system (1.1) by fixing the centre of mass at the origin. Due to the complicated nature of the first integrals of the angular momentum and the total energy, further reduction by fixing the values of these integrals is not convenient. In fact, it is not necessary. Instead of this we fix the values of two coordinates in the initial configuration, because it seems that there is at least a two parameter system degeneracy. A vector with these coordinates will be denoted by c, and a vector with the remaining coordinates will be denoted by z. Then we will show the existence of a fixed point of G on the z-coordinates. It is equivalent to the existence of a trajectory such that the configurations at times $t = 0$ and $t = T/N$ are (after a cyclical permutation of the bodies) the same on z-coordinates. Let

(z_0, c_0) and (z_0, c_1) be two such configurations. The last step is to show that we return exactly to the initial condition in the full space, so that $c_0 = c_1$. By $J(z, c)$ we will denote a function of the first integrals (the total energy and the angular momentum). We will show that for fixed $z = z_0$ the function J is injective in some set C which contains c_0 and c_1. Because of the conservation of the first integrals we have $J(z_0, c_0) = J(z_0, c_1)$, and this implies that $c_0 = c_1$.

Before going into details of the reduction method, we will state and prove some general result.

Definition 4.1. Let $X \subset \mathbb{R}^n$. We represent a point $x \in X$ as a pair $(z, c) \in \mathbb{R}^k \times \mathbb{R}^m$, where $n = k + m$. For such a representation we define projections

$$\pi_z(z, c) = z \qquad \pi_c(z, c) = c.$$

Theorem 4.2. *Let $X \subset \mathbb{R}^n$ and $(z_0, c_0) \in \mathbb{R}^k \times \mathbb{R}^m$. Let $G : X \longrightarrow \mathbb{R}^n$ and $J : X \longrightarrow \mathbb{R}^m$ be C^1 functions such that $\pi_z G(z_0, c_0) = z_0$ and $J(z_0, c_0) = J(G(z_0, c_0))$. Let Z and C be interval sets such that $z_0 \in Z$, $c_0 \in C$ and $[\pi_c(G(Z, c_0))] \subset C$. If the interval matrix $\left[\frac{\partial J}{\partial c}(Z, C)\right]$ is invertible, then $G(z_0, c_0) = (z_0, c_0)$.*

To prove that theorem we need the following lemma

Lemma 4.3. *Let $F : \mathbb{R}^m \to \mathbb{R}^m$ be a C^1 function. If the interval matrix $[DF(X)]$ is invertible for some interval set X, then F is injective on X.*

Proof. For any two points $x_1, x_2 \in X$ we have

$$F(x_2) - F(x_1) = \int_0^1 \frac{dF}{dt}(x_1 + t(x_2 - x_1))dt = \int_0^1 \frac{\partial F}{\partial x}(x_1 + t(x_2 - x_1))dt \cdot (x_2 - x_1) \tag{4.1}$$

Let $N(x_2, x_1) = \int_0^1 \frac{\partial F}{\partial x}(x_1 + t(x_2 - x_1))dt$. Because $N(x_2, x_1) \in [DF(X)]$, one has that $N(x_2, x_1)$ is invertible for any choice of $x_2, x_1 \in X$.

We can rewrite (4.1) as

$$F(x_2) - F(x_1) = N(x_2, x_1)(x_2 - x_1) \quad \text{for each } x_1, x_2 \in X \tag{4.2}$$

If $F(x_1) = F(x_2)$ for some $x_1, x_2 \in X$, then from (4.2) it follows that $N(x_2, x_1)$ $(x_2 - x_1) = 0$. Hence from invertibility of $N(x_2, x_1)$ we have $x_2 - x_1 = 0$, what means that the function F is injective on X. ∎

Proof of Theorem 4.2 Let $\pi_c G(z_0, c_0) = c_1$. It is enough to show that $c_1 = c_0$. For any fixed $z \in Z$ let $J_z(c) = J(z, c)$. The interval matrix $[DJ_z(C)]$ is invertible because $[DJ_z(C)] \subset \left[\frac{\partial J}{\partial c}(Z, C)\right]$. Hence from Lemma 4.3 for each $z \in Z$ the function J_z is injective on the set C. Directly from the assumptions of Theorem 4.2, we have that $c_0, c_1 \in C$ and $J_{z_0}(c_0) = J_{z_0}(c_1)$. Hence from injectivity of J_{z_0} it follows that $c_0 = c_1$. ∎

If in the application of Theorem 4.2 we use as J a function of all the first integrals of the N-body problem, then we will face a problem due to the special nature of the first integrals coming from the centre of mass condition. To calculate its value we need to know not only the configuration of the bodies at a given moment, but also the time spent from the beginning of the motion. To overcome this, we fix the centre of mass at the origin and we reduce system (1.1) by removing the equation and coordinates corresponding to the last body. We obtain

$$\left\{\begin{array}{ll} \ddot{q}_i = \displaystyle\sum_{i\neq j=1}^{N} \frac{(q_j - q_i)}{||q_i - q_j||^3} & \text{for } i = 1, 2, \ldots, N-1, \\ \text{where } q_N = -\displaystyle\sum_{i=1}^{N-1} q_i, & p_N = -\displaystyle\sum_{i=1}^{N-1} p_i. \end{array}\right. \tag{4.3}$$

For a point $x = (q_1, p_1, \ldots, q_{N-1}, p_{N-1})$ let us denote by $\bar{\varphi}(x,t)$ the flow generated by the reduced system (4.3) and let

$$\bar{\sigma}(x) = \left(-\sum_{i=1}^{N-1} q_i, -\sum_{i=1}^{N-1} p_i, q_1, p_1, \ldots, q_{N-2}, p_{N-2} \right).$$

Then the reduced version of the function G is defined as follows

$$\bar{G}(x) = \bar{\sigma} \circ \bar{\varphi}\left(x, \frac{T}{N}\right). \tag{4.4}$$

Lemma 4.4. *If $\bar{x} = (q_1, p_1, \ldots, q_{N-1}, p_{N-1})$ is a fixed point of the map $\bar{G}$, then $x = \left(q_1, p_1, \ldots, q_{N-1}, p_{N-1}, -\sum_{i=1}^{N-1} q_i, -\sum_{i=1}^{N-1} p_i\right)$ is a fixed point of the map G.*

Proof. The centre of mass and the total linear momentum of the configuration x are the zero vectors, so the centre of mass of the configuration $\varphi(x,t)$ will be at the origin. Hence $q_N(t) = -\sum_{i=1}^{N-1} q_i(t)$ and $p_N(t) = -\sum_{i=1}^{N-1} p_i(t)$ for all t. Using the fact that $\bar{x}$ is a fixed point of $\bar{G}$ at time $t = \frac{T}{N}$ we have

$$\begin{aligned} \varphi\left(x, \frac{T}{N}\right) &= \left(q_2, p_2, \ldots, -\sum_{i=1}^{N-1} q_i, -\sum_{i=1}^{N-1} p_i, \sum_{i=1}^{N-1} q_i - \sum_{i=2}^{N-1} q_i, \sum_{i=1}^{N-1} p_i - \sum_{i=2}^{N-1} p_i\right) \\ &= \left(q_2, p_2, \ldots, q_{N-1}, p_{N-1}, -\sum_{i=1}^{N-1} q_i, -\sum_{i=1}^{N-1} p_i, q_1, p_1\right) \\ &= \sigma^{-1}(x). \end{aligned}$$

Hence x is a fixed point of map G. ∎

Fixed points of $\bar{G}$, and thereby zeroes of a function

$$\bar{F}(x) = \bar{G}(x) - x, \tag{4.5}$$

still are not isolated because of the first integrals of the total energy and the angular momentum. In fact, this degeneracy seems to have at least three parameters: one coming from scaling (it is connected also with period changing), the second one from a rotation around the origin and the third one connected with the freedom on choosing an initial point on the trajectory. One of degeneracies is removed by fixing the period T. This is equivalent to fix the scaling due to Kepler's third law. We usually choose T to be the period of our numerical approximation of the choreography. To remove the other two degeneracies we assume that at $t = 0$ the first body is on the X axis and has a velocity orthogonal to that axis. It is not restrictive, because for every choreography we can rotate the coordinate frame and choose an appropriate initial point to fulfil this assumption (see [S1]). One can even assume that at $t = 0$ the first body is at maximal distance from the origin. In such initial configuration the second coordinate of the position of the first body and the first coordinate of the velocity are equal to 0. These will be our constrains for the initial conditions.

We deal with the planar N-body problem, so let $q_i = (x_i, y_i)$ and $p_i = (\dot{x}_i, \dot{y}_i)$. We want also to follow the notation of Theorem 4.2. A point $x = (q_1, p_1, \ldots, q_{N-1}, p_{N-1})$ in the phase space is split into a pair (z, c), where $z = (x_1, \dot{y}_1, q_2, p_2, \ldots, q_{N-1}, p_{N-1})$ and $c = (y_1, \dot{x}_1)$.

From previous assumptions it follows that at initial time we have $c_0 = (0, 0)$. We also define

$$\hat{F}(z) = \pi_z(\bar{F}(z, c_0)) = \pi_z\bar{G}(z, c_0) - z. \tag{4.6}$$

As J we take a function of the first integrals of total energy and angular momentum. More precisely it is given by

$$J(q_1, p_1, \ldots, q_{N-1}, p_{N-1}) = \left(\sum_{i=1}^{N} \frac{||p_i||^2}{2} - \sum_{1\le i<j\le N} \frac{1}{||q_i - q_j||}, \quad \sum_{i=1}^{N} q_i \times p_i \right), \tag{4.7}$$

where again $q_N = -\sum_{i=1}^{N-1} q_i$ and $p_N = -\sum_{i=1}^{N-1} p_i$. As function J is constant along the trajectories of the system (4.3) and invariant under permutation of the bodies, we have $J(z, c) = J(\bar{G}(z, c))$.

Now we are ready to state the main theorem

Theorem 4.5. *Let Z and C be two interval sets such that $z_0 \in Z$, $[\pi_c(\bar{G}(Z, c_0))] \subset C$ and $c_0 \in C$. If $K(z_0, Z, \hat{F}) \subset \operatorname{int} Z$ and an interval matrix $\left[\frac{\partial J}{\partial c}(Z, C)\right]$ is invertible, then there exists a fixed point of the map G.*

Proof. Because of Lemma 4.4 it is enough to show the existence of the fixed point of the map $\bar{G}$. Because $K(z_0, Z, \hat{F}) \subset \operatorname{int} Z$, from Theorem 3.1 it follows that there exists a point $z^* \in Z$ such that $\hat{F}(z^*) = 0$. Hence $\pi_z\bar{G}(z^*, c_0) = z^*$. We have also $J(z^*, c_0) = J(\bar{G}(z^*, c_0))$ and the matrix $\left[\frac{\partial J}{\partial c}(Z, C)\right]$ is invertible. Therefore from Theorem 4.2 we get that (z^*, c_0) is a fixed point of the map $\bar{G}$. ∎

Theorem 4.5 allows us to prove the existence of choreographies, because each fixed point of the function G is an initial configuration for some choreography. Observe that the assumptions of this theorem can be verified using rigorous computations. First of all, we need $x_0 = (z_0, c_0)$, a very good initial guess for an initial condition of the choreography to be proved to exist. It can be obtained by methods described in [S1]. The period T can be obtained from numerical simulations. Next, for a good chosen interval set Z we try to get an inclusion $K(z_0, Z, \hat{F}) \subset \operatorname{int} Z$. We can use different sizes of the set Z and various parameters of the computations such as time step or the order of the Taylor method. If one of our tries succeeds and we have the required inclusion, then as set C we take upper estimates for $[\pi_c(\bar{G}(Z, c_0))]$. When computing Krawczyk operator we get also these estimates. The last step is to check if $c_0 \in C$ and if the interval matrix $\left[\frac{\partial J}{\partial c}(Z, C)\right]$ is invertible. If all these conditions all fulfilled, then from Theorem 4.5 we are sure that in the set $Z \times \{c_0\}$ there is an initial condition for the choreography. Moreover, as the set Z is usually very small, the shape of the proved choreography is very similar to our first approximation. For example we are able to rigorously estimate the maximal distance between them.

4.1. **The parallel shooting method.** When we want to solve an equation

$$\bar{G}(x) = x, \tag{4.8}$$

we are searching for a configuration $x = (q_1, p_1, \ldots, q_{N-1}, p_{N-1})$ such that after time $\frac{T}{N}$ the bodies cyclically interchange their positions. One of the main difficulties in applications of the method described in the previous section is the strong instability of some solutions for large N. Strong dependence on initial conditions appears also when we pass near a collision. Additionally, a region where the Krawczyk operator is a contraction can be very small because of possible presence of other nearby solutions. To overcome these difficulties we can search not only for the configuration at time $t_0 = 0$, but also for intermediate configurations at times $t_1, t_2, \ldots, t_k$, where $0 < t_1 < t_2 < \cdots < t_k = \frac{T}{N}$. Let x^m be the configuration of the bodies at time t_m. Then instead of the equation (4.8) we have to solve the system of equations

$$\begin{cases} \varphi(t_{m+1} - t_m, x^m) = x^{m+1} \text{ for } m = 0, 1, \ldots, k-1, \\ \text{where } x^k = \bar{\sigma}^{-1} x^0. \end{cases} \tag{4.9}$$

The new system has a bigger dimension, but it has less dependence on initial conditions. That is, we are using the very well known parallel shooting method (see, e.g., [SB]).

If the interval $[0, \frac{T}{N}]$ is divided uniformly, i.e., if $t_i = i \cdot \frac{T}{k \cdot N}$ for $i = 0, 1, \ldots, k$ (despite this requirement is not strictly necessary) then the system (4.9) can be rewritten as:

$$\hat{G}(x) := \hat{\sigma} \circ \hat{\varphi}(x, \frac{T}{k \cdot N}) = x \tag{4.10}$$

where $x = (x^0, x^1, \ldots, x^{k-1})$, $\hat{\sigma}(x) = (\bar{\sigma} x^{k-1}, x^0, x^1, \ldots, x^{k-2})$ and $\hat{\varphi}(x, t)$ is the flow generated by k copies of the reduced N-body equation (4.3). The function $\hat{\sigma}(x)$ can be seen as a right shift of the particles

$$(x_1^0, x_1^1, \ldots, x_1^{k-1}, x_2^0, x_2^1, \ldots, x_2^{k-1}, \ldots, x_{N-1}^0, x_{N-1}^1, \ldots, x_{N-1}^{k-1}),$$

where x_i^j denotes the position and the velocity of the i-th body in the j-th configuration.

To prove the existence of a fixed point of the map $\hat{G}$ we use again Theorem 4.2. Like in the previous section we split the point x in the full space into z and c. As c we take the second coordinate of position and the first coordinate of velocity of the first body in the first configuration x^0. Again we fix the value of c_0 to be $(0, 0)$. This means that the first body in the first configuration is on the X axis with velocity orthogonal to that axis. The function $\hat{J}$ is defined as follows

$$\hat{J}(x^0, x^1, \ldots, x^{k-1}) = J(x^0), \tag{4.11}$$

where J is given by the equation (4.7), i.e. $\hat{J}$ returns the total energy and the angular momentum of the first configuration x^0. Let

$$\hat{F}(z) = \pi_z(\hat{G}(z, c_0)) - z. \tag{4.12}$$

With the above definitions and notations we have

Theorem 4.6. *Let Z and C be interval sets such that $\bar{z} \in Z$, $[\pi_c(\hat{G}(Z, c_0))] \subset C$ and $c_0 \in C$. If $K(\bar{z}, Z, \hat{F}) \subset \operatorname{int} Z$ and the interval matrix $\left[\frac{\partial \hat{J}}{\partial c}(Z, C)\right]$ is invertible, then there exists a fixed point of the map G.*

Proof. As $K(\bar{z}, Z, \hat{F}) \subset \operatorname{int} Z$, Theorem 3.1 guarantees the existence of a point $z^* \in Z$ such that $\pi_z \hat{G}(z^*, c_0) = z^*$. Let $c_1 = \pi_c(\hat{G}(z^*, c_0))$. In general we do not have $\hat{J}(x) = \hat{J}(\hat{G}(x))$, because the map $\hat{\sigma}$ exchanges configurations. But we will show that $\hat{J}(z^*, c_0) = \hat{J}(z^*, c_1)$. Let $z^* = (z^0, x^1, \ldots, x^{k-1})$, where z^0 is a "z-part" of the first configuration. Then for $\bar{t} = \frac{T}{k \cdot N}$ we have

$$
\begin{cases} \varphi(\bar{t}, (z^0, c_0)) = x^1, \\ \varphi(\bar{t}, x^m) = x^{m+1} \text{ for } m = 1, 2, \ldots, k-2, \\ \bar{\sigma} \circ \varphi(\bar{t}, x^{k-1}) = (z^0, c_1). \end{cases} \tag{4.13}
$$

We have also $J(\varphi(t, x)) = J(x)$ for each configuration x and all times $t \in \mathbb{R}$. Then from (4.13) it follows that $J(z^0, c_0) = J(x^1) = \cdots = J(x^{k-1}) = J(z^0, c_1)$. Hence we have $\hat{J}(z^*, c_0) = \hat{J}(z^*, c_1)$. Now applying Theorem 4.2 to functions $\hat{G}$ and $\hat{J}$ we get $c_1 = c_0$. Finally, (z^*, c_0) is a fixed point of the map $\hat{G}$. This means that (z^0, c_0) is a fixed point of $\bar{G}$ and Lemma 4.4 gives us the conclusion of Theorem 4.6. ∎

5. Examples

Compared with methods from papers [KZ, Ka] the method presented in this paper does not assume any symmetry of the orbit. Because of that it can be used to prove the existence of any planar choreography, even non symmetric ones. Using Theorem 4.5 or 4.6 we do not get an exact initial configuration of the proved choreography but only a set X that contains that configuration. With this information usually we are not able to show symmetry of the choreography, but it gives us an upper bound for the image of its trajectory. This can be used to exclude a given symmetry. We can also display an approximate shape of the orbit.

We will present three examples. The first one is the very famous figure eight orbit (Fig. 1). We chose this choreography because of its simplicity and also to test the implementation of the method. The small dimension of the phase space allow us to present results of rigorous computations. Next two examples are nonsymmetric choreographies with 6 and 7 bodies shown in Figures 2 and 3. See also [S, S1, S2].

To write real numbers we use the `C` language convention. For example -1.34e-09 represents the number -1.34×10^{-9}. Also for intervals we use a special notation to make them more readable. Instead of [1.076143732238492,1.076143734466012] we write $1.07614373^{2238492}_{4466012}$. The only exceptions to this rule are intervals having ends with opposite signs, in which case we leave them in the standard form. When rounding interval to a desired precision we always round down the left end and round up the right end. For example the above interval rounded to 4 decimal places is 1.076^{1}_{2}.

All computations were performed on computer equipped with Intel Pentium IV 3.2 GHz processor and 960 MB RAM memory.

The source code of programs performing proofs described in this paper can be found on the webpage [KT].

5.1. **The Eight.**

Theorem 5.1. *Eight shaped choreography with 3 bodies exists and is locally unique up to symmetries of the equation (1.1).*

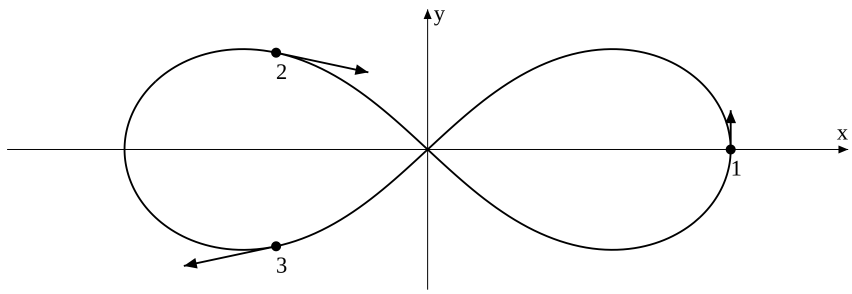

Figure 1. The Eight orbit - initial configuration.

Proof. To prove this we use Theorem 4.5. Using an interval arithmetic we verified its assumptions. Local uniqueness comes from the part (2) of Theorem 3.1. In Table 1 we present the initial values and the results of rigorous computations. ∎

Crucial to this proof is $\bar{z}$, a very good estimate for a zero of $\hat{F}$. Around $\bar{z}$ we build an interval set Z (we take an interval ball with centre $\bar{z}$ and radius 10^{-8}). Period T was obtained by a non rigorous simulation of a choreography and in fact it was almost 2π. Next, we use C^1-Lohner algorithm (see [ZLo]) to integrate the equation (4.3). First we estimate the image of the set Z after time $\frac{T}{3}$ (i.e., $\bar{\varphi}(\frac{T}{3}, Z \times c_0)$) and the matrix $\frac{\partial \bar{\varphi}}{\partial z}(\frac{T}{3}, Z \times c_0)$. Then we perform the same computation for a point $(\bar{z}, c_0)$. The results of these computations allow us to rigorously compute Krawczyk operator $K(\bar{z}, Z, \hat{F})$. This is the most time consuming part of the computation. If we succeed in showing the inclusion $K(\bar{z}, Z, \hat{F}) \subset \operatorname{int} Z$, then we additionally check if the interval matrix $\frac{\partial J(Z,C)}{\partial c}$ is invertible and if $c_0 \in C$. In our case, as it is shown in Table 1, all these conditions are fulfilled. Therefore using Theorem 4.5 we conclude that there exists exactly one T-periodic choreography of 3 bodies with initial conditions in the set $Z \times \{c_0\}$. In Lohner algorithm we used time step 0.01 and 10-th order Taylor method, when performing computations for the point $(\bar{z}, c_0)$, and time step 0.004 and 7-th order Taylor method for computations for the set $Z \times \{c_0\}$. Due to the good stability properties of this solution (see [S2]) there is no need of parallel shooting for the figure eight choreography.

5.2. **Nonsymmetric choreographies with 6 and 7 bodies.** The second example is a non symmetric choreography with $N = 7$ bodies shown in Figure 2. In this case two bodies pass quite close to each other and this requires small time steps. It causes the computation to be very long and the estimates of Krawczyk operator to be not good enough. Because of this it turns out that the use of the parallel shooting method is needed. Hence the proof is done using Theorem 4.6. In this example the number of intermediate configurations equals 2 ($k = 3$).

At the beginning we have x_0, an estimated initial condition for the choreography, and the period T. We set $\bar{T} = \frac{1}{N}T$. Next we integrate the equation (4.3), without producing bounds of the errors, to get estimated intermediate configurations x^1 and x^2 at times $t = \frac{\bar{T}}{3}$ and $t = \frac{2\bar{T}}{3}$, respectively. Alternatively, one can find the periodic solution by using also parallel shooting without interval arithmetic. We get a point (x^0, x^1, x^2), which we split into $(\bar{z}, c_0)$ in the way described in section 4.1. We define the set Z as the interval ball with centre at the point $\bar{z}$ and radius $2 \cdot 10^{-8}$.

Initial values	
$\bar{z}$	$(1.07614373351,\ 0.46826621840,\ -0.53807186675,\ -0.34370682775,\ -1.09960375207,\ -0.23413310920)$
Z	$\bar{z} + [\text{-1e-8},\ \text{1e-8}]^6$
c_0	$(0,0)$
T	2π
Results of computations	
$\hat{F}(\bar{z})$	$({}^{-1.28}_{+1.25}\text{e-12},\ {}^{-1.95}_{+1.96}\text{e-12},\ 5.{}^{08}_{11}\text{e-10},\ 1.0^{7}_{9}\text{e-10}, -1.6^{6}_{5}\text{e-10},\ -1.0^{2}_{1}\text{e-09})$
$\operatorname{diam}\hat{F}(\bar{z})$	(2.52e-12, 3.91e-12, 1.34e-12, 9.52e-13, 1.85e-12, 2.04e-12)
$\frac{\partial \hat{F}}{\partial z}(Z)$	$\begin{pmatrix} -0.2^{4}_{3} & 1.6^{7}_{8} & -0.6^{9}_{7} & -6.0^{4}_{2} & -3.2^{1}_{0} & -0.2^{8}_{6} \\ -1.0^{8}_{6} & -2.5^{2}_{1} & -1.1^{3}_{1} & 0.7^{4}_{7} & 0.5^{1}_{2} & -0.5^{4}_{3} \\ -1.8^{8}_{7} & -3.9^{9}_{8} & -6.8^{1}_{0} & -10.0^{9}_{7} & -4.1^{3}_{1} & -2.7^{5}_{4} \\ 6.2^{5}_{6} & 6.0^{3}_{4} & 6.8^{2}_{3} & 10.{}^{18}_{20} & 4.0^{1}_{2} & 3.3^{2}_{3} \\ -8.1^{1}_{0} & -9.0^{8}_{7} & -8.7^{6}_{4} & -13.{}^{4}_{3} & -6.{}^{10}_{08} & -4.0^{1}_{0} \\ 6.0^{6}_{7} & 10.6^{4}_{6} & 15.0^{2}_{3} & 30.8^{8}_{9} & 12.8^{3}_{4} & 6.1^{5}_{7} \end{pmatrix}$
$K(\bar{z}, Z, \hat{F})$	$(1.07614373^{223}_{447},\ 0.46826621^{793}_{895},\ -0.53807186^{965}_{370},\ -0.34370682^{835}_{706},\ -1.09960375^{371}_{060},\ -0.2341331^{166}_{019})$
$\operatorname{diam} K(\bar{z}, Z, \hat{F})$	(2.23e-9, 1.02e-9, 5.93e-9, 1.28e-9, 3.10e-9, 1.46e-8)
C	([-4.43e-7, 4.43e-7], [-8.80e-7, 8.81e-7])
$\frac{\partial J(Z,C)}{\partial c}$	$\begin{pmatrix} -2.192^{71}_{68} & -1.0996^{1}_{0} \\ 1.0996^{0}_{1} & 0.34370^{6}_{8} \end{pmatrix}$
$det(\frac{\partial J(Z,C)}{\partial c})$	0.4554^{72}_{99}
Computations time	$\hat{F}(\bar{z})$: 3.61 s. $\hat{F}(Z)$: 5.86 s.

TABLE 1. Data from the proof of the existence of the Eight orbit.

The rigorously computed Krawczyk operator $K(\bar{z}, Z, \hat{F})$ was only partially contained in the set Z. So we took the intersection of $K(\bar{z}, Z, \hat{F})$ and Z as a new set Z and computed again Krawczyk operator. The second iteration brought the needed inclusion $K(\bar{z}, Z, \hat{F}) \subset \operatorname{int} Z$. As set C we took upper estimates of $[\pi_c(G(z))]$. Next we checked that $c_0 \in C$ and that $\frac{\partial J(Z,C)}{\partial c}$ is invertible. Because the rigorous computations verified the assumptions of Theorem 4.6 we have following theorem

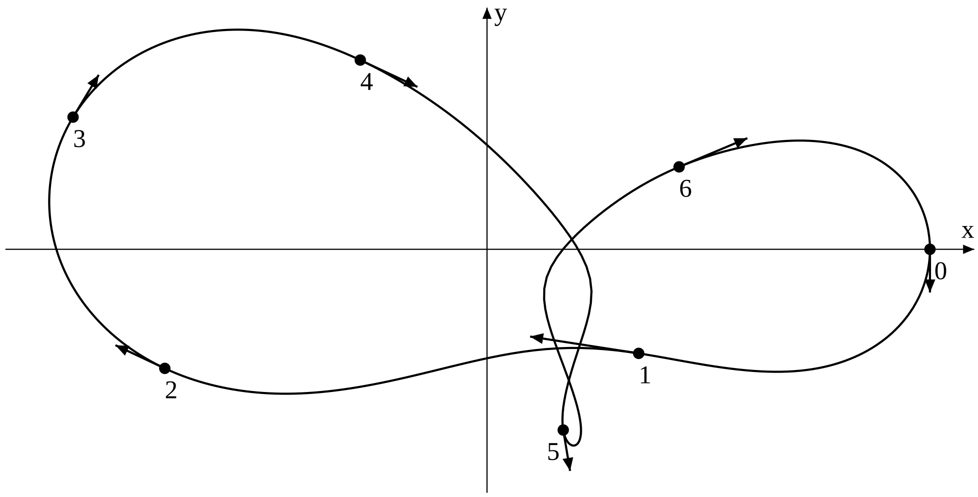

Figure 2. Non symmetric choreography with 7 bodies

Theorem 5.2. *A nonsymmetric choreography with 7 bodies exists.*

For 7-bodies choreography with 2 intermediate points the dimension of the phase space is 72. We do not present whole sets of vectors and matrices. Instead of this in Table 2 we give some general information on diameters, maximum values, etc.

The same procedure was applied to the third example: the 6 bodies choreography shown on Figure 3. Again we use parallel shooting with two additional intermediate configurations. Data from the proof are presented in Table 3.

Theorem 5.3. *A nonsymmetric choreography with 6 bodies exists.*

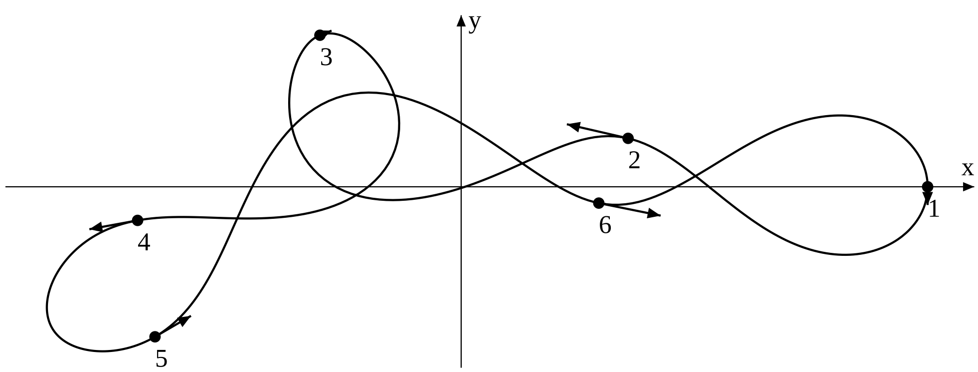

Figure 3. Non symmetric choreography with 6 bodies.

6. Linear stability of the Eight

The Eight was for a long time the only choreography believed to be linearly stable. Here we want to provide a proof of this fact. In Section 5.1 we proved existence of the Eight solution. As a byproduct we obtained the set $K(\bar{z}, Z, \hat{F})$

Initial values	
x_0	$(1.570000823,\ 0.000000000,\ 0.000000000,\ -1.020988487,$ $0.537599552,\ -0.369094762,\ -2.565831109,\ 0.397511202,$ $-1.142082898,\ -0.422251256,\ -1.166452823,\ 0.547404173,$ $-1.467257921,\ 0.468766346,\ 0.606469297,\ 0.998837740,$ $-0.449270290,\ 0.671112038,\ 1.351529425,\ -0.634305641,$ $0.270083191,\ -0.640890579,\ 0.161663517,\ -0.966669530,$ $0.680927544,\ 0.292358212,\ 1.612621693,\ 0.678210542)$
c_0	$(0,0)$
Z	$\bar{z}+$[-2e-8, 2e-8]^{70}
T	2π
Parameters of computations	
$\hat{F}(\bar{z})$	time step : 0.0009, order of Taylor method : 13
$\hat{F}(Z)$	time step : 0.000075, order of Taylor method : 7
Results of computations	
max diam $\hat{F}_i(\bar{z})$	5.29e-11
$\max \frac{\partial \hat{F}_i}{\partial z_j}(Z)$	16.7
max diam $\frac{\partial \hat{F}_i}{\partial z_j}(Z)$	3.62e-04
diam $\frac{\partial \hat{F}_i}{\partial z_j}(Z)$	1.71e-06 (average value)
max diam $K(\bar{z}, Z, \hat{F})$	6.03e-09
C	([-1.13e-08,1.14e-08],[-5.03e-08,5.16e-08])
$\frac{\partial \hat{J}}{\partial c}(Z,C)$	$\begin{pmatrix} -3.1^6_5 & -1.6^0_1 \\ 1.6^1_2 & 0.29^2_3 \end{pmatrix}$
$det\left(\frac{\partial \hat{J}}{\partial c}(Z,C)\right)$	1.6^7_8
Computation time	$\hat{F}(\bar{z})$: 13.3 min., $\hat{F}(Z)$: 69 min. (for one iteration)

Table 2. Data from the proof of the existence of non symmetric choreography with 7 bodies.

Initial values	
z_0	$(1.65664706, \quad -0.64689132,$ $0.59294259, \quad 0.17186502, \quad -2.17431535, \quad 0.50023185,$ $-0.50169146, \quad 0.53779281, \quad 0.41392017, \quad 0.16258946,$ $-1.14921834, \quad -0.11931360, \quad -1.71442918, \quad -0.31427360,$ $-1.08742527, \quad -0.53240619, \quad 1.27714030, \quad 0.74254988)$
c_0	$(0,0)$
Z	$\bar{z}+$[-9e-9, 9e-9]^{58}
T	2π
Parameters of computations	
$\hat{F}(\bar{z})$ $\hat{F}(Z)$	time steps: 2^{-9}, order of Taylor method : 15 time steps : 2^{-14}, order of Taylor method : 7
Results of computations	
$\max \operatorname{diam} \hat{F}_i(\bar{z})$	4.00e-11
$\max \frac{\partial \hat{F}_i}{\partial z_j}(Z)$ $\max \operatorname{diam} \frac{\partial \hat{F}_i}{\partial z_j}(Z)$ $\operatorname{diam} \frac{\partial \hat{F}_i}{\partial z_j}(Z)$	23.4 4.78e-3 6.77e-5 (average value)
$\max \operatorname{diam} K(\bar{z}, Z, \hat{F})$	1.40e-8
C	([-3.50e-08,3.50e-08], [-2.36e-07,2.36e-07])
$\frac{\partial \hat{J}}{\partial c}(Z,C)$	$\begin{pmatrix} 14.^{4}_{5} & -2.^{19}_{20} \\ 2.^{19}_{20} & -5.^{60}_{79}e-2 \end{pmatrix}$
$det\left(\frac{\partial \hat{J}}{\partial c}(Z,C)\right)$	$3.^{99}_{40}$
Computation time	2.1 hours (four iterations)

TABLE 3. Data from the proof of the existence of non symmetric choreography with 6 bodies.

which contains initial data for this choreography. In this section we use these data to show the linear stability of the Eight when we consider planar motion on the zero level of angular momentum. We prove that all eigenvalues of the monodromy matrix $\frac{\partial\varphi}{\partial x}(x_0, T)$ are on the unit circle.

In the case of the Eight solution we have an 8×8 monodromy matrix A (last body is removed by fixing the center of the mass). From Hamiltonian Systems theory we know that at least 4 eigenvalues, that correspond to first integrals, are equal to 1. They are associated to (energy-time shift) and (angular momentum-rotation). Let $\lambda_j, j = 1, \ldots, 4$ be the remaining or "relevant" eigenvalues. We know that $\lambda \in \mathrm{Spec}(A)$ implies $\bar{\lambda} \in \mathrm{Spec}(A)$ and $\lambda^{-1} \in \mathrm{Spec}(A)$. The possible situations (up to index exchange) are:

(S1) some of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are real and $\lambda_1\lambda_2 = 1$, $\lambda_3\lambda_4 = 1$,
(S2) all $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are not real and $\lambda_1 = \lambda_2^{-1} = \bar{\lambda}_3 = \bar{\lambda}_4^{-1}$,
(S3) all $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are different, not real and $\lambda_1 = \lambda_2^{-1} = \bar{\lambda}_2$,$\lambda_3 = \lambda_4^{-1} = \bar{\lambda}_4$.

From the third case it follows immediately that all eigenvalues have to be on the unit circle. We want to avoid the first and the second situations.

In all these three cases the characteristic polynomial has the form

$$(\lambda - 1)^4(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) \tag{6.1}$$

$$= (\lambda - 1)^4(\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2)(\lambda^2 - (\lambda_3 + \lambda_4)\lambda + \lambda_3\lambda_4) \tag{6.2}$$

$$= (\lambda - 1)^4(\lambda^2 - T_1\lambda + 1)(\lambda^2 - T_2\lambda + 1) \tag{6.3}$$

$$= \lambda^8 - (T_1 + T_2 + 4)\lambda^7 + (T_1T_2 + 4(T_1 + T_2) + 8)\lambda^6 + \ldots \tag{6.4}$$

where $T_1 = \lambda_1 + \lambda_2$, $T_2 = \lambda_3 + \lambda_4$. On the other hand if $A = (a_{ij})$ is any 8×8 matrix then

$$det(A - \lambda I) \quad = \quad \lambda^8 - \alpha\lambda^7 + \beta\lambda^6 + \ldots \tag{6.5}$$

where $\alpha = \text{ trace } (A) = \sum_{i=1}^{8} a_{ii}$, $\beta = \sum_{1 \le i < j \le 8} (a_{ii}a_{jj} - a_{ij}a_{ji})$.

From (6.4) and (6.5) we get the system of equations

$$\begin{cases} T_1 + T_2 + 4 = \alpha, \\ T_1T_2 + 4(T_1 + T_2) + 8 = \beta. \end{cases} \tag{6.6}$$

Finally, from (6.6) it follows that T_1, T_2 are solutions of the equation

$$T^2 - (\alpha - 4)T + \beta - 4\alpha + 8 = 0. \tag{6.7}$$

With the notation introduced above we have

Theorem 6.1. *Let T_1, T_2 be solutions of the equation (6.7). If*

$$\Delta = (\alpha - 4)^2 - 4(\beta - 4\alpha + 8) > 0, \tag{6.8}$$

$$|T_1| < 2, |T_2| < 2, \tag{6.9}$$

then all eigenvalues of the monodromy matrix A are different and they are on the unit circle (the situation S3).

Proof. From (6.8) we have that T_1 and T_2 are two different real solutions. The situation S2 is not possible, because in this case either $T_1 = T_2$ or $Im\, T_1 = Im\,(\lambda_1 + \lambda_2) \neq 0$. In the case S1 we have $|T_1| > 2$ or $|T_2| > 2$ which is opposite to 6.9. Hence the only possibility is the case S3. ∎

Now we will use Theorem 6.1 to prove stability of the Eight. First, we need to improve our estimates of initial configuration and monodromy matrix of the Eight obtained in Section 5.1. We can do this in several ways. First of all, we can try to carry out the proof with smaller initial set Z. We can also iterate computations of interval Krawczyk operator $Z_{n+1} := K(\bar{z}, Z_n, \hat{F})$ with $Z_0 = Z$. With fixed precision of floating point numbers (roughly 16 decimal digits) this iteration does not converge to a unique fixed point as it should in real arithmetic. Due to computational errors improvements stop at some moment. We overcome this using multiple precision interval arithmetic. Finally, we end up with a good rigorous approximation of initial condition of the Eight solution. This allows us to obtain an approximation of the monodromy matrix A good enough to show stability of the Eight. Also this computation was made with multiple precision interval arithmetic. Numerical computations made by Simó (see [S3]) suggested that the relevant eigenvalues are $\lambda_{1,2} \approx 0.99859998 \pm 0.05289683i$, $\lambda_{3,4} \approx -0.29759667 \pm 0.95469169i$. The need for a greater accuracy is caused mainly by eigenvalues λ_1 and λ_2, because they are very close to 1. Using double precision we were not able to check conditions of Theorem 6.1 even if the set Z is taken equal to a single point.

Theorem 6.2. *The Eight solution is linearly stable when motion is restricted to the plane and to the zero angular momentum level.*

Proof. Using rigorous computations we checked that the assumptions of Theorem 6.1 are satisfied, hence all eigenvalues are on the unit circle. The nontrivial eigenvalues are different. This implies the desired stability. Details from this proof are presented in the Table 4. ∎

Remark. We note that the real parts of the non-rigorous eigenvalues given in [S3] are extremely close to the center of the rigorous intervals obtained for $T_1/2$ and $T_2/2$.

7. Conclusions

The method presented in Section 4 allows us to verify computer simulations of choreographies. Once we have a careful approximation of the initial configuration we can use Theorem 4.5 to prove that close to that approximation exists an initial configuration of some true choreography.

The main difficulty to make this process automatic is the quality of the rigorous estimates of the Krawczyk operator. There are several well known sources of overestimation such as round off errors, wrapping effect, errors coming from numerical integration of ODE. One can improve estimates using multiple precision, high order of Taylor method in ODE integration, various set representation methods e.g. those proposed by Lohner or Taylor models. So we have several parameters (such as order of Taylor method, time step during integration, mantissa bits in floating point number representation, set representation method) to set up for a specific problem. The most time consuming process to produce the proofs presented in this paper was the search for suitable parameter values which yield estimates good enough to fulfill theorem assumptions. On the one hand we want to obtain very good estimates but on the other hand we want to make this in a reasonable computational time. Computations which take several days for one choreography can be worthless in practical applications. We hope that this issue can be cured in the future by still increasing the power of modern computers.

Initial values	
$\bar{z}$, c_0	the same as in Table 1
max diam Z	1.29e-16
precision	100 bits in mantissa
Results of computations	
Δ	6.720^{458965}_{547070}
T_1	1.997^{195667}_{204261}
T_2	-0.5951^{97631}_{89038}
Computation time: 999 sec. (16.65 min.)	

Table 4. Data from the proof of the stability of the Eight.

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