LINKING COMBINATORIAL AND CLASSICAL DYNAMICS: CONLEY INDEX AND MORSE DECOMPOSITIONS

BOGDAN BATKO, TOMASZ KACZYNSKI, MARIAN MROZEK, AND THOMAS WANNER

Abstract. We prove that every combinatorial dynamical system in the sense of Forman, defined on a family of simplices of a simplicial complex, gives rise to a multivalued dynamical system F on the geometric realization of the simplicial complex. Moreover, F may be chosen in such a way that the isolated invariant sets, Conley indices, Morse decompositions, and Conley-Morse graphs of the two dynamical systems are in one-to-one correspondence.

1. Introduction

In the years since Forman [13, 14] introduced combinatorial vector fields on simplicial complexes, they have found numerous applications in such areas as visualization and mesh compression [20], graph braid groups [12], homology computation [16, 23], astronomy [32], the study of Čech and Delaunay complexes [6], and many others. One reason for this success has its roots in Forman's original motivation. In his papers, he sought to transfer the rich dynamical theories due to Morse [24] and Conley [9] from the continuous setting of a continuum (connected compact metric space) to the finite, combinatorial setting of a simplicial complex. This has proved to be extremely useful for establishing finite, combinatorial results via ideas from dynamical systems. In particular, Forman's theory yields an alternative when studying sampled dynamical systems. The classical approach consists in the numerical study of the dynamics of the differential equation constructed from the sample. The construction uses the data in the sample either to discover the natural laws governing the dynamics [33] in order to write the

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equations or to interpolate or approximate directly the unknown right-hand-side of the equations [7]. In the emerging alternative one can eliminate differential equations and study directly the combinatorial dynamics defined by the sample [13, 14, 29, 19, 28].

The two approaches are essentially distinct. On the one hand, dynamical systems defined by differential equations on a differentiable manifold arise in a wide variety of applications and show an extreme wealth of observable dynamical behavior, at the expense of fairly involved mathematical techniques which are needed for their precise description. On the other hand, the discrete simplicial complex setting makes the study of many phenomena simple, due to the availability of fast combinatorial algorithms. This leads to the natural question of which approach should be chosen when for a given problem.

In order to answer this question it may be helpful to go beyond the exchange of abstract underlying ideas present in much of the existing work and look for the precise relation between the two theories. In our previous paper [18] we took this path and studied the formal ties of multivalued dynamics in the combinatorial and continuum settings. The choice of multivalued dynamics is natural, because the combinatorial vector fields generate multivalued dynamics in a natural way. Moreover, in the finite setting such dynamical phenomena as homoclinic or heteroclinic connections are not possible in single-valued dynamics. The choice of multivalued dynamics on continua is not a restriction. This is a broadly studied and well understood theory. The theory originated in the middle of the 20th century from the study of contingent equations and differential inclusions [35, 31, 3] and control theory [30]. At the end of the 20th century it was successfully applied to computer assisted proofs in dynamics [22, 27]. In particular, the Conley theory for multivalued dynamics was studied by several authors [26, 17, 34, 10, 11, 5, 4].

In [18] we proved that for any combinatorial vector field on the collection of simplices of a simplicial complex one can construct an acyclic-valued and upper semicontinuous map on the underlying geometric realization whose dynamics on the level of invariant sets exhibits the same complexity. More precisely, by introducing the notion of isolated invariant sets in the discrete setting, we established a correspondence between isolated invariant sets in the combinatorial and classical multivalued settings. We also presented a link on the level of individual dynamical trajectories.

In the present paper we complete the program started in [18] by showing that the formal correspondence established there extends to Conley indices of the corresponding isolated invariant set as well as Morse decompositions and Conley-Morse graphs [2, 8], a global descriptor of dynamics capturing its gradient structure.

The organization of the paper is as follows. In Section 2 we present the main result of the paper and illustrate it with some examples. In Section 3 we recall the basics of the Conley theory for multivalued dynamics. In Section 4 we recall from [18] the construction of a multivalued self-map $F : X \multimap X$ associated

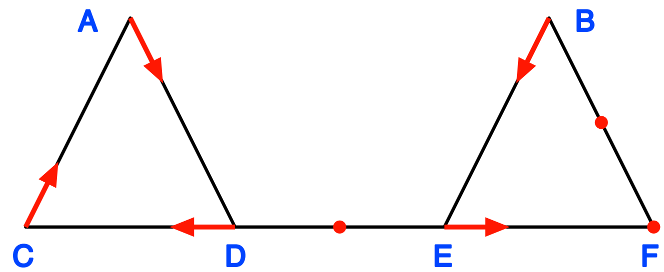


FIGURE 1. Sample discrete vector field. This figure shows a simplicial complex $\mathcal{X}$ which is a graph on six vertices with seven edges. Critical cells are indicated by red dots, vectors of the vector field are shown as red arrows.

with a combinatorial vector field $\mathcal{V}$ on a simplicial complex $\mathcal{X}$ with the geometric realization $X := |\mathcal{X}|$. In Section 5 we use this construction to outline the proof of the main result of the paper in a series of auxiliary theorems. The remaining sections are devoted to the proofs of these theorems.

2. Main result

Let $\mathcal{X}$ denote the family of simplices of a finite abstract simplicial complex. The face relation on $\mathcal{X}$ defines on $\mathcal{X}$ the T_0 Alexandroff topology [1]. A subset $\mathcal{A} \subseteq \mathcal{X}$ is *open* in this topology if all cofaces of any element of $\mathcal{A}$ are also in $\mathcal{A}$. The closure of $\mathcal{A}$ in this topology, denoted $\operatorname{Cl}\mathcal{A}$, is the family of all faces of all simplices in $\mathcal{A}$ (see Section 3.1 for more details). A *combinatorial vector field* $\mathcal{V}$ on $\mathcal{X}$ is a partition of $\mathcal{X}$ into singletons and doubletons such that each doubleton consists of a simplex and one of its cofaces of codimension one. The singletons are referred to as *critical cells*. The doubleton considered as a pair with lower dimensional simplex coming first is referred to as a *vector*.

The elementary example in Figure 1 presents a one-dimensional simplicial complex $\mathcal{X}$ consisting of six vertices $\{A, B, C, D, E, F\}$ and seven edges $\{AC, AD, BE, BF, CD, DE, EF\}$, and the combinatorial vector field consisting of three singletons (critical cells) $\{\{BF\}, \{DE\}, \{F\}\}$ and five doubletons (vectors) $\{\{A, AD\}, \{B, BE\}, \{C, AC\}, \{D, CD\}, \{E, EF\}\}$. With a combinatorial vector field $\mathcal{V}$ we associate multivalued dynamics given as iterates of a multivalued map $\Pi_{\mathcal{V}} : \mathcal{X} \multimap \mathcal{X}$ sending each critical simplex to all of its faces, each source of a vector to the corresponding target, and each target of a vector to all faces of the target other than the corresponding source and the target itself. In the case of the example in Figure 1 the map is (we skip the braces in the case of singletons

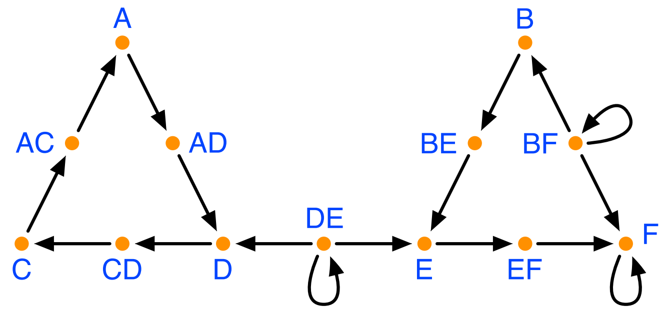


FIGURE 2. The directed graph $G_{\mathcal{V}}$ for the combinatorial vector field in Figure 1.

to keep the notation simple)

$$\begin{aligned}\Pi_{\mathcal{V}} \;=\; & \{(A,AD),(AD,D),(B,BE),(BE,E),(BF,\{B,BF,F\}),\\ & (C,AC),(AC,A),(CD,C)(D,CD),(DE,\{D,DE,E\}),\\ & (E,EF),(EF,F),(F,F)\}\;.\end{aligned}$$

The multivalued map $\Pi_{\mathcal{V}}$ may be considered as a directed graph $G_{\mathcal{V}}$ with vertices in $\mathcal{X}$ and an arrow from a simplex σ to a simplex τ whenever $\tau \in \Pi_{\mathcal{V}}(\sigma)$. The directed graph $G_{\mathcal{V}}$ for the combinatorial vector field in Figure 1 is presented in Figure 2. A subset $\mathcal{A} \subseteq \mathcal{X}$ is *invariant* with respect to $\mathcal{V}$ if every element of $\mathcal{A}$ is both a head and a tail of an arrow in $G_{\mathcal{V}}$ which joins vertices in $\mathcal{A}$. An element $\sigma \in \operatorname{Cl}\mathcal{A} \setminus \mathcal{A}$ is an *internal tangency* of $\mathcal{A}$ if it admits an arrow originating in σ with its head in $\mathcal{A}$, as well as an arrow terminating in σ with its tail in $\mathcal{A}$. The set $\operatorname{Ex}\mathcal{A} := \operatorname{Cl}\mathcal{A} \setminus \mathcal{A}$ is referred to as the *exit set* of $\mathcal{A}$ (see [18, Definition 3.4]) or *mouth* of $\mathcal{A}$ (see [28, Section 4.4]). An invariant $\mathcal{S}$ set is an *isolated invariant set* if the exit set $\operatorname{Ex}\mathcal{S}$ is closed and it admits no internal tangencies. Note that $\mathcal{X}$ itself is an isolated invariant set if and only if it is invariant. The *(co)homological Conley index* of an isolated invariant set $\mathcal{S}$ is the relative singular (co)homology of the pair $(\operatorname{Cl}\mathcal{S}, \operatorname{Ex}\mathcal{S})$. Note that $(\operatorname{Cl}\mathcal{S}, \operatorname{Ex}\mathcal{S})$ is a pair of simplicial subcomplexes of the simplicial complex $\mathcal{X}$. Therefore, by McCord's Theorem [21], the singular (co)homology of the pair $(\operatorname{Cl}\mathcal{S}, \operatorname{Ex}\mathcal{S})$ isomorphic to the simplicial homology of the pair $(\operatorname{Cl}\mathcal{S}, \operatorname{Ex}\mathcal{S})$.

The singleton $\{BF\}$ in Figure 1 is an example of an isolated invariant set of $\mathcal{V}$. Its exit set is $\{B,F\}$ and its Conley index is the (co)homology of the pointed circle. Another example is the set $\{A,AC,AD,C,CD,D\}$ with an empty exit set and the Conley index equal to the (co)homology of the circle. Both these examples are minimal isolated invariant sets, that is, none of their proper non-empty subsets is an isolated invariant set.

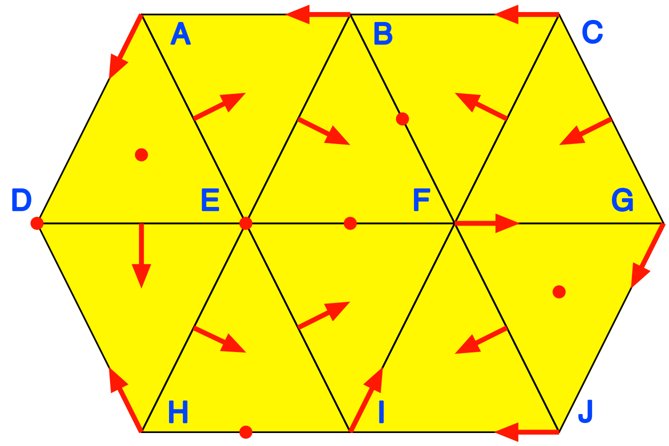


FIGURE 3. Sample discrete vector field. This figure shows a simplicial complex $\mathcal{X}$ which triangulates a hexagon (shown in yellow), together with a discrete vector field. Critical cells are indicated by red dots, vectors of the vector field are shown as red arrows. This example will be discussed throughout the paper.

The two-dimensional example depicted in Figure 3 presents a simplicial complex which is built from 10 triangles, 19 edges and 10 vertices, and a combinatorial vector field consisting of 7 critical cells and a total of 16 vectors. The set $\{ADE, DE, DEH, EF, EFI, EH, EHI, EI, F, FG, FI, G, GJ, HI, I, IJ, J\}$ is an example of an isolated invariant set for this combinatorial vector field. It is presented in Figure 4. Its exit set is $\{A, AD, AE, D, DH, E, H\}$ and its Conley index is the (co)homology of the pointed circle. This isolated invariant set is not minimal. For instance, the singleton $\{EF\}$ is a subset which itself is an isolated invariant set.

A *connection* from an isolated invariant set $\mathcal{S}_1$ to an isolated invariant set $\mathcal{S}_2$ is a sequence of vertices on a walk in $G_{\mathcal{V}}$ originating in $\mathcal{S}_1$ and terminating in $\mathcal{S}_2$. A family $\mathcal{M} = \{\mathcal{M}_p \,|\, p \in \mathbb{P}\}$ indexed by a poset $\mathbb{P}$ and consisting of mutually disjoint isolated invariant subsets of an isolated invariant set $\mathcal{S}$ is a *Morse decomposition* of $\mathcal{S}$ if any connection between elements in $\mathcal{M}$ which is not contained entirely in one of the elements of $\mathcal{M}$ originates in $\mathcal{M}_{q'}$ and terminates in $\mathcal{M}_q$ with $q' > q$. The associated *Conley-Morse graph* is the partial order induced on $\mathcal{M}$ by the existence of connections, and represented as a directed graph labelled with the Conley indices of the isolated invariant sets in $\mathcal{M}$. Typically, the labels are written as Poincaré polynomials, that is, polynomials whose ith coefficient equals the ith Betti number of the Conley index.

An example of a Morse decomposition for the combinatorial vector field in Figure 1 is

$$\mathcal{M} := \{\{BF\}, \{F\}, \{DE\}, \{A, AD, C, CA, CD, D\}\} ,$$

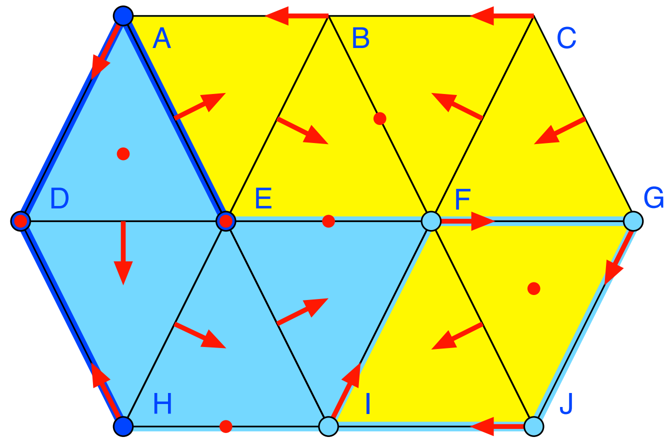


FIGURE 4. Sample isolated invariant set for the discrete vector field shown in Figure 3. The simplices which belong to the isolated invariant set $\mathcal{S}$ are indicated in light blue, and are given by four vertices, nine edges, and four triangles. Its exit set $\operatorname{Ex}\mathcal{S}$ is shown in dark blue, and it consists of four vertices and three edges.

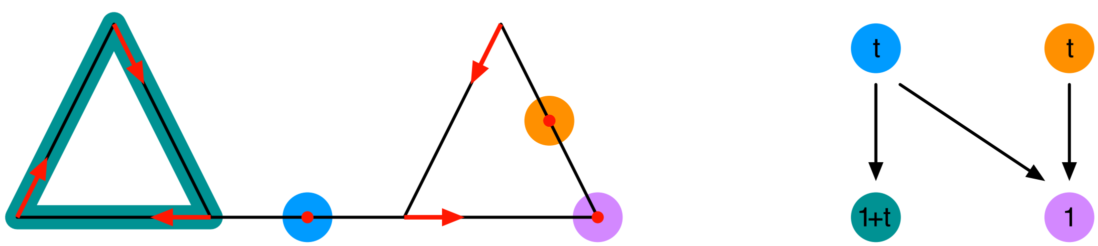


FIGURE 5. Morse decomposition for the example shown in Figure 1. For this example, one can find four minimal Morse sets, which are indicated in the left image in different colors. The right image shows the associated Morse graph.

and the corresponding Conley-Morse graph is presented in Figure 5. A Morse decomposition of the example in Figure 3 together with the associated Conley-Morse graph is presented in Figure 6.

The main result of this paper is the following theorem.

Theorem 2.1. *For every combinatorial vector field $\mathcal{V}$ on a simplicial complex $\mathcal{X}$ there exists an upper semicontinuous, acyclic, and inducing identity in homology multivalued map $F : |\mathcal{X}| \multimap |\mathcal{X}|$ on the geometric realization $|\mathcal{X}|$ of $\mathcal{X}$ such that*

(i) *for every Morse decomposition $\mathcal{M}$ of $\mathcal{V}$ there exists a Morse decomposition M of the semidynamical system induced by F,*

(ii) *the Conley-Morse graph of M is isomorphic to the Conley-Morse graph of $\mathcal{M}$,*

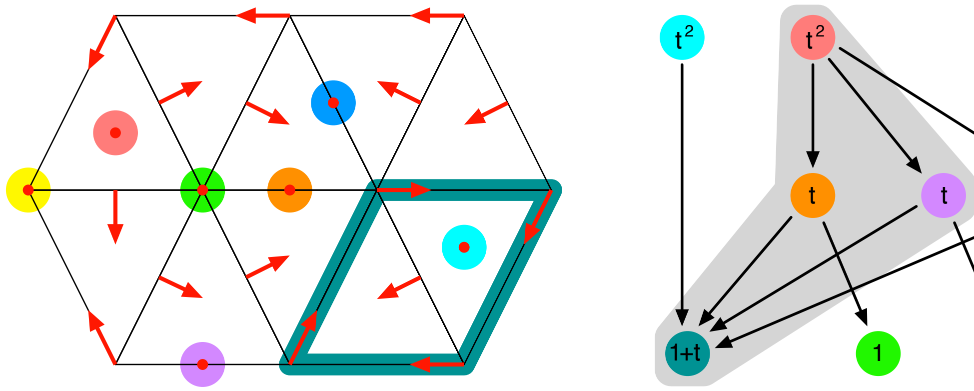


FIGURE 6. Morse decomposition for the example shown in Figure 3. For this example, one can find eight minimal Morse sets, which are indicated in the left image in different colors. The right image shows the associated Morse graph. The isolated invariant set shown in Figure 4 corresponds to the subgraph indicated by the gray shaded area in the Morse graph.

(iii) *each element of M is contained in the geometric representation of the corresponding element of $\mathcal{M}$.*

This theorem is an immediate consequence of the much more detailed theorems presented in Section 5. The multivalued map F guaranteed by Theorem 2.1 for the example in Figure 1 is presented in Figure 7.

3. Conley theory for multivalued topological dynamics

In this section we recall the main concepts of Conley theory for multivalued dynamics in the combinatorial and classical setting: isolated invariant sets, index pairs, Conley index and Morse decompositions.

3.1. **Preliminaries.** We write $f : X \nrightarrow Y$ to denote a *partial function*, that is, a function whose *domain*, denoted $\operatorname{dom} f$, is a subset of X. We write $\operatorname{im} f := f(X)$ to denote the *image* of f and $\operatorname{Fix} f := \{\, x \in \operatorname{dom} f \mid f(x) = x \,\}$ to denote the set of *fixed points* of f.

Given a topological space X and a subset $A \subseteq X$, we denote by $\operatorname{cl} A$, $\operatorname{int} A$ and $\operatorname{bd} A$ respectively the *closure*, the *interior* and the *boundary* of A. We often use the set $\operatorname{ex} A := \operatorname{cl} A \setminus A$ which we call the *exit set* or *mouth* of A. Whenever applying an operator like cl or ex to a singleton, we drop the braces to keep the notation simple.

The *singular cohomology* of the pair (X, A) is denoted $H^*(X, A)$. Note that in this paper we apply cohomology only to polyhedral pairs or pairs weakly homotopy equivalent to polyhedral pairs. Hence, the singular cohomology is the same as Alexander-Spanier cohomology. In particular, all but a finite number of Betti

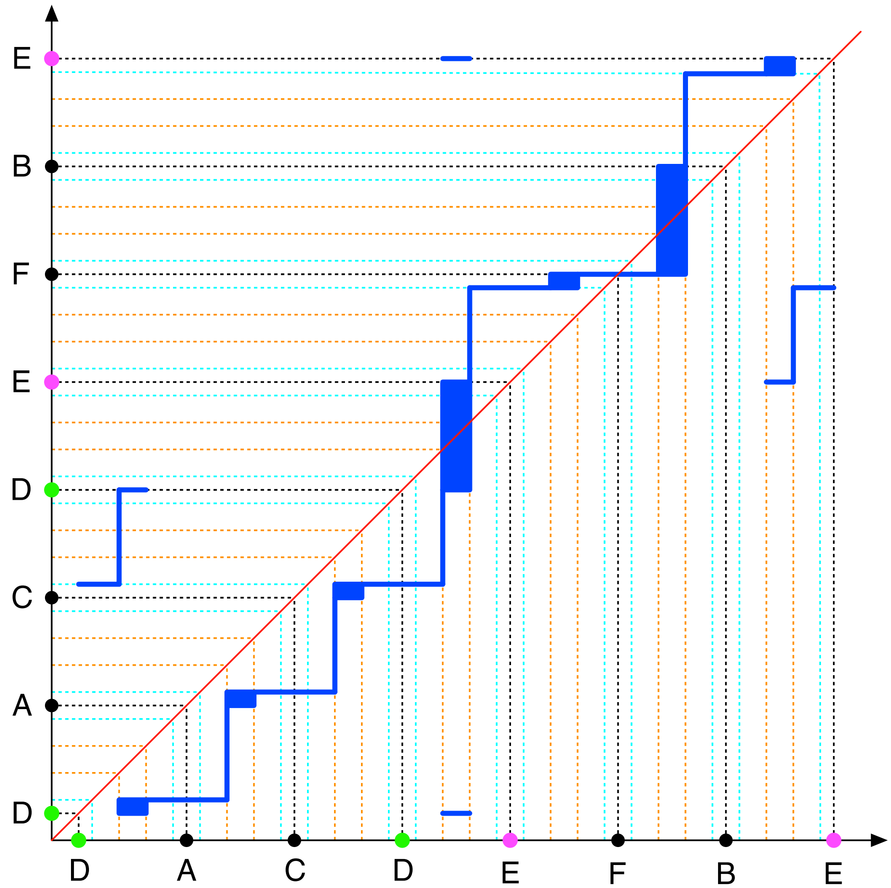


Figure 7. The multivalued map F for the combinatorial vector field shown in Figure 1. For visualization purposes the domain of F is straightened to a segment in which vertices D (marked in green) and E (marked in magenta) are represented twice. The graph of F is shown in blue. The edge DE in the middle corresponds to the center edge in Figure 1. To its left, the three line segments correspond to the cycle in the combinatorial vector field. Note that the two green vertices are identified. The three edges to the right of the center correspond to the right triangle in Figure 1. Also here the two magenta vertices are identified.

numbers of the pair (X, A) are zero. The corresponding *Poincaré polynomial* is the polynomial whose ith coefficient is the ith Betti number.

By a multivalued map $F : X \multimap X$ we mean a map from X to the family of non-empty subsets of X. We say that F is *upper semicontinuous* if for any open $U \subseteq X$ the set $\{x \in X \mid F(x) \subseteq U\}$ is open. We say that F is *strongly upper semicontinuous* if for any $x \in X$ there exists a neighborhood U of X such that $x' \in U$ implies $F(x') \subseteq F(x)$. Note that every strongly upper semicontinuous multivalued map is upper semicontinuous. We say that F is *acyclic-valued* if $F(x)$ is acyclic for any $x \in X$.

We consider a *simplicial complex* as a finite family $\mathcal{X}$ of finite sets such that any non-empty subset of a set in $\mathcal{X}$ is in $\mathcal{X}$. We refer to the elements of $\mathcal{X}$ as simplices. By the *dimension* of a simplex we mean one less than its cardinality. We denote by $\mathcal{X}_k$ the set of simplices of dimension k. A *vertex* is a simplex of dimension zero. If $\sigma, \tau \in \mathcal{X}$ are simplices and $\tau \subseteq \sigma$ then we say that τ is a *face* of σ and σ is a *coface* of τ. An $(n-1)$-dimensional face of an n-dimensional simplex is called a *facet*. We say that a subset $\mathcal{A} \subseteq \mathcal{X}$ is *open* if all cofaces of any element of $\mathcal{A}$ are also in $\mathcal{A}$. It is easy to see that the family of all open sets of $\mathcal{X}$ is a T_0 topology on $\mathcal{X}$, called *Alexandroff topology*. It corresponds to the face poset of $\mathcal{X}$ via the Alexandroff Theorem [1]. In particular, the *closure* of $\mathcal{A} \subseteq \mathcal{X}$ in the Alexandroff topology consists of all faces of simplices in $\mathcal{A}$. To avoid confusion, in the case of Alexandroff topology we write $\operatorname{Cl}\mathcal{A}$ and $\operatorname{Ex}\mathcal{A}$ for the closure and the exit of $\mathcal{A} \subseteq \mathcal{X}$.

By identifying vertices of an n-dimensional simplex σ with a collection of $n+1$ linearly independent vectors in $\mathbb{R}^d$ with $d > n$ we obtain a *geometric realization* of σ. We denote it by $|\sigma|$. However, whenever the meaning is clear from the context we drop the bars to keep the notation simple. By choosing the identification in such a way that all vectors corresponding to vertices of $\mathcal{X}$ are linearly independent we obtain a *geometric realization* of $\mathcal{X}$ given by

$$|\mathcal{X}| := \bigcup_{\sigma \in \mathcal{X}} |\sigma|.$$

Note that up to a homeomorphism the geometric realization does not depend on a particular choice of the identification. In the sequel we assume that a simplicial complex $\mathcal{X}$ and its geometric realization $X := |\mathcal{X}|$ are fixed. Given a vertex $v \in \mathcal{X}_0$, we denote by $t_v : |\mathcal{X}| \to [0,1]$ the map which assigns to each point $x \in |\mathcal{X}|$ its barycentric coordinate with respect to the vertex v. For a simplex $\sigma \in \mathcal{X}$ the *open cell* of σ is

$$\overset{\circ}{\sigma} := \{ x \in |\sigma| \mid t_v(x) > 0 \text{ for } v \in \sigma \}.$$

For $\mathcal{A} \subseteq \mathcal{X}$ we write

$$\langle \mathcal{A} \rangle := \bigcup_{\sigma \in \mathcal{A}} \overset{\circ}{\sigma}.$$

One easily verifies the following proposition.

Proposition 3.1. *We have the following properties*

(i) *if* $\mathcal{A}$ *is closed in* $\mathcal{X}$ *then* $|\mathcal{A}| = \langle \mathcal{A} \rangle$,
(ii) *if* $\operatorname{Ex}\mathcal{A}$ *is closed in* $\mathcal{X}$ *then* $\langle \mathcal{A} \rangle = |\mathcal{A}| \setminus |\operatorname{Ex}\mathcal{A}|$.

□

3.2. **Combinatorial case.** The concept of a combinatorial vector field was introduced by Forman [14]. There are a few equivalent ways of stating its definition. The definition introduced in Section 2 is among the simplest: a *combinatorial vector field* on a simplicial complex $\mathcal{X}$ is a partition $\mathcal{V}$ of $\mathcal{X}$ into singletons and doubletons such that each doubleton consists of a simplex and one of its facets. The partition induces an injective partial map which sends the element of each singleton to itself and each facet in a doubleton to its coface in the same doubleton. This leads to the following equivalent definition which will be used in the rest of the paper.

Definition 3.2. *(see* [18, Definition 3.1]*) An injective partial self-map $\mathcal{V} : \mathcal{X} \nrightarrow \mathcal{X}$ of a simplicial complex $\mathcal{X}$ is called a* combinatorial vector field*, or also a* discrete vector field *if*

(i) *For every simplex $\sigma \in \operatorname{dom} \mathcal{V}$ either $\mathcal{V}(\sigma) = \sigma$, or σ is a facet of $\mathcal{V}(\sigma)$.*
(ii) $\operatorname{dom} \mathcal{V} \cup \operatorname{im} \mathcal{V} = \mathcal{X}$*,*
(iii) $\operatorname{dom} \mathcal{V} \cap \operatorname{im} \mathcal{V} = \operatorname{Fix} \mathcal{V}$*.*

Note that every combinatorial vector field is a special case of a combinatorial multivector field introduced and studied in [28].

Given a combinatorial vector field $\mathcal{V}$ on $\mathcal{X}$, we define the associated *combinatorial multivalued flow* as the multivalued map $\Pi_{\mathcal{V}} : \mathcal{X} \multimap \mathcal{X}$ given by

$$\Pi_{\mathcal{V}}(\sigma) := \begin{cases} \operatorname{Cl} \sigma & \text{if } \sigma \in \operatorname{Fix} \mathcal{V}\ , \\ \operatorname{Ex} \sigma \setminus \{\mathcal{V}^{-1}(\sigma)\} & \text{if } \sigma \in \operatorname{im} \mathcal{V} \setminus \operatorname{Fix}(\mathcal{V})\ , \\ \{\mathcal{V}(\sigma)\} & \text{if } \sigma \in \operatorname{dom} \mathcal{V} \setminus \operatorname{Fix}(\mathcal{V})\ . \end{cases} \tag{1}$$

For the rest of the paper we assume that $\mathcal{V}$ is a fixed combinatorial vector field on $\mathcal{X}$ and $\Pi_{\mathcal{V}}$ denotes the associated combinatorial multivalued flow.

A *solution* of the flow $\Pi_{\mathcal{V}}$ is a partial function $\varrho : \mathbb{Z} \nrightarrow \mathcal{X}$ such that $\varrho(i+1) \in \Pi_{\mathcal{V}}(\varrho(i))$ whenever $i, i+1 \in \operatorname{dom} \varrho$. The solution ϱ is *full* if $\operatorname{dom} \varrho = \mathbb{Z}$. The *invariant part* of $\mathcal{S} \subseteq \mathcal{X}$, denoted $\operatorname{Inv} \mathcal{S}$, is the collection of those simplices $\sigma \in \mathcal{S}$ for which there exists a full solution $\varrho : \mathbb{Z} \to \mathcal{S}$ such that $\varrho(0) = \sigma$. A set $\mathcal{S} \subseteq \mathcal{X}$ is *invariant* if $\operatorname{Inv} \mathcal{S} = \mathcal{S}$.

Definition 3.3. *(see* [18, Definition 3.4]*) A subset $\mathcal{S} \subseteq \mathcal{X}$, invariant with respect to a combinatorial vector field $\mathcal{V}$, is called an* isolated invariant set *if the* exit set $\operatorname{Ex} \mathcal{S} = \operatorname{Cl} \mathcal{S} \setminus \mathcal{S}$ *is closed and there is no solution $\varrho : \{-1, 0, 1\} \to \mathcal{X}$ such that $\varrho(-1), \varrho(1) \in \mathcal{S}$ and $\varrho(0) \in \operatorname{Ex} \mathcal{S}$. The closure* $\operatorname{Cl} \mathcal{S}$ *is called an* isolating block *for the isolated invariant set $\mathcal{S}$.*

Proposition 3.4. *(see* [18, Proposition 3.7]*) An invariant set $\mathcal{S} \subseteq \mathcal{X}$ is an isolated invariant set if* $\operatorname{Ex}\ \mathcal{S}$ *is closed and for every $\sigma \in \mathcal{X}$ we have $\sigma^- \in \mathcal{S}$ if and only if $\sigma^+ \in \mathcal{S}$, where*

$$\sigma^+ := \begin{cases} \mathcal{V}(\sigma) & \textit{if } \sigma \in \operatorname{dom} \mathcal{V} \\ \sigma & \textit{otherwise} \end{cases} \qquad \textit{and} \qquad \sigma^- := \begin{cases} \sigma & \textit{if } \sigma \in \operatorname{dom} \mathcal{V} \\ \mathcal{V}^{-1}(\sigma) & \textit{otherwise} \end{cases}. \tag{2}$$

An immediate consequence of Proposition 3.4 is the following corollary.

Corollary 3.5. *If $\mathcal{S}$ is an isolated invariant set then for any $\tau \in \mathcal{X}$ and $\sigma, \sigma' \in \mathcal{S}$ we have*

$$\sigma \subseteq \tau \subseteq \sigma' \;\Rightarrow\; \tau \in \mathcal{S}.$$

□

A pair $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2)$ of closed subsets of $\mathcal{X}$ such that $\mathcal{P}_2 \subseteq \mathcal{P}_1$ is an *index pair* for $\mathcal{S}$ if the following three conditions are satisfied

$$\mathcal{P}_1 \cap \Pi_{\mathcal{V}}(\mathcal{P}_2) \subseteq \mathcal{P}_2, \tag{3}$$

$$\Pi_{\mathcal{V}}(\mathcal{P}_1 \setminus \mathcal{P}_2) \subseteq \mathcal{P}_1, \tag{4}$$

$$\mathcal{S} = \operatorname{Inv}(\mathcal{P}_1 \setminus \mathcal{P}_2). \tag{5}$$

By [28, Theorem 7.11] the pair $(\operatorname{cl}\mathcal{S}, \operatorname{Ex}\mathcal{S})$ is an index pair for $\mathcal{S}$ and the (co)homology of the index pair of $\mathcal{S}$ does not depend on the particular choice of index pair but only on $\mathcal{S}$. Hence, by definition, it is the *Conley index* of $\mathcal{S}$. We denote it $\operatorname{Con}(\mathcal{S})$.

3.3. **Classical case.** The study of the Conley index for multivalued maps was initiated in [17] with a restrictive concept of the isolating neighborhood, limiting possible applications. In particular, that theory is not satisfactory for the needs of this paper. These limitations were removed by a new theory developed recently in [5, 4]. We recall the main concepts of the generalized theory below.

Let $F : X \multimap X$ be an upper semicontinuous map with compact, acyclic values. A partial map $\varrho : \mathbb{Z} \nrightarrow X$ is called a *solution for F through $x \in X$* if we have both $\varrho(0) = x$ and $\varrho(n+1) \in F(\varrho(n))$ for all $n, n+1 \in \operatorname{dom}\varrho$. Given $N \subseteq X$ we define its *invariant part* by

$$\operatorname{Inv} N := \{x \in N \mid \exists\, \varrho : \mathbb{Z} \to N \text{ which is a solution for } F \text{ through } x\}.$$

A compact set $N \subseteq X$ is an *isolating neighborhood* for F if $\operatorname{Inv} N \subseteq \operatorname{int} N$. The *$F$-boundary* of a given set $A \subseteq X$ is

$$\operatorname{bd}_F(A) := \operatorname{cl} A \cap \operatorname{cl}(F(A) \setminus A).$$

Definition 3.6. A pair $P = (P_1, P_2)$ of compact sets $P_2 \subseteq P_1 \subseteq N$ is a *weak index pair* for F in N if the following properties are satisfied.

(a) $F(P_i) \cap N \subseteq P_i$ for $i = 1, 2$,
(b) $\operatorname{bd}_F(P_1) \subseteq P_2$,
(c) $\operatorname{Inv} N \subseteq \operatorname{int}(P_1 \setminus P_2)$,
(d) $P_1 \setminus P_2 \subseteq \operatorname{int} N$.

For the weak index pair P we set

$$T(P) := T_N(P) := (P_1 \cup (X \setminus \operatorname{int} N), P_2 \cup (X \setminus \operatorname{int} N)).$$

and define the associated index map I_P as the composition $H^*(F_P) \circ H^*(i_P)^{-1}$, where $F_P : P \multimap T(P)$ is the restriction of F and $i_P : P \multimap T(P)$ is the inclusion map. The module $\operatorname{Con}(S,F) := L(H^*(P), I_P)$, where L is the Leray functor (see [25]) is the *cohomological Conley index* of the isolated invariant set S. The correctness of the definition is the consequence of the following two results.

Theorem 3.7. *(see* [5, Theorem 4.12]*) For every neighborhood W of* $\operatorname{Inv} N$ *there exists a weak index pair P in N such that $P_1 \setminus P_2 \subseteq W$.*

Theorem 3.8. *(see* [5, Theorem 6.4]*) The module $L(H^*(P), I_P)$ is independent of the choice of an isolating neighborhood N for S and of a weak index pair P in N.*

3.4. **Morse decompositions.** In order to formulate the definition of the Morse decomposition of an isolated invariant set we need the concepts of α- and ω-limit sets. We formulate theses definitions independently in the combinatorial and classical settings. Given a full solution $\varrho : \mathbb{Z} \to \mathcal{X}$ of the combinatorial dynamics $\Pi_{\mathcal{V}}$ on $\mathcal{X}$, the α- *and* ω*-limit sets* of ϱ are respectively the sets

$$\alpha(\varrho) := \bigcap_{n\in\mathbb{Z}} \{\varrho(k) \mid k \geq n\}, \qquad \omega(\varrho) := \bigcap_{n\in\mathbb{Z}} \{\varrho(k) \mid k \leq n\}.$$

Note that α- and ω-limit sets of $\Pi_{\mathcal{V}}$ are always non-empty invariant sets, because $\mathcal{X}$ is finite.

Now, given a solution $\varphi : \mathbb{Z} \to X$ of a multivalued upper semicontinuous map $F : X \multimap X$, we define its α- *and* ω*-limit sets* respectively by

$$\alpha(\varphi) := \bigcap_{k\in\mathbb{Z}} \operatorname{cl} \varphi((-\infty,-k]) \qquad \omega(\varphi) := \bigcap_{k\in\mathbb{Z}} \operatorname{cl} \varphi([k,+\infty)).$$

Definition 3.9. Let $\mathcal{S}$ be an isolated invariant set of $\Pi_{\mathcal{V}} : \mathcal{X} \multimap \mathcal{X}$. We say that the family $\mathcal{M} := \{\mathcal{M}_r \,|\, r \in \mathbb{P}\}$ indexed by a poset $\mathbb{P}$ is a *Morse decomposition of* $\mathcal{S}$ if the following conditions are satisfied:

(a) the elements of $\mathcal{M}$ are mutually disjoint isolated invariant subsets of $\mathcal{S}$,
(b) for every full solution φ in X there exist $r, r' \in \mathbb{P}$, $r \leq r'$, such that $\alpha(\varphi) \subseteq \mathcal{M}_{r'}$ and $\omega(\varphi) \subseteq \mathcal{M}_r$,
(c) if for a full solution φ in $\mathcal{X}$ and $r \in \mathbb{P}$ we have $\alpha(\varphi) \cup \omega(\varphi) \subseteq \mathcal{M}_r$, then $\operatorname{im} \varphi \subseteq \mathcal{M}_r$.

By replacing in the above definition the multivalued map $\mathcal{F}$ on $\mathcal{X}$ by an upper semicontinuous map $F : X \to X$ and adjusting the notation accordingly we obtain the definition of the *Morse decomposition* $M := \{M_r \,|\, r \in \mathbb{P}\}$ of an isolated invariant set S of F.

It is not difficult to observe that Definition 3.9 in the combinatorial setting is equivalent to the brief definition of Morse decomposition given in terms of connections in Section 2. Moreover, in the case of combinatorial vector fields Definition 3.9 coincides with the definition presented in [28, Section 9.1].

4. From combinatorial to classical dynamics

In this section, given a combinatorial vector field $\mathcal{V}$ on a simplicial complex $\mathcal{X}$, we recall from [18] the construction of a multivalued self-map $F = F_{\mathcal{V}} : X \multimap X$ on the geometric realization $X := |\mathcal{X}|$ of $\mathcal{X}$. This map will be used to establish the correspondence of Conley indices, Morse decompositions and Conley-Morse graphs between the combinatorial and classical multivalued dynamics.

4.1. **Cellular decomposition.** We begin by recalling a special cellular complex representation of $X = |\mathcal{X}|$ used in the construction of the multivalued map F. For this we need some terminology. Let d denote the maximal dimension of the simplices in $\mathcal{X}$. Fix a $\lambda \in \mathbb{R}$ such that $0 \leq \lambda < \frac{1}{d+1}$ and a point $x \in X$. The λ-*signature* of x is the function

$$\operatorname{sign}^{\lambda} x : \mathcal{X}_0 \ni v \mapsto \operatorname{sgn}(t_v(x) - \lambda) \in \{-1, 0, 1\}, \tag{6}$$

where $\operatorname{sgn} : \mathbb{R} \to \{-1, 0, 1\}$ is the standard sign function. Then a simplex $\sigma \in \mathcal{X}$ is a λ-*characteristic simplex* of x if both $\operatorname{sign}^{\lambda} x|_{\sigma} \geq 0$ and $(\operatorname{sign}^{\lambda} x)^{-1}(\{1\}) \subseteq \sigma$ are satisfied. We denote the family of λ-characteristic simplices of x by

$$\mathcal{X}^{\lambda}(x) := \{\, \sigma \in \mathcal{X} \;\mid\; (\operatorname{sign}^{\lambda} x)^{-1}(\{1\}) \subseteq \sigma \text{ and } \operatorname{sign}^{\lambda} x(v) \geq 0 \text{ for all } v \in \sigma \,\}\,.$$

For any $\lambda \geq 0$, the set $(\operatorname{sign}^{\lambda} x)^{-1}(\{1\})$ is a simplex. We call it the *minimal characteristic simplex* of x and we denote it by $\sigma^{\lambda}_{min}(x)$. Note that

$$\sigma = \sigma^{0}_{min}(x) \;\Leftrightarrow\; x \in \overset{\circ}{\sigma}. \tag{7}$$

If $\lambda > 0$, then the set $(\operatorname{sign}^{\lambda} x)^{-1}(\{0, 1\})$ is also a simplex. We call it the *maximal characteristic simplex* of x and we denote it by $\sigma^{\lambda}_{max}(x)$.

Lemma 4.1. *(see* [18, Lemma 4.2]*) If $0 \leq \varepsilon < \lambda < \frac{1}{1+d}$, then $\sigma^{\lambda}_{max}(x) \subseteq \sigma^{\varepsilon}_{min}(x)$ for any $x \in X = |\mathcal{X}|$.*

Lemma 4.2. *(see* [18, Lemma 4.3]*) For all $x \in X = |\mathcal{X}|$ we have $\mathcal{X}^{\lambda}(x) \neq \varnothing$. Moreover, there exists a neighborhood U of the point x such that $\mathcal{X}^{\lambda}(y) \subseteq \mathcal{X}^{\lambda}(x)$ for all $y \in U$.*

Given a $\sigma \in \mathcal{X}$ by a λ-*cell generated by* σ we mean

$$\langle \sigma \rangle_{\lambda} := \left\{ x \in X \;\mid\; \mathcal{X}^{\lambda}(x) = \{\sigma\} \right\}.$$

We recall (cf. [18, Formulas (12) and (13)]) the following characterizations of $\langle \sigma \rangle_{\lambda}$ and its closure in terms of barycentric coordinates:

$$\langle \sigma \rangle_{\lambda} = \{ x \in X \mid t_v(x) > \lambda \text{ for } v \in \sigma \text{ and } t_v(x) < \lambda \text{ for } v \notin \sigma \}, \tag{8}$$

$$\operatorname{cl} \langle \sigma \rangle_{\lambda} = \{ x \in X \mid t_v(x) \geq \lambda \text{ for } v \in \sigma \text{ and } t_v(x) \leq \lambda \text{ for } v \notin \sigma \}. \tag{9}$$

Then the following proposition follows easily from (8).

Proposition 4.3. *For λ satisfying $0 < \lambda < \frac{1}{d+1}$ the λ-cells are open in $|\mathcal{X}|$ and mutually disjoint.* □

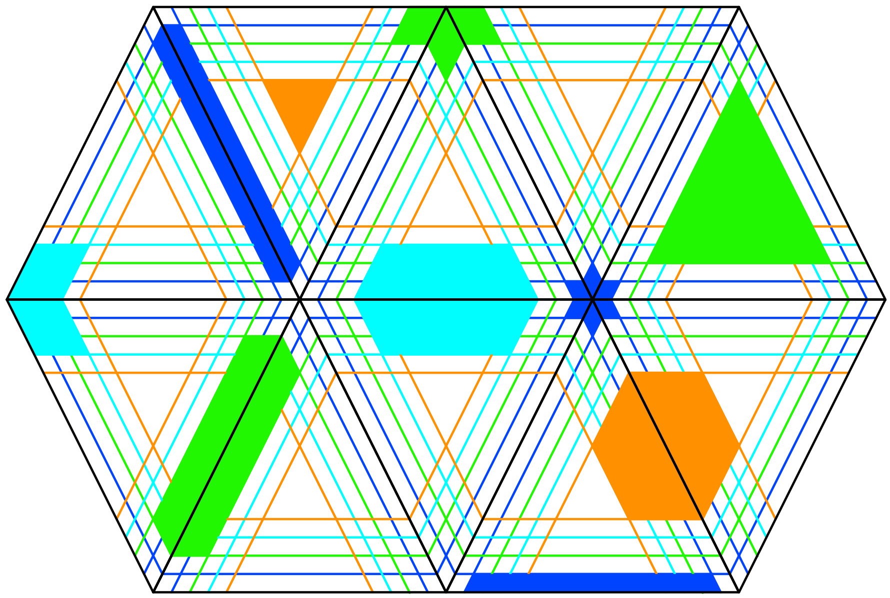


FIGURE 8. Sample cell decomposition boundaries for the simplicial complex $\mathcal{X}$ from Figure 3. The colored lines indicate the boundaries of ε-cells (orange), γ-cells (cyan), δ-cells (green), and δ'-cells (blue). Throughout the paper, we assume that $0 < \delta' < \delta < \gamma < \varepsilon$. The figure also contains ten sample cells: Two orange ε-cells which are associated with a 2-simplex (upper left) and a 1-simplex (lower right); two cyan γ-cells which correspond to a 1-simplex (middle) and a 0-simplex (left); three green δ-cells for a 2-simplex (upper right), a 1-simplex (lower left), and a 0-simplex (top middle); as well as three blue δ'-cells for two 1-simplices (upper left and bottom right) and a 0-simplex (right middle). All of these cells are open subsets of $|\mathcal{X}|$.

Another characterization of $\operatorname{cl}\langle\sigma\rangle_\lambda$ is given by the following corollary.

Corollary 4.4. *(see* [18, Corollary 4.6]*) The following statements are equivalent:*

(i) $\sigma \in \mathcal{X}^\lambda(x)$,
(ii) $\sigma^\lambda_{min}(x) \subseteq \sigma \subseteq \sigma^\lambda_{max}(x)$,
(iii) $x \in \operatorname{cl}\langle\sigma\rangle_\lambda$.

Note that for a simplicial complex $\mathcal{X}$ we have the following easy to verify formula:

$$|\mathcal{X}| = \bigcup_{\sigma\in\mathcal{X}} \operatorname{cl}\langle\sigma\rangle_\delta. \tag{10}$$

The cells $\langle\sigma\rangle_\lambda$ for various values of λ are visualized in Figure 8. They are the building blocks for the multivalued map F.

4.2. **The maps F_σ and the map F.** We now recall from [18] the construction of the strongly upper semi-continuous map F associated with a combinatorial vector field. For this, we fix two constants

$$0 < \gamma < \varepsilon < \frac{1}{d+1} \tag{11}$$

and for any $\sigma \in \mathcal{X}$ we set

$$\begin{aligned} A_\sigma &:= \left\{ x \in \sigma^+ \mid t_v(x) \geq \gamma \text{ for all } v \in \sigma^- \right\} \cup \sigma^- , \\ B_\sigma &:= \left\{ x \in \sigma^+ \mid \text{there exists a } v \in \sigma^- \text{ with } t_v(x) \leq \gamma \right\} , \\ C_\sigma &:= A_\sigma \cap B_\sigma . \end{aligned} \tag{12}$$

Then the following lemma is an immediate consequence of [18, Lemma 4.8].

Lemma 4.5. *For any simplex $\sigma \in \mathcal{X} \setminus \operatorname{Fix}\mathcal{V}$ the sets A_σ, B_σ and C_σ are contractible.*

For every simplex $\sigma \in \mathcal{X}$ we define a multivalued map $F_\sigma : X \multimap X$ by

$$F_\sigma(x) := \begin{cases} \varnothing & \text{if } \sigma \notin \mathcal{X}^\varepsilon(x) , \\ A_\sigma & \text{if } \sigma \in \mathcal{X}^\varepsilon(x) , \ \sigma \neq \sigma^\varepsilon_{max}(x)^+ , \text{ and } \sigma \neq \sigma^\varepsilon_{max}(x)^- , \\ B_\sigma & \text{if } \sigma = \sigma^\varepsilon_{max}(x)^+ \neq \sigma^\varepsilon_{max}(x)^- , \\ C_\sigma & \text{if } \sigma = \sigma^\varepsilon_{max}(x)^- \neq \sigma^\varepsilon_{max}(x)^+ , \\ \sigma & \text{if } \sigma = \sigma^\varepsilon_{max}(x)^- = \sigma^\varepsilon_{max}(x)^+ , \end{cases} \tag{13}$$

and the multivalued map $F : X \multimap X$ by

$$F(x) := \bigcup_{\sigma \in \mathcal{X}} F_\sigma(x) \qquad \text{for all} \qquad x \in X = |\mathcal{X}| . \tag{14}$$

Figure 7 shows the graph $\{(x, y) \in X \times X \mid y \in F(x)\}$ of the so-constructed map F for the vector field in Figure 1.

One of main results proved in [18] is the following theorem.

Theorem 4.6. *(see* [18, Theorem 4.12]*) The map F is strongly upper semicontinuous and for every $x \in X$ the set $F(x)$ is non-empty and contractible.* □

5. The correspondence between combinatorial and classical dynamics

In this section we present the constructions and theorems establishing the correspondence between the multivalued dynamics of a combinatorial vector field $\mathcal{V}$ on the simplicial complex $\mathcal{X}$ and the associated multivalued dynamics of the multivalued map $F = F_{\mathcal{V}}$ constructed in Section 4. The theorems presented in this section provide the proof of Theorem 2.1.

Throughout the section we assume that d is the maximal dimension of the simplices in $\mathcal{X}$.

5.1. **Correspondence of isolated invariant sets.** In order to establish the correspondence on the level of isolated invariant sets we fix a constant δ satisfying

$$0 < \delta < \gamma < \varepsilon < \frac{1}{d+1} \;,$$

where γ and ε are the constants chosen in Section 4.2 (see (11)). For $\mathcal{A} \subseteq \mathcal{X}$ and any constant β satisfying $0 < \beta < \frac{1}{d+1}$ we further set

$$N_\beta(\mathcal{A}) := \bigcup_{\sigma \in \mathcal{A}} \operatorname{cl} \langle \sigma \rangle_\beta. \tag{15}$$

Let $\mathcal{S} \subseteq \mathcal{X}$ be an isolated invariant set for the combinatorial vector field $\mathcal{V}$ in the sense of Definition 3.3. The following theorem associates with $\mathcal{S}$ an isolating block for F, and it was proved in [18].

Theorem 5.1. *(see* [18, Theorem 5.7]*) The set*

$$N := N_\delta := N_\delta(\mathcal{S}) \tag{16}$$

is an isolating block for F. In particular, it is an isolating neighborhood for F. □

A sample of an isolating block for the map F given by (14) which corresponds to the combinatorial isolated invariant set in Figure 4 is presented in Figure 9.

Theorem 5.1 lets us associate with $\mathcal{S}$ an isolated invariant set

$$S(\mathcal{S}) := \operatorname{Inv} N_\delta$$

given as the invariant part of N_δ with respect to F.

5.2. **The Conley index of $S(\mathcal{S})$.** In order to compare the Conley indices of $\mathcal{S}$ and $S(\mathcal{S})$ we need to construct a weak index pair for F in N_δ. To define such a weak index pair we fix another constant δ' such that

$$0 < \delta' < \delta < \gamma < \varepsilon < \frac{1}{d+1} \;, \tag{17}$$

and set

$$P_1 := N_\delta \cap N_{\delta'} \qquad \text{and} \qquad P_2 := N_{\delta'} \cap \operatorname{bd} N_\delta \;. \tag{18}$$

Clearly $P_2 \subseteq P_1 \subseteq N := N_\delta$ are compact sets. We have the following theorem.

Theorem 5.2. *The pair $P = (P_1, P_2)$ defined by (18) is a weak index pair for F and the isolating neighborhood $N = N_\delta$.*

The proof of Theorem 5.2 will be presented in Section 6. A weak index pair for the isolating block given in Figure 9 is presented in Figure 10. As recalled in Section 3.3 the Conley index of $S(\mathcal{S})$ with respect to F is

$$\operatorname{Con}(S(\mathcal{S}), F) := L(H^*(P), I_P),$$

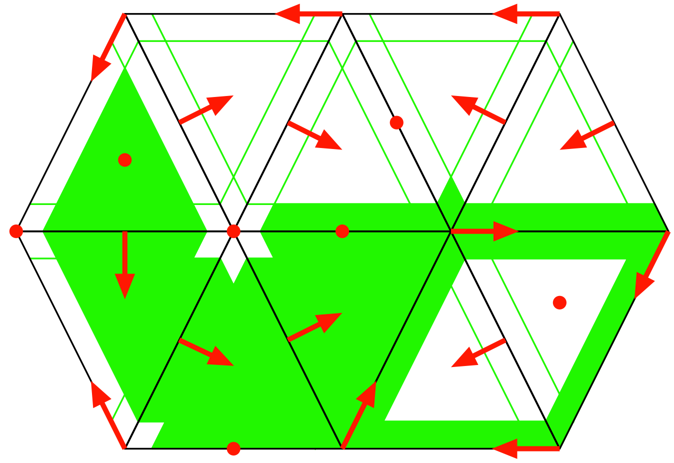


FIGURE 9. Isolating block N_δ for the isolated invariant set $\mathcal{S}$ shown in Figure 4. Notice that the block is the union of closed δ-cells. For reference, we also show the δ-cell boundaries outside N_δ, but these are not part of the isolating block. The block is homeomorphic to a closed annulus.

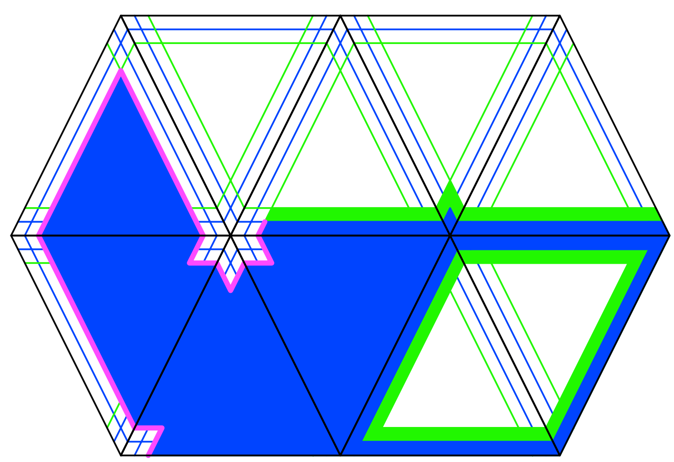


FIGURE 10. The weak index pair $P = (P_1, P_2)$ associated with the isolating block N_δ from Figure 9. The set P_1 is shown in dark blue, and the part of its boundary which comprises P_2 is indicated in magenta. Notice that the parts of the δ-cells shown in green are cut from the isolating block N_δ when passing to P_1.

where L is the Leray reduction of the relative cohomology graded module $H^*(P) = H^*(P_1, P_2)$ of P, and I_P is the index map on $H^*(P)$. In Section 7 we prove the following theorem.

Theorem 5.3. *We have*

$$\operatorname{Con}(S(\mathcal{S}), F) \cong (H^*(P), \operatorname{id}_{H^*(P)}),$$

where $\operatorname{id}_{H^*(P)}$ *denotes the identity map. In other words, as in the case of flows, the Conley index of* $S(\mathcal{S})$ *with respect to* F *can be simply defined as the relative cohomology* $H^*(P)$*.*

5.3. **Correspondence of Conley indices.** As recalled in Section 3.2 the Conley index of $\mathcal{S}$ with respect to $\Pi_{\mathcal{V}}$ is

$$\operatorname{Con}(\mathcal{S}) := H^*(\operatorname{Cl}\mathcal{S}, \operatorname{Ex}\mathcal{S}) .$$

In Section 8 we prove the following theorem.

Theorem 5.4. *We have*

$$H^*(P_1, P_2) \cong H^*(\operatorname{Cl}\mathcal{S}, \operatorname{Ex}\mathcal{S}) .$$

As a consequence,

$$\operatorname{Con}(S(\mathcal{S})) \cong \operatorname{Con}(\mathcal{S}) .$$

Theorem 5.4 extends the correspondence between the isolated invariant sets $\mathcal{S}$ and $S(\mathcal{S})$ to the respective Conley indices.

5.4. **Correspondence of Morse decompositions.** Given $\mathcal{M} = \{\mathcal{M}_r \,|\, r \in \mathbb{P}\}$, a Morse decomposition of $\mathcal{X}$ with respect to the combinatorial flow $\Pi_{\mathcal{V}}$, we define the sets

$$M_r := N_\varepsilon^r \cap \langle \mathcal{M}_r \rangle ,$$

where $N_\varepsilon^r := N_\varepsilon(\mathcal{M}_r)$ is given by (15), that is, we have

$$N_\varepsilon^r = \bigcup_{\sigma \in \mathcal{M}_r} \operatorname{cl} \langle \sigma \rangle_\varepsilon .$$

In Section 9 we prove the following theorem, which establishes the correspondence between Morse decompositions of $\mathcal{V}$ and of F.

Theorem 5.5. *The collection* $M := \{M_r \,|\, r \in \mathbb{P}\}$ *is a Morse decomposition of* X *with respect to* F*. Moreover, for each* $r \in \mathbb{P}$ *we have*

$$\operatorname{Con}(\mathcal{M}_r) = C(M_r) ,$$

and the Conley-Morse graphs for the Morse decompositions $\mathcal{M}$ *and* M *coincide.*

The reader can immediately see that Theorem 2.1, the main result of the paper, is now an easy consequence of Theorems 5.4 and 5.5.

6. Proof of Theorem 5.2

In this section we prove Theorem 5.2. The proof is split into six auxiliary lemmas and the verification that the pair P defined by (18) satisfies the conditions (a) through (d) of Definition 3.6.

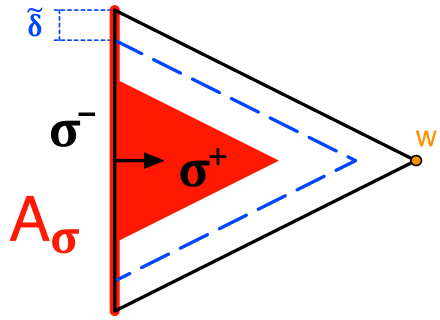


FIGURE 11. The sets σ^+, A_σ and vertex w in the proof of Lemma 6.1.

6.1. **Auxiliary lemmas.**

Lemma 6.1. *Consider a $\tilde{\delta}$ satisfying $0 < \tilde{\delta} < \gamma$ and assume $A_\sigma \cap \operatorname{cl}\langle\tau\rangle_{\tilde{\delta}} \neq \varnothing$ for all simplices $\tau, \sigma \in \mathcal{X}$. Then either τ is a face of σ^- or $\tau = \sigma^+$.*

Proof: Choose an $x \in A_\sigma \cap \operatorname{cl}\langle\tau\rangle_{\tilde{\delta}}$. Accordingly to (12) we have

$$A_\sigma = \left\{\tilde{x} \in \sigma^+ \;\middle|\; t_v(\tilde{x}) \geq \gamma \text{ for all } v \in \sigma^-\right\} \cup \sigma^- .$$

If τ is a face of σ^-, we are done. Suppose that this does not hold. Then τ has to contain the vertex w of σ^+ complementing σ^- as shown in Figure 11 and $x \notin \sigma^-$. This implies that $t_v(x) \geq \gamma > \tilde{\delta}$ for all $v \in \sigma^-$. Since

$$\operatorname{cl}\langle\tau\rangle_{\tilde{\delta}} = \left\{\tilde{x} \in X \;\middle|\; t_v(\tilde{x}) \geq \tilde{\delta} \text{ for all } v \in \tau \text{ and } t_v(x) \leq \tilde{\delta} \text{ for all } v \notin \tau\right\},$$

this implies that all vertices of σ^- have to be in τ. Hence $\tau = \sigma^+$ and the claim is proved. ☐

Lemma 6.2. *Suppose that $x \in \operatorname{cl}\langle\tau\rangle_\delta$ for some $\tau \in \mathcal{S}$ and that $\sigma := \sigma^\varepsilon_{max}(x) \notin \mathcal{S}$. Then for every $\tilde{\delta}$ satisfying $0 < \tilde{\delta} < \gamma$ we have*

$$F(x) \cap N_{\tilde{\delta}} = \varnothing .$$

Proof: Suppose that $F(x) \cap N_{\tilde{\delta}} \neq \varnothing$. Hence, there exists a simplex $\hat{\tau} \in \mathcal{S}$ and a point $y \in F(x) \cap \operatorname{cl}\langle\hat{\tau}\rangle_{\tilde{\delta}}$. Then Lemma 4.1 and Corollary 4.4 imply that

$$\sigma = \sigma^\varepsilon_{max}(x) \subseteq \sigma^\delta_{min}(x) \subseteq \tau.$$

In other words, σ is a face of τ. Since $\mathcal{S}$ is an isolated invariant set, $\tau \in \mathcal{S}$ implies the inclusions $\tau^\pm \in \mathcal{S}$.

Since $y \in F(x)$, we have $y \in F_\varrho(x)$ for some simplex $\varrho \in \mathcal{X}^\varepsilon(x)$. There are four possible cases to consider.

First, assume that $\varrho \neq \sigma^+$ and $\varrho \neq \sigma^-$. Then $F_\varrho(x) = A_\varrho$ and $y \in A_\varrho \cap \operatorname{cl}\langle\hat{\tau}\rangle_{\tilde{\delta}}$. According to Lemma 6.1, $\hat{\tau}$ has to be a face of ϱ^- or $\hat{\tau} = \varrho^+$. Recall that we assumed $\sigma \notin \mathcal{S}$. Since $\varrho \in \mathcal{X}^\varepsilon(x)$, we get

$$\varrho \subseteq \sigma \subseteq \tau.$$

Since $\sigma \notin \mathcal{S}$ and $\tau \in \mathcal{S}$, Corollary 3.5 implies that $\varrho \notin \mathcal{S}$ and Proposition 3.4 implies that $\varrho^{\pm} \notin \mathcal{S}$. Since we assumed $\hat{\tau} \in \mathcal{S}$, Lemma 6.1 shows that we cannot have $\hat{\tau} = \varrho^{+}$. Thus, $\hat{\tau}$ has to be a face of ϱ^{-}. The inclusions

$$\hat{\tau} \subseteq \varrho^{-} \subseteq \varrho \subseteq \sigma \subseteq \tau,$$

with $\hat{\tau}, \tau \in \mathcal{S}$ and $\sigma \notin \mathcal{S}$ contradict the closedness of $\operatorname{Ex} \mathcal{S}$ in Definition 3.3.

Now assume that $\varrho = \sigma^{+} \neq \sigma^{-}$. Since $\varrho \in \mathcal{X}^{\varepsilon}(x)$, ϱ has to be a face of σ, so we get $\sigma^{+} = \sigma \notin \mathcal{S}$ as well as $\sigma^{-} \notin \mathcal{S}$. Moreover, in this case,

$$y \in F_{\varrho}(x) = B_{\varrho} = B_{\sigma^{+}} \subseteq \sigma^{+} = \sigma.$$

Hence, $\sigma^{0}_{min}(y) \subseteq \sigma$. Given that $y \in \operatorname{cl} \langle \hat{\tau} \rangle_{\tilde{\delta}}$, we obtain

$$\hat{\tau} \subseteq \sigma^{\tilde{\delta}}_{max}(y) \subseteq \sigma^{0}_{min}(y) \subseteq \sigma \subseteq \tau \; .$$

This contradicts Corollary 3.5, because $\hat{\tau}, \tau \in \mathcal{S}$ and $\sigma \notin \mathcal{S}$.

Next assume that $\varrho = \sigma^{-} \neq \sigma^{+}$. Then $y \in F_{\varrho}(x) = C_{\varrho} \subseteq A_{\varrho}$. Hence, the inclusion $y \in A_{\varrho} \cap \operatorname{cl} \langle \hat{\tau} \rangle_{\tilde{\delta}}$ holds, and we get a contradiction as in the first case.

The last possible case is $\varrho = \sigma^{-} = \sigma \in \operatorname{Fix} \mathcal{V}$. Then $y \in \varrho$. Since,

$$\hat{\tau} \subseteq \sigma^{0}_{min}(y) \subseteq \sigma,$$

we get the inclusions

$$\hat{\tau} \subseteq \varrho \subseteq \sigma \subseteq \tau,$$

and a contradiction is reached as before. □

Lemma 6.3. *(see* [18, Lemma 4.10]*) The image $F(x)$ can be expressed alternatively as*

$$F(x) \; = \; F_{\sigma^{\varepsilon}_{max}(x)} \cup \bigcup_{\tau \in \mathcal{T}^{\varepsilon}(x)} F_{\tau}(x) \; , \tag{19}$$

where

$$\mathcal{T}^{\varepsilon}(x) := \left\{ \tau \in \mathcal{X}^{\varepsilon}(x) \setminus \{\sigma^{\varepsilon}_{max}(x)\} \;\middle|\; \tau = \tau^{-} \;\; \text{and} \;\; \tau^{+} \notin \operatorname{Cl} \sigma^{\varepsilon}_{max}(x) \right\} \; . \tag{20}$$

Furthermore, every $\tau \in \mathcal{T}^{\varepsilon}(x)$ automatically satisfies $\tau \in \operatorname{dom} \mathcal{V} \setminus \operatorname{Fix} \mathcal{V}$.

Lemma 6.4. *We have*

$$F(N_{\delta}) \cap N_{\delta} \subseteq \langle \mathcal{S} \rangle.$$

Consequently, $\operatorname{Inv} N_{\delta} \subseteq \langle \mathcal{S} \rangle$*.*

Proof: We begin by showing that

$$F(N_{\delta}) \cap N_{\delta} \subseteq |\mathcal{S}|. \tag{21}$$

Let $y \in F(N_{\delta}) \cap N_{\delta}$ with $y \in F(x)$ for some $x \in N_{\delta}$. Then there exists a $\tau \in \mathcal{S}$ and a $\hat{\tau} \in \mathcal{S}$ such that $x \in \operatorname{cl} \langle \tau \rangle_{\delta}$ and $y \in \operatorname{cl} \langle \hat{\tau} \rangle_{\delta}$. Let $\sigma_0 := \sigma^{0}_{min}(x)$ and $\sigma := \sigma^{\varepsilon}_{max}(x)$. By Lemma 4.1 and by Corollary 4.4 we have

$$\tau \subseteq \sigma^{\delta}_{max}(x) \subseteq \sigma^{0}_{min}(x) = \sigma_0,$$
$$\sigma = \sigma^{\varepsilon}_{max}(x) \subseteq \sigma^{0}_{min}(x) = \sigma_0,$$

and

$$\sigma^{\varepsilon}_{max}(x) \subseteq \sigma^{\delta}_{min}(x) \subseteq \tau \ .$$

This implies that $\sigma \subseteq \tau \subseteq \sigma_0$. We have $F(x) \cap N_\delta \neq \varnothing$, because $y \in F(x) \cap N_\delta$. From Lemma 6.2 we get $\sigma \in \mathcal{S}$. Hence $\sigma^{\pm} \in \mathcal{S}$. Since $y \in F(x)$, we now consider the following cases resulting from Lemma 6.3.

Assume first that $y \in F_\varrho(x)$ for a simplex $\varrho \in \mathcal{T}^{\varepsilon}(x)$. Then $\varrho \neq \sigma$, $\varrho = \varrho^-$, and $\varrho^+ \notin \operatorname{Cl}\sigma$. We will show that $\varrho \in \mathcal{S}$. To see this assume the contrary. Then one obtains $\varrho \neq \sigma^-$, $\varrho \neq \sigma^+$, so $F_\varrho(x) = A_\varrho$ and $y \in A_\varrho \cap \operatorname{cl}\langle\hat{\tau}\rangle_\delta$ with $\hat{\tau} \in \mathcal{S}$. Lemma 6.1 implies that $\hat{\tau} = \varrho^+$ or $\hat{\tau}$ is a face of ϱ. If $\hat{\tau} = \varrho^+$, then $\varrho^+ \in \mathcal{S}$, which implies that $\varrho \in \mathcal{S}$, a contradiction. If $\hat{\tau}$ is a face of ϱ, we get the inclusions

$$\hat{\tau} \subseteq \varrho \subseteq \sigma^{\varepsilon}_{max}(x) = \sigma \in \mathcal{S}$$

with $\hat{\tau}, \sigma \in \mathcal{S}$ and $\varrho \notin \mathcal{S}$. As in the proof of Lemma 6.2, this contradicts that $\mathcal{S}$ is an isolated invariant set. Thus, $\varrho \in \mathcal{S}$. Then $\varrho^+ \in \mathcal{S}$ and $y \in F_\varrho(x) \subseteq \varrho^+ \subseteq |\mathcal{S}|$.

Now assume that $y \in F_\sigma(x) = F_{\sigma^{\varepsilon}_{max}(x)}(x)$. Then $F_\sigma(x)$ can either be B_σ, or C_σ, or σ. All these sets are contained in $\sigma^+ \in \mathcal{S}$, hence also in this case $y \in |\mathcal{S}|$ and (21) is proved. By Proposition 3.1(ii), in order to conclude the proof it suffices to show that

$$F(N_\delta) \cap N_\delta \cap |\operatorname{Ex}\mathcal{S}| = \varnothing.$$

Assume the contrary. Then there exists a point $z \in F(N_\delta) \cap N_\delta$ with $z \in \mathring{\tilde{\sigma}}$ for some $\tilde{\sigma} \in \operatorname{Ex}\mathcal{S}$. Since $z \in N_\delta$, there exists a $\tilde{\tau} \in \mathcal{S}$ such that $z \in \operatorname{cl}\langle\tilde{\tau}\rangle_\delta$. We get the inclusions

$$\tilde{\tau} \subseteq \sigma^{\delta}_{max}(z) \subseteq \sigma^{0}_{min}(z) = \tilde{\sigma},$$

with $\tilde{\tau} \in \mathcal{S}$. Since $\tilde{\sigma} \in \operatorname{Ex}\mathcal{S}$, this contradicts the closedness of $\operatorname{Ex}\mathcal{S}$ and completes the proof. □

Lemma 6.5. *Assume $\mathcal{S}$ is an isolated invariant set for $\Pi_{\mathcal{V}}$ in the sense of Definition 3.3, and consider the set $N = N_\delta \subseteq X = |\mathcal{X}|$ given by* (16)*. Then $x \in \operatorname{bd} N_\delta$ if and only if*

$$\mathcal{X}^{\delta}(x) \cap \mathcal{S} \neq \varnothing \qquad \text{and} \qquad \mathcal{X}^{\delta}(x) \setminus \mathcal{S} \neq \varnothing \ . \tag{22}$$

Proof: The fact that $x \in \operatorname{bd} N_\delta$ implies (22) is shown in [18, Lemma 5.5]. The reverse implication is an easy consequence of Proposition 4.3. □

Lemma 6.6. *For any $x \in N_{\delta'} \cap \operatorname{bd} N_\delta$, we have $\sigma^{\varepsilon}_{max}(x) \notin \mathcal{S}$.*

Proof: Since $x \in \operatorname{bd} N_\delta$, Lemma 6.5 implies that there exists a $\sigma_1 \in \mathcal{X}^{\delta}(x) \setminus \mathcal{S}$. Moreover, let $\sigma_0 = \sigma^{0}_{min}(x)$ and $\sigma = \sigma^{\varepsilon}_{max}(x)$. By Corollary 4.4 we then obtain the inclusions $\sigma = \sigma^{\varepsilon}_{max}(x) \subseteq \sigma^{\delta}_{min}(x) \subseteq \sigma_1 \subseteq \sigma^{\delta}_{max}(x)$. In addition, we further have $x \in N_{\delta'}$. Hence, there exists a simplex $\tau' \in \mathcal{S}$ with $x \in \operatorname{cl}\langle\tau'\rangle_{\delta'}$. This implies that $\sigma^{\delta}_{max}(x) \subseteq \sigma^{\delta'}_{min}(x) \subseteq \tau' \subseteq \sigma^{\delta'}_{min}(x)$. Together, this gives $\sigma \subseteq \sigma_1 \subseteq \tau'$, where both $\sigma_1 \notin \mathcal{S}$ and $\tau' \in \mathcal{S}$ hold. Therefore, Corollary 3.5 implies that $\sigma \notin \mathcal{S}$, which completes the proof. □

6.2. **Property (a).**

Lemma 6.7. *The sets* P_1, P_2 *given by* (18) *and* $N = N_\delta$ *given by* (16) *satisfy property (a) in Definition 3.6.*

Proof: Let $i = 1$. We know from Lemma 6.4 that

$$F(P_1) \cap N_\delta \subseteq F(N_\delta) \cap N_\delta \subseteq \langle \mathcal{S} \rangle, \tag{23}$$

and obviously $F(P_1) \cap N_\delta \subseteq N_\delta$. We argue by contradiction. Suppose that there exists an $x \in F(P_1) \cap N_\delta$ with $x \notin N_{\delta'}$.

Since $x \in N_\delta$, we have $x \in \operatorname{cl} \langle \tau \rangle_\delta$ for some $\tau \in \mathcal{S}$. Moreover, since $x \notin N_{\delta'}$, we have $x \in \operatorname{cl} \langle \tau' \rangle_{\delta'}$ for some $\tau' \notin \mathcal{S}$. Now let $\sigma_0 = \sigma^0_{min}(x)$. Then one obtains the inclusions $\tau \subseteq \sigma^\delta_{max}(x) \subseteq \sigma^{\delta'}_{min}(x) \subseteq \tau' \subseteq \sigma^{\delta'}_{max}(x) \subseteq \sigma_0$, where $\tau \in \mathcal{S}$ and $\tau' \notin \mathcal{S}$. Therefore, we get from Corollary 3.5 that $\sigma_0 \notin \mathcal{S}$. This shows that $x \notin \langle \mathcal{S} \rangle$, because by (7) we also have $x \in \overset{\circ}{\sigma}_0$. Since $x \in F(P_1) \cap N_\delta$, this contradicts (23) and proves the claim for $i = 1$.

Consider now the case $i = 2$, and let $x \in P_2 = N_{\delta'} \cap \operatorname{bd} N_\delta$. Then Lemma 6.6 implies $\sigma^\varepsilon_{max}(x) \notin \mathcal{S}$. Since $x \in N_\delta$, Lemma 6.2 shows that $F(x) \cap N_\delta = \varnothing \subseteq P_2$, and the conclusion follows. ☐

6.3. **Property (d).**

Lemma 6.8. *We have*

$$P_1 \setminus P_2 = N_{\delta'} \cap \operatorname{int} N_\delta \, .$$

As a consequence, property (d) in Definition 3.6 is satisfied.

Proof: Let $x \in P_1 \setminus P_2$ be arbitrary. Since $x \in P_1$, we have $x \in N_\delta$ and $x \in N_{\delta'}$. Since $x \notin P_2$, either $x \notin \operatorname{bd} N_\delta$ or $x \notin N_{\delta'}$. The second case is excluded, hence we have $x \notin \operatorname{bd} N_\delta$. It follows that $x \in \operatorname{int} N_\delta$, and therefore also that $x \in N_{\delta'} \cap \operatorname{int} N_\delta$.

Conversely, let $x \in N_{\delta'} \cap \operatorname{int} N_\delta \subseteq P_1$. Then both $x \notin \operatorname{bd} N_\delta$ and $x \notin P_2$ are satisfied. It follows that $x \in P_1 \setminus P_2$.

Now, property (d) trivially follows from the inclusion $N_{\delta'} \cap \operatorname{int} N_\delta \subseteq \operatorname{int} N_\delta$. ☐

6.4. **Property (c).**

Lemma 6.9. *We have*

$$\operatorname{Inv} N_\delta \subseteq \operatorname{int}(P_1 \setminus P_2).$$

In other words, property (c) in Definition 3.6 is satisfied.

Proof: According to Theorem 5.1 we have

$$\operatorname{Inv} N_\delta \subseteq \operatorname{int} N_\delta \, . \tag{24}$$

We will show in the following that also

$$\operatorname{Inv} N_\delta \subseteq \operatorname{int} N_{\delta'} \, . \tag{25}$$

We argue by contradiction. Suppose that $\operatorname{Inv} N_\delta \setminus \operatorname{int} N_{\delta'} \neq \varnothing$ and let $x \in \operatorname{Inv} N_\delta$ be such that $x \notin \operatorname{int} N_{\delta'}$. If $x \in \operatorname{bd} N_{\delta'}$, then Lemma 6.5 shows that there exists a simplex $\tau' \notin \mathcal{S}$ with $x \in \operatorname{cl}\langle\tau'\rangle_{\delta'}$. It is clear that if $x \notin N_{\delta'}$ such a τ' also exists. According to Lemma 6.4, we have $x \in \langle\mathcal{S}\rangle$, that is, there exists a simplex $\sigma \in \mathcal{S}$ with $x \in \mathring{\sigma}$. By (7) we have $\sigma^0_{min}(x) = \sigma$. As before, we obtain inclusions

$$\sigma^{\varepsilon}_{max}(x) \subseteq \sigma^{\delta'}_{min}(x) \subseteq \tau' \subseteq \sigma^{\delta'}_{max}(x) \subseteq \sigma^0_{min}(x) = \sigma,$$

with $\tau' \notin \mathcal{S}$ and $\sigma \in \mathcal{S}$, and Corollary 3.5 implies that $\sigma^{\varepsilon}_{max}(x) \notin \mathcal{S}$. Due to our assumption, we have $x \in \operatorname{Inv} N_\delta \subseteq N_\delta$, and Lemma 6.2 gives $F(x) \cap N_\delta = \varnothing$. Hence $x \notin \operatorname{Inv} N_\delta$, which is a contradiction and thus proves (25).

The inclusions (24) and (25) give

$$\operatorname{Inv} N_\delta \subseteq \operatorname{int} N_{\delta'} \cap \operatorname{int} N_\delta \subseteq \operatorname{int}(N_{\delta'} \cap \operatorname{int} N_\delta) = \operatorname{int}(P_1 \setminus P_2),$$

which completes the proof. □

For the next result we need the following two simple observations. If A and B are closed subsets of X, then

$$\operatorname{bd}(A \cap B) \subseteq (\operatorname{bd} A \cap B) \cup (A \cap \operatorname{bd} B)\ . \tag{26}$$

If A is a closed subset of X, then

$$\operatorname{bd}_F(A) \subseteq A \cap F(A)\ . \tag{27}$$

The first observation is straightforward. In order to verify the second one, it is clear that $\operatorname{bd}_F(A) \subseteq \operatorname{cl} A = A$. Since $F(A) \setminus A \subseteq F(A)$, we get

$$\operatorname{cl}(F(A) \setminus A) \subseteq \operatorname{cl} F(A) = F(A),$$

because the map F is upper semi-continuous and the set X is compact. These two inclusions immediately give (27).

6.5. **Property (b).**

Lemma 6.10. *For P_1 and P_2 defined by (18), we have*

$$\operatorname{bd}_F(P_1) \subseteq P_2\ .$$

In other words, property (b) in Definition 3.6 is satisfied.

Proof: One can easily see that $\operatorname{bd}_F(P_1) \subseteq \operatorname{bd}(P_1)$. Together with (26) this further implies

$$\begin{aligned} \operatorname{bd}_F(P_1) &\subseteq \operatorname{bd}(P_1) = \operatorname{bd}(N_\delta \cap N_{\delta'}) \\ &\subseteq (N_{\delta'} \cap \operatorname{bd} N_\delta) \cup (N_\delta \cap \operatorname{bd} N_{\delta'}) \\ &= P_2 \cup (N_\delta \cap \operatorname{bd} N_{\delta'})\ . \end{aligned}$$

Thus, if we can show that

$$\operatorname{bd}_F(P_1) \cap (N_\delta \cap \operatorname{bd} N_{\delta'}) = \varnothing, \tag{28}$$

then the proof is complete. We prove this by contradiction. Assume that there exists an $x \in \operatorname{bd}_F(P_1) \cap (N_\delta \cap \operatorname{bd} N_{\delta'})$. Since $x \in \operatorname{bd}_F(P_1)$, by (27) we get

$$x \in P_1 \cap F(P_1) \subseteq N_\delta \cap F(N_\delta)\ .$$

Thus, due to Lemma 6.4, we have the inclusion $x \in \langle \mathcal{S} \rangle$. It follows that there exists a simplex $\sigma \in \mathcal{S}$ with $x \in \overset{\circ}{\sigma}$ and by (7) $\sigma = \sigma^0_{min}(x)$.

Since $x \in N_\delta$, we get a simplex $\tau \in \mathcal{S}$ such that $x \in \operatorname{cl} \langle \tau \rangle_\delta$ and since $x \in \operatorname{bd} N_{\delta'}$, by Lemma 6.5 we also get a simplex $\tau' \notin \mathcal{S}$ such that $x \in \operatorname{cl} \langle \tau' \rangle_{\delta'}$. Now, Lemma 4.1 and Corollary 4.4 imply

$$\tau \subseteq \sigma^\delta_{max}(x) \subseteq \sigma^{\delta'}_{min}(x) \subseteq \tau' \subseteq \sigma^{\delta'}_{max}(x) \subseteq \sigma^0_{min}(x) = \sigma \ .$$

Since $\tau, \sigma \in \mathcal{S}$ and $\tau' \notin \mathcal{S}$, this contradicts, in combination with Corollary 3.5, the fact that $\mathcal{S}$ is an isolated invariant set — and the proof is complete. □

Theorem 5.2 is now an immediate consequence of Lemmas 6.7, 6.8, 6.9, 6.10.

7. Proof of Theorem 5.3

In this section we prove Theorem 5.3. Since the Leray reduction of an identity is clearly the same identity, it suffices to prove that the index map I_P is the identity map. We achieve this by constructing an acyclic-valued and upper semicontinuous map G whose graph contains both the graph of F and the graph of the identity. The map G is constructed by gluing two multivalued and acyclic maps. One of these, the map $\tilde{F}$ defined below, is a modification of our map F, while the second map D contains the identity.

7.1. **The map $\tilde{F}$.** For $x \in X$ and $\sigma \in \mathcal{X}^\varepsilon(x)$ we define

$$\tilde{F}_\sigma(x) := F_\sigma(x) \cup A_\sigma.$$

One can then easily verify that

$$\tilde{F}_\sigma(x) = \begin{cases} A_\sigma & \text{if } F_\sigma(x) = C_\sigma \\ \sigma^+ & \text{if } F_\sigma(x) = B_\sigma \\ F_\sigma(x) & \text{otherwise,} \end{cases} \tag{29}$$

and that the inclusion

$$\sigma \subseteq \tilde{F}_\sigma(x) \tag{30}$$

holds. We will show that the auxiliary map $\tilde{F} : X \multimap X$ given by

$$\tilde{F}(x) := \bigcup_{\sigma \in \mathcal{X}^\varepsilon(x)} \tilde{F}_\sigma(x). \tag{31}$$

is acyclic-valued. For this we need a few auxiliary results.

Lemma 7.1. *For any $x \in X$ and $\sigma := \sigma^\varepsilon_{max}(x)$ we have*

$$\tilde{F}(x) = A_\sigma \cup F(x). \tag{32}$$

Proof: It is straightforward to observe that the right-hand side of (32) is contained in the left-hand side. To prove the opposite inclusion, take a $y \in \tilde{F}(x)$ and select a simplex $\tau \in \mathcal{X}^\varepsilon(x)$ such that $y \in \tilde{F}_\tau(x)$. In particular, $\tau \subseteq \sigma$. Note that if $\tau = \sigma$, then

$$\tilde{F}_\tau(x) = \tilde{F}_\sigma(x) = F_\sigma(x) \cup A_\sigma \subseteq F(x) \cup A_\sigma. \tag{33}$$

According to (13) the value $F_\tau(x)$ may be A_τ, B_τ, C_τ or τ. Assume first that we have $F_\tau(x) = A_\tau$. Then,

$$\tilde{F}_\tau(x) = F_\tau(x) \cup A_\tau = F_\tau(x) \subseteq F(x) \subseteq F(x) \cup A_\sigma.$$

Next, consider the case $F_\tau(x) = B_\tau$. Then, $\tau = \sigma^+ \neq \sigma^-$. Since $\tau \subseteq \sigma$, we cannot have $\sigma = \sigma^-$. Hence, $\sigma = \sigma^+ = \tau$ and estimation (33) applies. Assume in turn that $F_\tau(x) = C_\tau$. Then, $\tau = \sigma^- \neq \sigma^+$, $A_\tau = A_\sigma$ and

$$\tilde{F}_\tau(x) = C_\tau \cup A_\tau = A_\tau = A_\sigma \subseteq F(x) \cup A_\sigma.$$

Finally, if $F_\tau(x) = \tau$, then $\tau = \sigma^- = \sigma^+ = \sigma$ and again (33) applies. □

Proposition 7.2. *Let $x \in X$ and let $\sigma := \sigma^\varepsilon_{max}(x)$. For any simplex $\tau \in \mathcal{T}^\varepsilon(x)$, where $\mathcal{T}^\varepsilon(x)$ is given as in* (20)*, we have*

$$A_\tau \cap A_\sigma = \tau \cap \sigma^-. \tag{34}$$

Proof: We have $\tau \cap \sigma^- \subseteq \sigma^- \subseteq A_\sigma$ and $\tau \cap \sigma^- \subseteq \tau = \tau^- \subseteq A_\tau$, which shows that the right-hand side of (34) is contained in the left-hand side. Observe that for any simplex $\varrho \neq \sigma^+$ we have $\varrho \cap A_\sigma \subseteq \sigma^-$. We cannot have $\tau^+ = \sigma^+$, because then either $\sigma = \sigma^- = \tau^- = \tau$ or $\tau^+ = \sigma^+ = \sigma \in \operatorname{Cl} \sigma$, in both cases contradicting the inclusion $\tau \in \mathcal{T}^\varepsilon(x)$. Thus, $A_\tau \cap A_\sigma \subseteq \tau^+ \cap A_\sigma \subseteq \sigma^-$. It follows that we must have $A_\tau \cap A_\sigma \subseteq \sigma^- \cap \tau^+ \subseteq \sigma \cap \tau^+$. The simplex $\sigma \cap \tau^+$ must be a proper face of τ^+, because otherwise τ^+ is a face of σ which contradicts $\tau^+ \notin \operatorname{Cl} \sigma$. But, $\tau \subseteq \sigma \cap \tau^+$ and $\tau = \tau^-$ is a face of τ^+ of codimension one. It follows that $\sigma \cap \tau^+ = \tau$ which proves (34). □

The following proposition is implicitly proved in the second to last paragraph of the proof of [18, Theorem 4.12].

Proposition 7.3. *For any $\tau \in \mathcal{T}^\varepsilon(x)$ we have $F_\tau(x) = A_\tau$.* □

Lemma 7.4. *For an $x \in X$ and $\sigma := \sigma^\varepsilon_{max}(x)$ we have*

$$A_\sigma \cap F(x) = A_\sigma \cap F_\sigma(x). \tag{35}$$

Proof: Obviously, the right-hand side is contained in the left-hand side. To prove the opposite inclusion, choose a $y \in F(x) \cap A_\sigma$ and select a simplex τ such that $y \in F_\tau(x)$. It suffices to show that $y \in F_\sigma(x)$. By Lemma 6.3 we may assume that either $\tau = \sigma$ or $\tau \in \mathcal{T}^\varepsilon(x)$. If $\tau = \sigma$, the inclusion is obvious. Hence, assume that $\tau \in \mathcal{T}^\varepsilon(x)$. By Corollary 4.4 we have $\tau \subseteq \sigma$. This means that $\tau \subsetneq \sigma$, $\tau = \tau^-$ and $\tau^+ \notin \operatorname{Cl} \sigma$. From Proposition 7.3 we get $F_\tau(x) = A_\tau$. Therefore,

$$y \in A_\tau \cap A_\sigma \subseteq \tau^+ \cap A_\sigma \subseteq \tau^+ \cap \sigma^+. \tag{36}$$

If $\sigma = \sigma^+ = \sigma^-$, then $y \in A_\sigma = \sigma = F_\sigma(x)$, hence the inclusion holds. Thus, consider the case $\sigma^+ \neq \sigma^-$. By Proposition 7.2 we get $y \in \tau \cap \sigma^-$. We cannot have $\tau \cap \sigma^- = \sigma^-$, because then $\sigma^- \subseteq \tau \subsetneq \sigma$, $\tau^- = \tau = \sigma^-$, $\tau^+ = \sigma^+$ and $\tau \neq \sigma$ implies $\tau^+ = \sigma^+ = \sigma$, $\tau^+ \in \operatorname{Cl}\sigma$, a contradiction. Hence, $\tau \cap \sigma^-$ is a proper face of the simplex σ^-. Therefore, $\tau \cap \sigma^- \subseteq C_\sigma \subseteq F_\sigma(x)$. □

Theorem 7.5. *The map $\tilde{F}$ is upper semicontinuous and acyclic-valued.*

Proof: The upper semicontinuity of the map $\tilde{F}$ is an immediate consequence of formula (31) and Lemma 4.2. To show that $\tilde{F}$ is acyclic-valued fix an $x \in X$. By (32), $\tilde{F}(x) = A_\sigma \cup F(x)$. The set A_σ is acyclic by Lemma 4.5 and the set $F(x)$ is acyclic by Theorem 4.6. Moreover, $A_\sigma \cap F(x) = A_\sigma \cap F_\sigma(x)$ by (35). Hence, due to (13) the intersection $A_\sigma \cap F(x)$ is either A_σ or C_σ, hence also acyclic. Therefore, it follows from the Mayer-Vietoris theorem that $\tilde{F}(x)$ is acyclic. □

Proposition 7.6. *The weak index pair P is positively invariant with respect to $\tilde{F}$ and N_δ, that is, we have*

$$\tilde{F}(P_i) \cap N_\delta \subseteq P_i \quad \text{for} \quad i = 1, 2.$$

Proof: The proof is analogous to the proof of Lemma 6.7. □

7.2. **The map $\tilde{D}$.** We define a multivalued map $D : X \multimap X$, by letting

$$D(x) := \operatorname{conv}(\{x\} \cup \sigma^\varepsilon_{max}(x)),$$

where $\operatorname{conv} A$ denotes the convex hull of A. Note that the above definition is well-posed, because both $\{x\}$ and σ are subsets of the same simplex $\sigma^0_{min}(x)$.

In order to show that D is upper semicontinuous we need the following lemma.

Lemma 7.7. *The mapping*

$$X \ni x \mapsto \sigma^\varepsilon_{max}(x) \subseteq X$$

is strongly upper semicontinuous, that is, for every $x \in X$ there exists a neighborhood V of x such that for each $y \in V$ we have $\sigma^\varepsilon_{max}(y) \subseteq \sigma^\varepsilon_{max}(x)$.

Proof: By Lemma 4.2, we can choose a neighborhood V of x in such a way that $\mathcal{X}^\varepsilon(y) \subseteq \mathcal{X}^\varepsilon(x)$ for $y \in V$. In particular, $\sigma^\varepsilon_{max}(y) \in \mathcal{X}^\varepsilon(y) \subseteq \mathcal{X}^\varepsilon(x)$. By Corollary 4.4, we obtain $\sigma^\varepsilon_{max}(y) \subseteq \sigma^\varepsilon_{max}(x)$. □

Proposition 7.8. *The mapping D is upper semicontinuous and has non-empty and contractible values.*

Proof: Since the values of D are convex, they are obviously contractible. To see that D is upper semicontinuous, fix $\varepsilon > 0$. By Lemma 7.7 we can find a neighborhood V of x such that $\sigma^\varepsilon_{max}(y) \subseteq \sigma^\varepsilon_{max}(x)$. Let $B(x, \varepsilon)$ denote the ε-ball around x, let $y \in B(x, \varepsilon) \cap V$, and fix a point $z \in D(y)$. Then, $z = ty + (1-t)\bar{y}$ for a $t \in [0, 1]$ and $\bar{y} \in \sigma^\varepsilon_{max}(y)$. Let $z' := tx + (1-t)\bar{y}$. Since $\sigma^\varepsilon_{max}(y) \subseteq \sigma^\varepsilon_{max}(x)$, we have $z' \in D(x)$. Moreover, the estimate $||z - z'|| = t||y - x|| \leq ||y - x|| < \varepsilon$ holds. It follows that $z \in B(D(x), \varepsilon)$. Hence, $D(y) \subseteq B(D(x), \varepsilon)$, which proves the upper semicontinuity of D. □

Lemma 7.9. *Let $0 < \tilde{\delta} < \varepsilon$. For any $x \in X$ and any $y \in D(x)$, $y \neq x$, we have*

$$\sigma^{\tilde{\delta}}_{max}(y) \subseteq \sigma^{\tilde{\delta}}_{min}(x).$$

Proof: Let $x \in X$ and $y \in D(x)$, with $y \neq x$, be fixed. Then $y = \alpha x + (1-\alpha)x_\sigma$ for some $x_\sigma \in \sigma^{\varepsilon}_{max}(x)$ and $\alpha \in [0, 1)$. Consider a vertex $v \notin \sigma^{\tilde{\delta}}_{min}(x)$. By Lemma 4.1 we have $\sigma^{\varepsilon}_{max}(x) \subseteq \sigma^{\tilde{\delta}}_{min}(x)$, which shows that $v \notin \sigma^{\varepsilon}_{max}(x)$. Therefore,

$$\begin{aligned} t_v(y) &= \alpha t_v(x) + (1 - \alpha)t_v(x_\sigma) \\ &= \alpha t_v(x) < t_v(x) \leq \tilde{\delta}, \end{aligned}$$

which implies $v \notin \sigma^{\tilde{\delta}}_{max}(y)$, and the inclusion $\sigma^{\tilde{\delta}}_{max}(y) \subseteq \sigma^{\tilde{\delta}}_{min}(x)$ follows. □

Proposition 7.10. *The weak index pair P is positively invariant with respect to D and N_δ, that is, we have*

$$D(P_i) \cap N_\delta \subseteq P_i \quad \textit{for} \quad i = 1, 2. \tag{37}$$

Proof: We begin with the proof for the case $i = 1$. Fix an $x \in P_1 = N_\delta \cap N_{\delta'}$. Then $x \in \operatorname{cl}\langle\tau\rangle_\delta$ and $x \in \operatorname{cl}\langle\tau'\rangle_{\delta'}$ for some $\tau, \tau' \in \mathcal{S}$. Consider a $y \in D(x) \cap N_\delta$. For $y = x$ inclusion (37) trivially holds. Therefore, assume $y \neq x$. Since $y \in N_\delta$, there exists a simplex $\eta \in \mathcal{S}$ with $y \in \operatorname{cl}\langle\eta\rangle_\delta$. Consider a simplex $\eta' \in \mathcal{X}$ such that $y \in \operatorname{cl}\langle\eta'\rangle_{\delta'}$. By Lemma 7.9 we obtain $\eta' \subseteq \sigma^{\delta'}_{max}(y) \subseteq \sigma^{\delta'}_{min}(x) \subseteq \tau'$ and Lemma 4.1 together with Corollary 4.4 implies $\eta \subseteq \sigma^{\delta}_{max}(y) \subseteq \sigma^{\delta'}_{min}(y) \subseteq \eta'$. Thus, $\eta \subseteq \eta' \subseteq \tau'$. Since $\eta \in \mathcal{S}$ and $\tau' \in \mathcal{S}$, we get from Corollary 3.5 that $\eta' \in \mathcal{S}$. Thus, $y \in N_{\delta'}$, which completes the proof for $i = 1$.

To proceed with the proof for $i = 2$, fix an $x \in P_2$ and consider a $y \in D(x) \cap N_\delta$. Inclusion (37) is trivial when $y = x$. Hence, assume $y \neq x$. Note that $P_2 \subseteq N'_\delta$, therefore $x \in \operatorname{cl}\langle\tau'\rangle_{\delta'}$ for some $\tau' \in \mathcal{S}$. Since $x \in \operatorname{bd} N_\delta$, by Lemma 6.5 there exists a simplex $\tau \notin \mathcal{S}$ such that $x \in \operatorname{cl}\langle\tau\rangle_\delta$. According to Lemma 4.1 we then have the inclusion $\tau \subseteq \sigma^{\delta}_{max}(x) \subseteq \sigma^{\delta'}_{min}(x) \subseteq \tau'$, and this yields $\tau \in \operatorname{Ex}\mathcal{S}$. Consider a simplex η such that $y \in \operatorname{cl}\langle\eta\rangle_\delta$. By Lemma 7.9 we have $\eta \subseteq \sigma^{\delta}_{max}(y) \subseteq \sigma^{\delta}_{min}(x) \subseteq \tau$. Now, the closedness of $\operatorname{Ex}\mathcal{S}$ implies $\eta \in \operatorname{Ex}\mathcal{S}$. Hence, $y \in \operatorname{bd} N_\delta$ by Lemma 6.5. Observe that by case $i = 1$ we also have $y \in P_1 \subseteq N'_\delta$. Therefore, $y \in P_2$. □

7.3. **The map G.**

Define the multivalued map $G : X \multimap X$ by

$$G(x) := D(x) \cup \tilde{F}(x). \tag{38}$$

Proposition 7.11. *The following conditions hold:*

- (i) *G is upper semicontinuous,*
- (ii) *P is positively invariant with respect to G and N_δ,*

(iii) *G is acyclic-valued.*

Proof: The map G inherits properties (i) and (ii) directly from its summands $\tilde{F}$ and D (see Theorem 7.5, Proposition 7.6, Proposition 7.8 and Proposition 7.10).

To prove (iii), fix an $x \in X$ and note that $\tilde{F}(x)$ is acyclic by Theorem 7.5, and that $D(x)$ is acyclic by Proposition 7.11(iii). Let $\sigma := \sigma^{\varepsilon}_{max}(x)$ and $\sigma^0 := \sigma^0_{min}(x)$. Obviously either $x \in \sigma$ or $x \notin \sigma$. To begin with, we consider the case $x \in \sigma$. Then one has $D(x) = \sigma$. We will show that $D(x) \subseteq \tilde{F}(x)$. Indeed, if $\sigma = \sigma^-$ then we have $D(x) = \sigma = \sigma^- \subseteq A_\sigma \subseteq \tilde{F}(x)$ by Lemma 7.1. If $\sigma \neq \sigma^-$, then one has the equality $\sigma = \sigma^+$. In that case $F_\sigma(x) = B_\sigma$ and by (29) we get $\tilde{F}_\sigma(x) = \sigma^+$, which shows that $D(x) = \sigma = \sigma^+ = \tilde{F}_\sigma(x) \subseteq \tilde{F}(x)$. Consequently, if $x \in \sigma$, then we have the equality $G(x) = \tilde{F}(x)$, and this set is acyclic by Theorem 7.5.

Thus, consider the case $x \notin \sigma$. By the Mayer-Vietoris theorem it suffices to show that $D(x) \cap \tilde{F}(x)$ is acyclic, because both sets $D(x)$ and $\tilde{F}(x)$ are acyclic. To this end we use the following representation

$$\tilde{F}(x) = F_\sigma(x) \cup A_\sigma \cup \bigcup_{\tau \in \mathcal{T}^{\varepsilon}(x)} F_\tau(x),$$

which follows immediately from Lemma 7.1 and (19). First, we will show that for every simplex $\tau \in \mathcal{T}^{\varepsilon}(x)$ we have

$$D(x) \cap F_\tau(x) \subseteq D(x) \cap (F_\sigma(x) \cup A_\sigma)\,. \tag{39}$$

To see this observe that since $x \in \sigma^0 \setminus \sigma$, it is evident that σ is a proper face of the simplex σ^0. Moreover, one can easily observe that

$$D(x) \subseteq \sigma^0 \text{ and } D(x) \cap |\operatorname{Ex}\sigma^0| = \sigma. \tag{40}$$

Note that from the definition of the collection $\mathcal{T}^{\varepsilon}(x)$ (cf. Lemma 6.3) it follows that any simplex $\tau \in \mathcal{T}^{\varepsilon}(x)$ is a proper face of σ. Therefore, σ^0 is a coface of τ of codimension greater than one. Hence, we cannot have $\tau^+ = \sigma^0$. By Proposition 7.3 we have $F_\tau(x) = A_\tau \subseteq \tau^+$. Thus, from $\tau = \tau^- \subseteq A_\tau$ and (40), we obtain

$$D(x) \cap F_\tau(x) \subseteq \sigma \text{ for any } \tau \in \mathcal{T}^{\varepsilon}(x). \tag{41}$$

We will now show that

$$\sigma \subseteq D(x) \cap (F_\sigma(x) \cup A_\sigma)\,. \tag{42}$$

Obviously, $\sigma \subseteq D(x)$. If $\sigma = \sigma^-$, then $\sigma \subseteq A_\sigma \subseteq F_\sigma(x) \cup A_\sigma$. If $\sigma = \sigma^+$, then by (13) we have $\sigma \subseteq \sigma^+ \subseteq F_\sigma(x) \cup A_\sigma$. Hence, (42) is proved. Formula (39) follows now from (41) and (42). From (39) we immediately obtain that

$$D(x) \cap \tilde{F}_\tau(x) = D(x) \cap (F_\sigma(x) \cup A_\sigma)\,. \tag{43}$$

Now we distinguish the two complementary cases: $\sigma = \sigma^+$ and $\sigma = \sigma^- \neq \sigma^+$. First of all, if $\sigma = \sigma^+$, then

$$D(x) \cap (F_\sigma(x) \cup A_\sigma) \subseteq \sigma^0 \cap \sigma^+ = \sigma^0 \cap \sigma = \sigma.$$

It follows from (42) and (39) that in this case

$$D(x) \cap \tilde{F}(x) = D(x) \cap (F_\sigma(x) \cup A_\sigma) = \sigma$$

is an acyclic set. We show that the same is true in the second case $\sigma = \sigma^- \neq \sigma^+$. Now one has $F_\sigma(x) \cup A_\sigma = C_\sigma \cup A_\sigma = A_\sigma$. Observe that $A_\sigma = A^+ \cup \sigma^-$, where

$$A^+ := \{y \in \sigma^+ \mid t_v(y) \geq \gamma \text{ for } v \in \sigma^-\}$$

is a convex set. This, together with (43), shows that

$$D(x) \cap \tilde{F}(x) = D(x) \cap A_\sigma = D(x) \cap (A^+ \cup \sigma^-) = D(x) \cap A^+ \cup \sigma^-. \tag{44}$$

The acyclicity of the right-hand-side of (44) follows from the Mayer-Vietoris theorem, because $D(x) \cap A^+$, σ^-, and $D(x) \cap A^+ \cap \sigma^- = A^+ \cap \sigma^-$ are all convex. Therefore, by (44) also in this case the set $D(x) \cap \tilde{F}(x)$ is acyclic. This completes the proof. ☐

We are now able to prove Theorem 5.3.

Proof of Theorem 5.3: By Lemma 7.6 and Proposition 7.10 we can consider the map G, given by (38), as a map of pairs

$$G : (P_1, P_2) \multimap (T_1(P), T_2(P)).$$

Directly from the definition of G it follows that both the inclusion $i : P \to T(P)$ and $F : P \multimap T(P)$ are selectors of G, that is, for any $x \in P_1$ we have

$$x \in G(x) \qquad \text{and} \qquad F(x) \subseteq G(x).$$

Moreover, all of the above maps are acyclic-valued (cf. again Proposition 7.11 and Theorem 4.6). Therefore, it follows from [15, Proposition 32.13(i)] that the identities $H^*(F) = H^*(G) = H^*(i)$ are satisfied. As a consequence we obtain the desired equality $I_P = \mathrm{id}_{H^*(P)}$, which completes the proof. ☐

8. Proof of Theorem 5.4

In order to prove Theorem 5.4 we first construct an auxiliary pair (Q_1, Q_2) and show that $H^*(P_1, P_2) \cong H^*(Q_1, Q_2)$. As a second step, we then construct a continuous surjection $\psi : (Q_1, Q_2) \to (|\operatorname{Cl}\mathcal{S}|, |\operatorname{Ex}\mathcal{S}|)$ with contractible preimages and apply the Vietoris-Begle theorem to complete the proof.

8.1. **The pair** (Q_1, Q_2)**.** Consider the pair (Q_1, Q_2) consisting of the two sets

$$Q_1 := N_\delta(\operatorname{Cl}\mathcal{S}) \cap N_{\delta'}(\operatorname{Cl}\mathcal{S}),$$
$$Q_2 := N_\delta(\operatorname{Ex}\mathcal{S}) \cap N_{\delta'}(\operatorname{Cl}\mathcal{S}),$$

where $N_\delta(\mathcal{A})$ is given by (15). Figure 12 shows an example of such a pair for the isolated invariant set $\mathcal{S}$ presented in Figure 9.

Proposition 8.1. *We have*

$$H^*(Q_1, Q_2) \cong H^*(P_1, P_2). \tag{45}$$

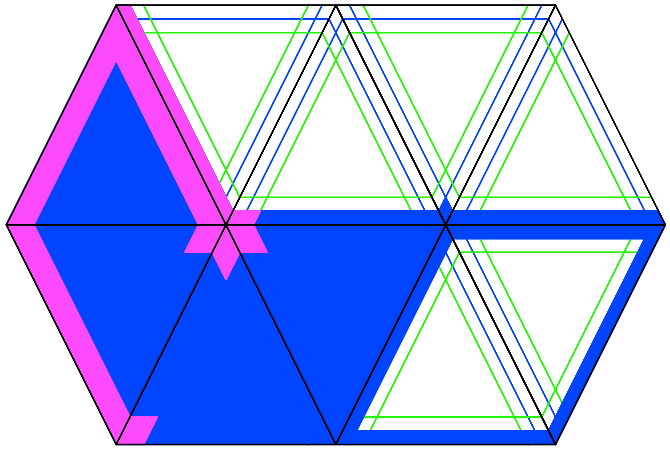


FIGURE 12. The pair $Q = (Q_1, Q_2)$ associated with the isolating block N_δ from Figure 9 and the weak index pair $P = (P_1, P_2)$ from Figure 10. The set Q_1 is the union of the dark blue and magenta regions, while the subset $Q_2 \subseteq Q_1$ is only the magenta part.

Proof: We begin by verifying the two inclusions

$$P_i \subseteq Q_i \quad \text{for} \quad i = 1, 2. \tag{46}$$

It is clear that $P_1 \subseteq Q_1$, therefore we shall verify (46) for $i = 2$.

Let $x \in P_2$. Then $x \in N_\delta$ and by Lemma 6.5, there exists a simplex $\sigma \notin \mathcal{S}$ such that $x \in \operatorname{cl}\langle\sigma\rangle_\delta$. On the other hand, $x \in P_1$ implies $x \in N_{\delta'}(\mathcal{S})$, so we can take $\tau \in \mathcal{S}$ with $x \in \operatorname{cl}\langle\tau\rangle_{\delta'}$. For any vertex $v \in \sigma$ we have $t_v(x) \geq \delta > \delta'$, which shows that $\sigma \subseteq \tau \in \mathcal{S}$. Consequently, $\sigma \in \operatorname{Cl}\mathcal{S}$, which along with $\sigma \notin \mathcal{S}$ implies the inclusions $\sigma \in \operatorname{Ex}\mathcal{S}$ and $x \in N_\delta(\operatorname{Ex}\mathcal{S})$. Observe now that we also have $x \in N_{\delta'}(\operatorname{Cl}\mathcal{S})$, according to $P_2 \subseteq N_{\delta'}(\mathcal{S}) \subseteq N_{\delta'}(\operatorname{Cl}\mathcal{S})$. Thus, $x \in Q_2$. The proof of (46) is now complete.

Note that P_1, P_2, Q_1, Q_2 are compact and $Q_2 \subseteq Q_1$ and $P_2 \subseteq P_1$. Therefore, by the strong excision property of Alexander-Spanier cohomology, in order to prove (46), it suffices to verify that $Q_1 \setminus Q_2 = P_1 \setminus P_2$.

For this, consider an $x \in Q_1 \setminus Q_2$. Then $x \in N_\delta(\operatorname{Cl}\mathcal{S})$ and $x \notin N_\delta(\operatorname{Ex}\mathcal{S})$. Hence, there exists a $\sigma \in \operatorname{Cl}\mathcal{S} \setminus \operatorname{Ex}\mathcal{S} = \mathcal{S}$ with $x \in \operatorname{cl}\langle\sigma\rangle_\delta$. It follows that

$$x \in N_\delta(\mathcal{S}). \tag{47}$$

We also have $x \in N_{\delta'}(\operatorname{Cl}\mathcal{S})$. In order to show that $x \in P_1$, we need to verify the inclusion $x \in N_{\delta'}(\mathcal{S})$. Suppose to the contrary that there is a $\tau \in \operatorname{Cl}\mathcal{S} \setminus \mathcal{S} = \operatorname{Ex}\mathcal{S}$ with $x \in \operatorname{cl}\langle\tau\rangle_{\delta'}$. Then for each vertex v of σ we have $t_v(x) \geq \delta > \delta'$, which means that each vertex of σ is a vertex of τ. In other words $\sigma \subseteq \tau$. However, $\tau \in \operatorname{Ex}\mathcal{S}$ which, according to the closedness of $\operatorname{Ex}\mathcal{S}$, implies $\sigma \in \operatorname{Ex}\mathcal{S}$, a contradiction. Therefore, $x \in N_{\delta'}(\mathcal{S})$, and together with (47) this implies the inclusion $x \in P_1$.

Since $x \notin Q_2$, by (46), we further have $x \notin P_2$. Consequently, both $x \in P_1 \setminus P_2$ and $Q_1 \setminus Q_2 \subseteq P_1 \setminus P_2$ are satisfied.

In order to prove the reverse inclusion let $x \in P_1 \setminus P_2$ be arbitrary. It is clear that then $x \in Q_1$. We need to show that $x \notin Q_2$. Suppose the contrary. Then there exists a simplex $\sigma \in \operatorname{Ex} \mathcal{S}$ such that $x \in \operatorname{cl} \langle\sigma\rangle_\delta$. It follows that $\sigma \in \mathcal{X}^\delta(x) \setminus \mathcal{S}$ and $\mathcal{X}^\delta(x) \setminus \mathcal{S} \neq \varnothing$. Since $x \in P_1 \subseteq N_\delta(\mathcal{S})$, we also have $\mathcal{X}^\delta(x) \cap \mathcal{S} \neq \varnothing$. Therefore, by Lemma 6.5, we get $x \in \operatorname{bd} N_\delta(\mathcal{S})$. Yet, we also have $x \in N_{\delta'}(\mathcal{S})$ in view of $x \in P_1$. Consequently, $x \in \operatorname{bd} N_\delta(\mathcal{S}) \cap N_{\delta'}(\mathcal{S}) = P_2 \subseteq Q_2$, which is a contradiction. ∎

8.2. **Auxiliary maps φ_σ^λ.**

Proposition 8.2. *For any $\sigma, \tau \in \mathcal{X}$ we have*

$$\begin{aligned}
\operatorname{cl}\langle\sigma\rangle_\lambda \cap \operatorname{cl}\langle\tau\rangle_\lambda = \{x \in X \mid\ & t_v(x) = \lambda \quad \textit{for} \quad v \in (\tau \setminus \sigma) \cup (\sigma \setminus \tau), \\
& t_v(x) \geq \lambda \quad \textit{for} \quad v \in \tau \cap \sigma, \\
& t_v(x) \leq \lambda \quad \textit{for} \quad v \notin \tau \cup \sigma \,\}.
\end{aligned}$$

In particular

$$\operatorname{cl}\langle\sigma\rangle_\lambda \cap \operatorname{cl}\langle\tau\rangle_\lambda \subseteq \operatorname{cl}\langle\sigma \cap \tau\rangle_\lambda.$$

Proof: The proposition follows immediately from (9). ∎

For $\lambda \in [0,1)$ let

$$\varphi_\lambda : [0,1] \ni t \longmapsto \begin{cases} \frac{t-\lambda}{1-\lambda} & \text{for } t \geq \lambda \\ 0 & \text{for } t \leq \lambda \end{cases} \in [0,1]. \tag{48}$$

Given a simplex σ in $\mathcal{X}$ we define the map

$$\varphi_\sigma^\lambda : \operatorname{cl}\langle\sigma\rangle_\lambda \ni x \longmapsto \varphi_\sigma^\lambda(x) \in |\sigma|$$

by

$$\varphi_\sigma^\lambda(x) := \sum_{v \in \mathcal{X}_0} \frac{\varphi_\lambda(t_v(x))}{\sum_{w \in \mathcal{X}_0} \varphi_\lambda(t_w(x))} v\,. \tag{49}$$

Proposition 8.3. *The map φ_σ^λ is well-defined and continuous.*

Proof: Let $x \in \operatorname{cl}\langle\sigma\rangle_\lambda$. Then we have $t_v(x) \leq \lambda$ for $v \notin \sigma$, and consequently the identity $\varphi_\lambda(t_v(x)) = 0$ holds for $v \notin \sigma$. Hence, $\varphi_\sigma^\lambda(x) \in |\sigma|$, which means that φ_σ^λ is well-defined. The continuity of $\varphi_\sigma^\lambda(x)$ follows from the continuity of φ_λ and the continuity of the barycentric coordinates. ∎

For $\sigma \in \mathcal{X}$ let n_σ denote the number of vertices in σ. For $x \in \operatorname{cl}\langle\sigma\rangle_\lambda$ set

$$r_\sigma^\lambda(x) := \sum_{w \notin \sigma} t_w(x).$$

Lemma 8.4. *Let $\sigma \in \mathcal{X}$ and $\lambda \in [0,1)$. For any $x \in \operatorname{cl}\langle\sigma\rangle_\lambda$ and $v \in \mathcal{X}_0$ we have*

$$t_v(\varphi_\sigma^\lambda(x)) = \begin{cases} \dfrac{t_v(x)-\lambda}{1-\lambda n_\sigma - r_\sigma^\lambda(x)} & \text{if } v \in \sigma \\ 0 & \text{otherwise.} \end{cases} \tag{50}$$

Proof: It is clear from (48) and (49) that (50) is correct for $v \notin \sigma$. On the other hand, if $v \in \sigma$ then $t_v(x) \geq \lambda$, hence, by (49) and (48) we have

$$t_v(x) = t_v(\varphi_\sigma^\lambda(x))(1-\lambda)\sum_{w\in\mathcal{X}_0}\varphi_\lambda(t_w(x)) + \lambda. \tag{51}$$

Summing up the barycentric coordinates of x over all vertices in $\mathcal{X}_0$, and taking into account the above equalities, which are valid for all vertices of σ, we obtain

$$\begin{aligned} \sum_{v\in\mathcal{X}_0} t_v(x) &= \sum_{v\in\sigma} t_v(x) + \sum_{v\notin\sigma} t_v(x) \\ &= \sum_{v\in\sigma} t_v(\varphi_\sigma^\lambda(x))(1-\lambda)\sum_{w\in\mathcal{X}_0}\varphi_\lambda(t_w(x)) + \lambda n_\sigma + r_\sigma^\lambda(x). \end{aligned}$$

Since the barycentric coordinates sum to 1, we have $\sum_{v\in\mathcal{X}_0} t_v(x) = 1$. Moreover, since $\varphi_\sigma^\lambda(x) \in |\sigma|$, we also have $\sum_{v\in\sigma} t_v(\varphi_\sigma^\lambda(x)) = 1$. Therefore, the above equality reduces to

$$1 = (1-\lambda)\sum_{w\in\mathcal{X}_0}\varphi_\lambda(t_w(x)) + \lambda n_\sigma + r_\sigma^\lambda(x).$$

Consequently,

$$\sum_{w\in\mathcal{X}_0}\varphi_\lambda(t_w(x)) = \frac{1-\lambda n_\sigma - r_\sigma^\lambda(x)}{1-\lambda}.$$

Replacing the sum $\sum_{w\in\mathcal{X}_0}\varphi_\lambda(t_w(x))$ in (51) by the right-hand side of this equation and calculating $t_v(\varphi_\sigma^\lambda(x))$ we obtain (50) for $v \in \sigma$. This completes the proof. □

Proposition 8.5. *For any simplex $\sigma \in \mathcal{X}$ and $\lambda \in [0,1)$ we have*

$$\varphi_\sigma^\lambda(\sigma \cap \operatorname{cl}\langle\sigma\rangle_\lambda) = \sigma.$$

Proof: It is clear that $\varphi_\sigma^\lambda(\sigma \cap \operatorname{cl}\langle\sigma\rangle_\lambda) \subseteq \sigma$, therefore we verify the opposite inclusion. Take an arbitrary $y \in \sigma$ and define

$$x := \sum_{v\in\sigma}\left(t_v(y)(1-\lambda n_\sigma) + \lambda\right) v$$

It is easy to check that the above formula correctly defines a point $x \in \sigma$ via its barycentric coordinates. Moreover, we have $x \in \sigma \cap \operatorname{cl}\langle\sigma\rangle_\lambda$, as $t_v(x) = 0$ for $v \notin \sigma$ and $t_v(x) \geq \lambda$ for $v \in \sigma$. An easy calculation, with the use of Lemma 8.4, finally shows that $\varphi_\sigma^\lambda(x) = y$. □

The following proposition is an immediate consequence of Proposition 8.2.

Proposition 8.6. *For any simplices σ and τ in $\mathcal{X}$ and arbitrary $\lambda \in [0,1)$ the maps φ_σ^λ and φ_τ^λ coincide on $\operatorname{cl}\langle\sigma\rangle_\lambda \cap \operatorname{cl}\langle\tau\rangle_\lambda$* □

8.3. **Mapping ψ.** In view of (10) and Proposition 8.6, we have a well-defined continuous surjection $\varphi : |\mathcal{X}| \to |\mathcal{X}|$ given by

$$\varphi(x) := \varphi_\sigma^\delta(x) \quad \text{ where } \sigma \in \mathcal{X} \text{ is such that } \quad x \in \operatorname{cl}\langle\sigma\rangle_\lambda.$$

Let $\psi := \varphi_{|Q_1} : Q_1 \to X$ denote the restriction of φ to Q_1.

Proposition 8.7. *For each $y \in |\operatorname{Cl}\mathcal{S}|$ the fiber $\psi^{-1}(y)$ is non-empty and contractible.*

Proof: Let $y \in |\operatorname{Cl}\mathcal{S}| = \langle \operatorname{Cl}\mathcal{S}\rangle$ be arbitrary and let the simplex $\sigma \in \operatorname{Cl}\mathcal{S}$ be such that $y \in \mathring{\sigma}$. Furthermore, define the set

$$X_\sigma := \{\, x \in X \;\mid\; t_v(x) \le \delta \text{ if } v \notin \sigma \text{ and } t_v(x) = t_v(y)(1 - \delta n_\sigma - r_\sigma^\lambda(x)) + \delta \text{ for } v \in \sigma \,\}.$$

We first verify that the fiber of y under φ_σ^δ is given by

$$(\varphi_\sigma^\delta)^{-1}(y) = X_\sigma. \tag{52}$$

For this, fix an $x \in X_\sigma$, and recall that δ satisfies (17), in particular $\delta < 1/(d+1)$. Therefore, for $v \in \sigma$ we deduce from (50) the inequality

$$t_v(x) = t_v(y)(1 - \delta n_\sigma - r_\sigma^\lambda(x)) + \delta \ge t_v(y)(1 - \delta n_\sigma - \delta(d+1-n_\sigma) + \delta \ge \delta.$$

This, together with the obvious inequality $t_v(x) \le \delta$ for $v \notin \sigma$, shows that

$$X_\sigma \subseteq \operatorname{cl}\langle\sigma\rangle_\delta = \operatorname{dom}\varphi_\sigma^\delta. \tag{53}$$

Moreover, a straightforward calculation implies that for every point $x \in X_\sigma$ the identity $\varphi_\sigma^\delta(x) = y$ holds. This shows that $X_\sigma \subseteq (\varphi_\sigma^\delta)^{-1}(y)$. Since the converse inclusion is straightforward, the proof of (52) is finished. Now let

$$\bar{X}_\sigma := (\varphi_\sigma^\delta)^{-1}(y) \cap N_{\delta'}(\operatorname{Cl}\mathcal{S}) = X_\sigma \cap N_{\delta'}(\operatorname{Cl}\mathcal{S}). \tag{54}$$

We claim that

$$\psi^{-1}(y) = \bar{X}_\sigma. \tag{55}$$

Note that

$$\bar{X}_\sigma \subseteq \operatorname{cl}\langle\sigma\rangle_\delta \cap N_{\delta'}(\operatorname{Cl}\mathcal{S}) \subseteq N_\delta \cap N_{\delta'} = Q_1 = \operatorname{dom}\psi.$$

Since $\bar{X}_\sigma \subseteq (\varphi_\sigma^\delta)^{-1}(y)$, one obtains for $w \in \bar{X}_\sigma$ the identity $y = \varphi_\sigma^\delta(w) = \psi(w)$. Therefore, $\bar{X}_\sigma \subseteq \psi^{-1}(y)$. For the proof of the reverse inclusion it suffices to verify that the condition $x \notin \operatorname{cl}\langle\sigma\rangle_\delta$ implies $\psi(x) \ne y$. Suppose to the contrary that $x \notin \operatorname{cl}\langle\sigma\rangle_\delta$ and $\psi(x) = y$, and consider a simplex τ such that $x \in \operatorname{cl}\langle\tau\rangle_\delta$. Then $\psi(x) = \varphi_\tau^\delta(x)$. Directly from the definition of φ_τ^δ we infer that $\psi(x) \in \tau$. However $\psi(x) = y \in \mathring{\sigma}$, which means that σ is a face of τ. Then, taking into account the inclusion $x \in \operatorname{cl}\langle\tau\rangle_\delta \setminus \operatorname{cl}\langle\sigma\rangle_\delta$, we can find a vertex $v \in \tau \setminus \sigma$ such that $t_v(x) > \delta$. Consequently, by (50), we have $t_v(y) = t_v(\psi(x)) = t_v(\varphi^\delta_\tau(x)) > 0$, which contradicts $y \in \mathring{\sigma}$, and completes the proof of (55).

We still need to show that $\bar{X}_\sigma = \psi^{-1}(y)$ is contractible. To this end, we define the map $h : \bar{X}_\sigma \times [0,1] \to \bar{X}_\sigma$ by

$$h(x,s) := \sum_{v \in \mathcal{X}_0} t_{v,x,s} v,$$

where

$$t_{v,x,s} := \begin{cases} (1-s)t_v(x) & \text{if } v \notin \sigma, \\ t_v(y)(1 - \delta n_\sigma - (1-s) r^\lambda_\sigma(x)) + \delta & \text{if } v \in \sigma. \end{cases}$$

We will show that h is a well-defined homotopy between the identity on $\bar{X}_\sigma$ and a constant map on $\bar{X}_\sigma$.

To begin with, we verify that for any point $x \in \bar{X}_\sigma$ and arbitrary $s \in [0,1]$ we have $h(x,s) \in \bar{X}_\sigma$. The verification that the inclusion $h(x,s) \in \operatorname{cl}\langle\sigma\rangle_\delta$ holds, as well as $\varphi^\delta_\sigma(h(x,s)) = y$, which in turn shows that $h(x,s) \in (\varphi^\delta_\sigma)^{-1}(y)$, is tedious but straightforward. We still need to verify that $h(x,s) \in N_{\delta'}(\operatorname{Cl}\mathcal{S})$. For this, consider a simplex $\tau \in \operatorname{Cl}\mathcal{S}$ such that $x \in \operatorname{cl}\langle\sigma\rangle_\delta \cap \operatorname{cl}\langle\tau\rangle_{\delta'}$. Since for any $v \in \sigma$ we have $t_v(x) \geq \delta > \delta'$, we deduce that $\sigma \subseteq \tau$. Let

$$\eta := \{\, v \in \mathcal{X}_0 \mid t_v(h(x,s)) > \delta' \,\}.$$

Then we claim that the inclusions $\sigma \subseteq \eta \subseteq \tau$ hold. Indeed, if $v \notin \tau$ then $\sigma \subseteq \tau$ implies the inequalities $t_v(h(x,s)) = (1-s)t_v(x) \leq t_v(x) \leq \delta'$. Furthermore, if one has $v \in \sigma \subseteq \tau$, then $t_v(h(x,s)) \geq t_v(x) \geq \delta > \delta'$. Therefore, η is a simplex and it satisfies $h(x,s) \in \operatorname{cl}\langle\eta\rangle_{\delta'}$. Since $\sigma \in \operatorname{Cl}\mathcal{S}$ as well as $\tau \in \operatorname{Cl}\mathcal{S}$, the closedness of $\operatorname{Cl}\mathcal{S}$ implies that $\eta \in \operatorname{Cl}\mathcal{S}$. Consequently, $h(x,s) \in N_{\delta'}(\operatorname{Cl}\mathcal{S})$, and this proves that the map h is well-defined.

The continuity of h follows from the continuity of the barycentric coordinates. Verification that $h(\cdot,0) = \operatorname{id}_{\bar{X}_\sigma}$ as well as that $h(\cdot,1)$ is constant on $\bar{X}_\sigma$ is straightforward. This completes the proof. ☐

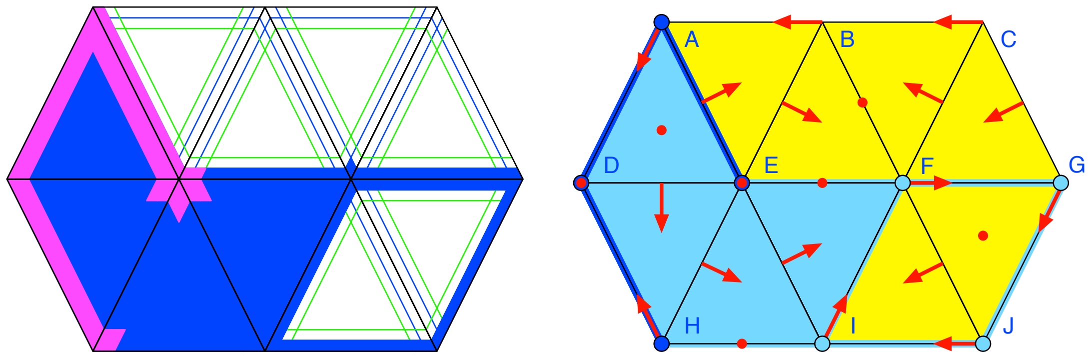
Proposition 8.8. *We have*

(i) $\psi(Q_1) = |\operatorname{Cl}\mathcal{S}|$,
(ii) $\psi(Q_2) = |\operatorname{Ex}\mathcal{S}|$,
(iii) $\psi^{-1}(|\operatorname{Ex}\mathcal{S}|) = Q_2$.

In particular, we can consider ψ as a map of pairs

$$\psi : (Q_1, Q_2) \to (|\operatorname{Cl}\mathcal{S}|, |\operatorname{Ex}\mathcal{S}|).$$

Proof: For the proof of (i), fix an arbitrary point $x \in Q_1$. Then there exists a simplex $\sigma \in \operatorname{Cl}\mathcal{S}$ with $x \in \operatorname{cl}\langle\sigma\rangle_\delta$. Thus, $\psi(x) = \varphi(x) = \varphi^\delta_\sigma(x) \in \sigma \subseteq |\operatorname{Cl}\mathcal{S}|$ and $\psi(x) \in |\operatorname{Cl}\mathcal{S}|$. This implies that the inclusion $\psi(Q_1) \subseteq |\operatorname{Cl}\mathcal{S}|$. The reverse inclusion is a consequence of Proposition 8.5, because for any simplex $\sigma \in \operatorname{Cl}\mathcal{S}$ we have $\sigma \cap \operatorname{cl}\langle\sigma\rangle_\delta \subseteq \operatorname{cl}\langle\sigma\rangle_\delta \cap \operatorname{cl}\langle\sigma\rangle_{\delta'} \subseteq Q_1$. The proof of (ii) is analogous to the proof of (i).



(A) The pair $Q = (Q_1, Q_2)$. Q_1 is the union of the dark blue and magenta regions, while Q_2 is only the magenta region.

(B) The pair $(|\operatorname{Cl}\mathcal{S}|, |\operatorname{Ex}\mathcal{S}|)$. $|\operatorname{Cl}\mathcal{S}|$ is the union of the light blue and dark blue regions, while $|\operatorname{Ex}\mathcal{S}|$ is only the dark blue set.

FIGURE 13. The pairs $Q = (Q_1, Q_2)$ and $(|\operatorname{Cl}\mathcal{S}|, |\operatorname{Ex}\mathcal{S}|)$.

In order to prove the remaining statement (iii), first observe that we have the inclusion $\psi^{-1}(|\operatorname{Ex}\mathcal{S}|) \subseteq Q_2$. Indeed, given a $y \in |\operatorname{Ex}\mathcal{S}|$ there exists a $\sigma \in \operatorname{Ex}\mathcal{S}$ such that $y \in \mathring{\sigma}$, and by (54), (55), and (53), we have

$$\psi^{-1}(y) \subseteq \operatorname{cl}\langle\sigma\rangle_\delta \cap N_{\delta'}(\operatorname{Cl}\mathcal{S}) \subseteq N_\delta(\operatorname{Ex}\mathcal{S}) \cap N_{\delta'}(\operatorname{Cl}\mathcal{S}) = Q_2,$$

which implies $\psi^{-1}(|\operatorname{Ex}\mathcal{S}|) \subseteq Q_2$. This, together with (ii), implies (iii). The last statement is a direct consequence of (i) and (ii). □

Proposition 8.9. *We have*

$$H^*(Q_1, Q_2) \cong H^*(|\operatorname{Cl}\mathcal{S}|, |\operatorname{Ex}\mathcal{S}|).$$

Proof: By Proposition 8.8 the mapping $\psi : (Q_1, Q_2) \to (|\operatorname{Cl}\mathcal{S}|, |\operatorname{Ex}\mathcal{S}|)$ is a continuous surjection with $\psi^{-1}(|\operatorname{Ex}\mathcal{S}|) = Q_2$. By Proposition 8.7, ψ has contractible, and hence acyclic fibers. Moreover, ψ is proper, that is, the counterimages of compact sets under ψ are compact. Therefore, the map ψ is a Vietoris map. By the Vietoris-Begle mapping theorem for the pair of spaces we conclude that

$$\psi^* : H^*(|\operatorname{Cl}\mathcal{S}|, |\operatorname{Ex}\mathcal{S}|) \to H^*(Q_1, Q_2)$$

is an isomorphism, which completes the proof. □

Figure 13 shows an example of the pairs (Q_1, Q_2) and $(|\operatorname{Cl}\mathcal{S}|, |\operatorname{Ex}\mathcal{S}|)$ in Proposition 8.9.

Proof of Theorem 5.4: Theorem 5.4 is an immediate consequence of Proposition 8.1, Proposition 8.9, and Theorem 5.3. □

9. Proof of Theorem 5.5

In order to prove Theorem 5.5 we first establish a few auxiliary lemmas. Then we recall some results concerning the correspondence of solutions for $\Pi_{\mathcal{V}}$ and F. We then use this correspondence to prove an auxiliary theorem and finally present the proof of Theorem 5.5.

9.1. Auxiliary lemmas.

First observe that Theorem 5.1 applies to the set $N_\beta(\mathcal{S})$ given by (15) for any β which satisfies $0 < \beta < 1/(d+1)$.

Lemma 9.1. *We have*

$$N_\varepsilon \cap \langle \mathcal{S} \rangle = N_\varepsilon \cap |\operatorname{Cl} \mathcal{S}|.$$

In particular, $N_\varepsilon \cap \langle \mathcal{S} \rangle$ *is closed.*

Proof: Clearly $N_\varepsilon \cap \langle \mathcal{S} \rangle \subseteq N_\varepsilon \cap |\operatorname{Cl} \mathcal{S}|$. To prove the opposite inclusion, assume to the contrary that there exists an $x \in N_\varepsilon \cap |\operatorname{Cl} \mathcal{S}|$ and $x \notin N_\varepsilon \cap \langle \mathcal{S} \rangle$. Then, by Proposition 3.1(ii), $x \in N_\varepsilon \cap |\operatorname{Ex} \mathcal{S}|$. Consider simplices $\sigma \in \operatorname{Ex} \mathcal{S}$ and $\tau \in \mathcal{S}$ such that $x \in \overset{\circ}{\sigma}$ and $x \in \operatorname{cl} \langle \tau \rangle_\varepsilon$. Since for any vertex $v \in \tau$ we have $t_v(x) \geq \varepsilon > 0$, the inclusion $v \in \sigma$ has to hold. Hence, $\tau \subseteq \sigma$. Therefore, by the closedness of $\operatorname{Ex} \mathcal{S}$ we get $\tau \in \operatorname{Ex} \mathcal{S}$, a contradiction. □

Lemma 9.2. *For any* $x \in N_\varepsilon \cap \langle \mathcal{S} \rangle$ *we have* $\sigma^\varepsilon_{max}(x) \in \mathcal{S}$.

Proof: Fix a point $x \in N_\varepsilon \cap \langle \mathcal{S} \rangle$. Then there exist simplices $\tau, \sigma \in \mathcal{S}$ such that $x \in \operatorname{cl} \langle \tau \rangle_\varepsilon$ and $x \in \overset{\circ}{\sigma}$. Clearly, one has $\sigma = \sigma^0_{min}(x)$. By Corollary 4.4 and Lemma 4.1 we then obtain $\tau \subseteq \sigma^\varepsilon_{max}(x) \subseteq \sigma$. Therefore, the closedness of $\operatorname{Ex} \mathcal{S}$ implies that $\sigma^\varepsilon_{max}(x) \in \mathcal{S}$. □

Lemma 9.3. *We have*

$$N_\varepsilon \cap \langle \mathcal{S} \rangle \subseteq N_\delta \cap \langle \mathcal{S} \rangle.$$

Proof: Fix a point $x \in N_\varepsilon \cap \langle \mathcal{S} \rangle$. Then we have $\sigma^0_{min}(x) \in \mathcal{S}$. By Lemma 9.2 we further obtain $\sigma^\varepsilon_{max}(x) \in \mathcal{S}$. In addition, Lemma 4.1 immediately implies the inclusions $\sigma^\varepsilon_{max}(x) \subseteq \sigma^\delta_{min}(x) \subseteq \sigma^\delta_{max}(x) \subseteq \sigma^0_{min}(x)$. Now the closedness of $\operatorname{Ex} \mathcal{S}$ yields $\sigma^\delta_{max}(x) \in \mathcal{S}$, and consequently, $x \in N_\delta$, which completes the proof. □

9.2. Solution correspondence.

In the sequel we need two results on the correspondence of solutions of the combinatorial flow $\Pi_{\mathcal{V}}$ and the associated multivalued dynamical system F. We recall them from [18]. We begin with a definition.

Definition 9.4. *(see* [18, Definition 5.2]*)*

(a) *Let* $\varrho : \mathbb{Z} \to \mathcal{X}$ *denote a full solution of the combinatorial flow* $\Pi_{\mathcal{V}}$*. Then the* reduced solution $\varrho^* : \mathbb{Z} \to \mathcal{X}$ *is obtained from* ϱ *by removing* $\varrho(k+1)$ *whenever* $\varrho(k+1)$ *is the target of an arrow of* $\mathcal{V}$ *whose source is* $\varrho(k)$*.*

(b) *Conversely, let* $\varrho^* : \mathbb{Z} \to \mathcal{X}$ *denote an arbitrary sequence of simplices in* $\mathcal{X}$*. Then its* arrowhead extension $\varrho : \mathbb{Z} \to \mathcal{X}$ *is defined as follows. If* $\varrho^*(k) \in \operatorname{dom} \mathcal{V} \setminus \operatorname{Fix} \mathcal{V}$ *and if* $\varrho^*(k+1) \neq \varrho^*(k)^+$*, then we insert* $\varrho^*(k)^+$ *between* $\varrho^*(k)$ *and* $\varrho^*(k+1)$*. In other words, the arrowhead extension* ϱ *is obtained from* ϱ^* *by inserting missing targets of arrows.*

Theorem 9.5. *(see* [18, Theorem 5.3]*) Let*

$$X^\varepsilon := \bigcup_{\sigma \in \mathcal{X}} \langle \sigma \rangle_\varepsilon \subseteq X$$

denote the union of all open ε*-cells of* X*. Then the following hold.*

(a) *Let* $\varrho : \mathbb{Z} \to \mathcal{X}$ *denote a full solution of the combinatorial flow* $\Pi_\mathcal{V}$*. Furthermore, let* $\varrho^* : \mathbb{Z} \to \mathcal{X}$ *denote the reduced solution as in Definition 9.4(a). Then there is a function* $\varphi : \mathbb{Z} \to X^\varepsilon$ *such that for* $k \in \mathbb{Z}$ *we have*

$$\varphi(k+1) \in F(\varphi(k)) \quad \text{and} \quad \varphi(k) \in \langle \varrho^*(k) \rangle_\varepsilon .$$

In other words, φ *is an orbit of* F *which follows the dynamics of the combinatorial simplicial solution* ϱ *after removing arrowheads.*

(b) *Conversely, let* $\varphi : \mathbb{Z} \to X^\varepsilon$ *denote a full solution of* F *which is completely contained in* X^ε*. Let* $\varrho^*(k) = \sigma^\varepsilon_{max}(\varphi(k))$ *for* $k \in \mathbb{Z}$*, and let* $\varrho : \mathbb{Z} \to \mathcal{X}$ *denote the arrowhead extension of* ϱ *as in Definition 9.4(b). Then* ϱ *is a solution of the combinatorial flow* $\Pi_\mathcal{V}$*.*

□

Lemma 9.6. *(see* [18, Lemma 4.9]*) For all simplices* $\sigma \in \mathcal{X}$ *and all points* $x \in X$ *we have* $F_\sigma(x) \subseteq \sigma^+$*.* □

Theorem 9.7. *(see* [18, Theorem 5.4]*) Let* $\varphi : \mathbb{Z} \to X$ *denote an arbitrary full solution of the multivalued map* F *and let*

$$\varrho^*(k) = \sigma^\varepsilon_{max}(\varphi(k)) \tag{56}$$

for $k \in \mathbb{Z}$*. Extend this sequence of simplices in the following way:*

(1) *For all* $k \in \mathbb{Z}$ *with* $\varphi(k) \notin |\varrho^*(k-1)^+|$*, we choose a face* $\tau \subseteq \varrho^*(k-1)$ *such that* $\varphi(k) \in |\operatorname{Cl} \tau^+ \setminus \{\tau\}|$*, and then insert* τ *between* $\varrho^*(k-1)$ *and* $\varrho^*(k)$*.*
(2) *Let* $\varrho : \mathbb{Z} \to \mathcal{X}$ *denote the arrowhead extension of the sequence created in* (1)*, according to Definition 9.4(b).*

Then the so-obtained simplex sequence $\varrho : \mathbb{Z} \to \mathcal{X}$ *is a solution of the combinatorial flow* $\Pi_\mathcal{V}$*.* □

9.3. **Invariance.**

Lemma 9.8. *The set* $N_\varepsilon \cap \langle \mathcal{S} \rangle$ *is negatively invariant with respect to* F*, that is*

$$\operatorname{Inv}^-_F(N_\varepsilon \cap \langle \mathcal{S} \rangle) = N_\varepsilon \cap \langle \mathcal{S} \rangle.$$

Proof: Obviously, it suffices to prove that for every $y \in N_\varepsilon \cap \langle \mathcal{S} \rangle$ there exists an $x \in N_\varepsilon \cap \langle \mathcal{S} \rangle$ such that $y \in F(x)$. To verify this, fix a $y \in N_\varepsilon \cap \langle \mathcal{S} \rangle$. Let $\sigma \in \mathcal{S}$ be such that $y \in \mathring{\sigma}$. We will consider several cases concerning the simplex σ. First assume that $\sigma \in \operatorname{Fix} \mathcal{V}$, that is, $\sigma = \sigma^- = \sigma^+$. Take any $x \in \langle \sigma \rangle_\varepsilon \cap \sigma \subseteq N_\varepsilon \cap \langle \mathcal{S} \rangle$. Since $\sigma^\varepsilon_{max}(x) = \sigma^- = \sigma^+$, the definition of F (see (14)) shows that $F_\sigma(x) = \sigma$. Hence, $y \in F_\sigma(x) \subseteq F(x)$.

Now assume that $\sigma^- \neq \sigma^+$. Note that if $\sigma = \sigma^+$ then we have $\sigma = A_\sigma \cup B_\sigma$, and if $\sigma = \sigma^-$, then $\sigma \subseteq A_\sigma$. Hence, either $y \in A_\sigma$ or $\sigma = \sigma^+$ and $y \in B_\sigma$. In the latter case we may take any point $x \in \langle \sigma \rangle_\varepsilon \cap \sigma \subseteq N_\varepsilon \cap \langle \mathcal{S} \rangle$, because in that case one has $\sigma = \sigma^\varepsilon_{max}(x) = \sigma^\varepsilon_{max}(x)^+ \neq \sigma^\varepsilon_{max}(x)^-$, which immediately yields the inclusion $y \in B_\sigma = F_\sigma(x) \subseteq F(x)$.

It remains to consider the case $y \in A_\sigma$. Since $\mathcal{S}$ is invariant with respect to $\Pi_\mathcal{V}$ and $\sigma^+ = \mathcal{V}(\sigma^-)$, there exists a trajectory ϱ of $\mathcal{V}$ in $\mathcal{S}$ which contains σ^- and σ^+ as consecutive simplices. Let τ denote the simplex in this solution which precedes the tail $\sigma^- \in \operatorname{dom} \mathcal{V}$. Then $\tau \in \mathcal{S}$ and, according to the definition of the multivalued flow $\Pi_\mathcal{V}$, we have $\sigma^- \subsetneq \tau \neq \sigma^+$. Now let k denote the number of vertices in $\tau \setminus \sigma$ and let $x \in X$ be the point with the barycentric coordinates given by

$$t_v(x) := \begin{cases} \varepsilon & \text{if } v \in \tau \setminus \sigma, \\ 1 - k\varepsilon & \text{if } v \in \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have both $x \in \operatorname{cl} \langle \tau \rangle_\varepsilon \cap \operatorname{cl} \langle \sigma^- \rangle_\varepsilon \cap \tau$ and $\sigma^\varepsilon_{max}(x) = \tau$, and this in turn implies $\sigma^\varepsilon_{max}(x)^+ \neq \sigma^-$ and $\sigma^\varepsilon_{max}(x)^- \neq \sigma^-$. Therefore, $F_{\sigma^-}(x) = A_{\sigma^-} = A_\sigma$, which shows that $y \in F_{\sigma^-}(x) \subseteq F(x)$. $\square$

Lemma 9.9. *The set $N_\varepsilon \cap \langle \mathcal{S} \rangle$ is positively invariant with respect to F, that is*

$$\operatorname{Inv}^+_F(N_\varepsilon \cap \langle \mathcal{S} \rangle) = N_\varepsilon \cap \langle \mathcal{S} \rangle.$$

Proof: For the proof it is enough to justify that for any point $x \in N_\varepsilon \cap \langle \mathcal{S} \rangle$ we have $F(x) \cap N_\varepsilon \cap \langle \mathcal{S} \rangle \neq \varnothing$. Let $x \in N_\varepsilon \cap \langle \mathcal{S} \rangle$ be fixed and let $\sigma := \sigma^\varepsilon_{max}(x)$. By Lemma 9.2 we have $\sigma \in \mathcal{S}$. Then $x \in \operatorname{cl} \langle \sigma \rangle_\varepsilon$. Since $F(\langle \sigma \rangle_\varepsilon) \subseteq F(\operatorname{cl} \langle \sigma \rangle_\varepsilon)$ and F is strongly upper semicontinuous by Theorem 4.6, without loss of generality we may assume that $x \in \langle \sigma \rangle_\varepsilon$. The set $\mathcal{S}$ is invariant with respect to the combinatorial flow $\Pi_\mathcal{V}$. Hence, there exists a solution $\varrho : \mathbb{Z} \to \mathcal{X}$ of $\Pi_\mathcal{V}$, which is contained in $\mathcal{S}$ and passes through σ. Furthermore, let $\varrho^* : \mathbb{Z} \to \mathcal{X}$ denote the reduced solution as defined in Definition 9.4(a). There are two possible complementary cases: $\sigma \in \operatorname{im} \varrho^*$ or $\sigma \notin \operatorname{im} \varrho^*$.

In the first case there exists a $k \in \mathbb{Z}$ with $\sigma = \varrho^*(k)$. Consider $\varphi : \mathbb{Z} \to \mathcal{X}^\varepsilon$, which is a corresponding solution with respect to F as constructed in Theorem 9.5(a). Then $\varphi(k) \in \langle \sigma \rangle_\varepsilon$ and $\varphi(k+1) \in F(\varphi(k)) \cap \langle \varrho^*(k+1) \rangle_\varepsilon$. Since the map F is constant on open ε-cells, we further obtain

$$\varphi(k+1) \in F(x) \cap \langle \varrho^*(k+1) \rangle_\varepsilon. \tag{57}$$

Due to $x \in \langle\sigma\rangle_\varepsilon$, by Lemma 9.6 we have $F(x) = F_\sigma(x) \subseteq \sigma^+ \in \mathcal{S}$, where the last inclusion follows from Proposition 3.4, as $\mathcal{S}$ is an isolated invariant set. This, along with (57), completes the proof in the case where $\sigma \in \operatorname{im}\varrho^*$.

Finally, we consider the case $\varrho(k) := \sigma \notin \operatorname{im}\varrho^*$, which immediately gives rise to the inclusion $\sigma \in \operatorname{im}\mathcal{V} \setminus \operatorname{Fix}\mathcal{V}$. In this case, the identity $\Pi_\mathcal{V}(\sigma) = \operatorname{Ex}\sigma \setminus \{\sigma^-\}$ implies $\varrho(k+1) \in \operatorname{Ex}\sigma \setminus \{\sigma^-\}$. However, we also have $F(x) = F_\sigma(x) = B_\sigma$, according to the fact that $\sigma = \sigma^\varepsilon_{max}(x)^+ \neq \sigma^\varepsilon_{max}(x)^-$. This readily furnishes the inclusions $|\varrho(k+1)| \subseteq |\operatorname{Ex}\sigma \setminus \{\sigma^-\}| \subseteq B_\sigma = F(x)$. In particular, the barycenter of $\varrho(k+1)$ belongs to $|\varrho(k+1)| \cap \langle\varrho(k+1)\rangle_\varepsilon \cap F(x) \subseteq F(x) \cap N_\varepsilon \cap \langle\mathcal{S}\rangle$. This completes the proof. □

As a straightforward consequence of Lemma 9.8 and Lemma 9.9 we obtain the following corollary.

Corollary 9.10. *The set $N_\varepsilon \cap \langle\mathcal{S}\rangle$ is invariant with respect to F, that is, we have*

$$\operatorname{Inv}_F(N_\varepsilon \cap \langle\mathcal{S}\rangle) = N_\varepsilon \cap \langle\mathcal{S}\rangle.$$

□

9.4. **An auxiliary theorem and lemma.** The following characterization of the set $S(\mathcal{S}) = \operatorname{Inv} N_\delta$ is needed in the proof of Theorem 5.5.

Theorem 9.11. *We have*

$$\operatorname{Inv} N_\delta = N_\varepsilon \cap \langle\mathcal{S}\rangle.$$

Proof: According to Lemma 9.3 and Corollary 9.10 we immediately obtain the inclusion $N_\varepsilon \cap \langle\mathcal{S}\rangle \subseteq \operatorname{Inv}_F N_\delta$. Therefore, it suffices to verify the opposite inclusion.

To accomplish this, take an $x \in N_\delta \setminus (N_\varepsilon \cap \langle\mathcal{S}\rangle)$. If $x \in N_\delta \setminus N_\varepsilon$ then $x \in \operatorname{cl}\langle\tau\rangle_\delta$ for some simplex $\tau \in \mathcal{S}$, and $\sigma^\varepsilon_{max}(x) \notin \mathcal{S}$. This, according to Lemma 6.2, implies $F(x) \cap N_\delta = \varnothing$. If $x \in N_\delta \setminus \langle\mathcal{S}\rangle$, then we again obtain $F(x) \cap N_\delta = \varnothing$ as a consequence of Lemma 6.4. Both cases show that there is no solution with respect to F passing through x and contained in N_δ, which means that $\operatorname{Inv}_F N_\delta \subseteq N_\varepsilon \cap \langle\mathcal{S}\rangle$, and therefore completes the proof. □

Note that by Theorem 9.11 the sets M_r can be alternatively expressed as

$$M_r = \operatorname{Inv}_F(N^r_\delta),$$

where $N^r_\delta = \bigcup_{\sigma \in \mathcal{M}_r} \operatorname{cl}\langle\sigma\rangle_\delta$.

Lemma 9.12. *Let $\varphi : \mathbb{Z} \to X$ be a solution for the multivalued map F. Assume that the sequence of simplices $\varrho^* : \mathbb{Z} \to \mathcal{X}$ and $\varrho : \mathbb{Z} \to \mathcal{X}$ define a corresponding solution of the combinatorial flow $\Pi_\mathcal{V}$, as introduced in Theorem 9.7. If $\mathcal{S}$ is an isolated invariant set with respect to $\Pi_\mathcal{V}$, and if there exists a integer $k \in \mathbb{Z}$ such that $\varrho^*(k), \varrho^*(k+1) \in \mathcal{S}$, and if each simplex in the extended solution ϱ between the simplices $\varrho^*(k)$ and $\varrho^*(k+1)$ belongs to $\mathcal{S}$, then $\varphi(k+1) \in \langle\mathcal{S}\rangle \cap N_\varepsilon$.*

Proof: Observe that by (56) we have the inclusion $\varphi(k+1) \in \langle \varrho^*(k+1) \rangle_\varepsilon$, and since $\varrho^*(k+1) \in \mathcal{S}$ we get $\varphi(k+1) \in N_\varepsilon$. We need to verify that $\varphi(k+1) \in \langle \mathcal{S} \rangle$. Let $\sigma_i = \sigma^\varepsilon_{max}(\varphi(i)) = \varrho^*(i)$ for $i \in \mathbb{Z}$. Then we have to consider the following two complementary cases: $\varphi(k+1) \in \sigma_k^+$ and $\varphi(k+1) \notin \sigma_k^+$.

The first case immediately shows that $\varphi(k+1) \in |\operatorname{Cl} \mathcal{S}|$, as $\sigma_k \in \mathcal{S}$ implies the inclusion $\sigma_k^+ \in \mathcal{S}$ according to the assertion that $\mathcal{S}$ is an isolated invariant set (cf. Proposition 3.3). Since we also have $\varphi(k+1) \in N_\varepsilon$, the inclusion $\varphi(k+1) \in \langle \mathcal{S} \rangle$ follows from Lemma 9.1.

Consider now the second case $\varphi(k+1) \notin \sigma_k^+$. According to Theorem 9.7(1) we have the inclusion $\varphi(k+1) \in |\operatorname{Cl} \tau^+|$, where τ is a simplex in the extended solution ϱ which lies between the simplices $\varrho^*(k)$ and $\varrho^*(k+1)$. According to our assumption τ belongs to $\mathcal{S}$, hence so does τ^+, as $\mathcal{S}$ is isolated and invariant. Consequently, $\varphi(k+1) \in |\operatorname{Cl} \mathcal{S}|$ which, along with the inclusion $\varphi(k+1) \in N_\varepsilon$ and Lemma 9.1, implies $\varphi(k+1) \in \langle \mathcal{S} \rangle$. This completes the proof. $\square$

9.5. **Proof of Theorem 5.5.** First note that the sets M_r of M are mutually disjoint, which is a consequence of the mutual disjointness of the sets in the family $\mathcal{M}$ and the definition of the sets M_r. Moreover, by Theorem 5.1 and Theorem 9.11, they are isolated invariant sets with respect to the map F. Hence, condition (a) of Definition 3.9 holds.

We now verify condition (b) of Definition 3.9. Let $\varphi : \mathbb{Z} \to X$ be an arbitrary solution of the multivalued map F. Let $\varrho : \mathbb{Z} \to \mathcal{X}$ denote a corresponding solution of the multivalued flow $\Pi_\mathcal{V}$ as constructed in Theorem 9.7. Since $\mathcal{M}$ is a Morse decomposition of $\mathcal{V}$, there exist two indices $r, r' \in \mathbb{P}$ with $r' \geq r$, such that the inclusions $\alpha(\varrho) \subseteq \mathcal{M}_{r'}$ and $\omega(\varrho) \subseteq \mathcal{M}_r$ are satisfied.

Let us first focus on the ω-limit set of φ. The inclusion $\omega(\varrho) \subseteq \mathcal{M}_r$ implies that there exists a $k' \in \mathbb{Z}^+$ such that $\varrho(n) \in \mathcal{M}_r$ for all $n \geq k'$. Passing to the reduced solution ϱ^* we infer that $\varrho^*(n) \in \mathcal{M}_r$ for large enough n. Let $k \in \mathbb{Z}^+$ be such that both $\varrho(n) \in \mathcal{M}_r$ and $\varrho^*(n) \in \mathcal{M}_r$ hold for $n \geq k$. Then Lemma 9.12 implies the inclusion $\varphi(k+1) \in M_r$. Applying Lemma 9.12 another time, we further obtain the inclusion $\varphi([k+1, +\infty)) \subseteq M_r$, which in combination with the closedness of M_r yields $\omega(\varphi) \subseteq M_r$.

For the set $\alpha(\varphi)$ a similar argument applies. Indeed, the inclusion $\alpha(\varrho) \subseteq \mathcal{M}_{r'}$ establishes the existence of $k \in \mathbb{Z}^+$ with $\varrho(-n) \in \mathcal{M}_{r'}$ and $\varrho^*(-n) \in \mathcal{M}_{r'}$ for all integers $n \geq k$. Applying Lemma 9.12, this time to the arguments $-(k+1)$ and $-k$, we can further deduce that $\varphi(-k) \in M_{r'}$. Now, by the reverse recurrence and Lemma 9.12, we have $\varphi((-\infty, -k]) \subseteq M_r$, and the inclusion $\alpha(\varphi) \subseteq M_{r'}$ follows.

Next, we verify condition (c) of Definition 3.9. Suppose that φ is a full solution of F such that $\alpha(\varphi) \cup \omega(\varphi) \subseteq M_r$ for some $r \in \mathbb{P}$. Consider the corresponding solution $\varrho : \mathbb{Z} \to \mathcal{X}$ of the multivalued flow $\Pi_\mathcal{V}$, as constructed in Theorem 9.7. Since $\mathcal{M}$ is a Morse decomposition with respect to $\Pi_\mathcal{V}$, we have $\alpha(\varrho) \subseteq \mathcal{M}_{r_1}$ and $\omega(\varrho) \subseteq \mathcal{M}_{r_2}$, for some $r_1, r_2 \in \mathbb{P}$. Then the argument used for the proof of condition (b) shows that the two inclusions $\alpha(\varphi) \subseteq M_{r_1}$ and $\omega(\varphi) \subseteq M_{r_2}$ are satisfied. This immediately yields $r_1 = r_2 = r$, as M is a family of disjoint sets. Thus, we have $\alpha(\varrho) \cup \omega(\varrho) \subseteq \mathcal{M}_r$. Since $\mathcal{M}$ is a Morse decomposition, the inclusion $\operatorname{im} \varrho \subseteq \mathcal{M}_r$ follows, and consequently $\varrho(k) \in \mathcal{M}_r$ and $\varrho^*(k) \in \mathcal{M}_r$ for all $k \in \mathbb{Z}$. Again, by the recurrent argument with respect to k in both forward and backward directions and Lemma 9.12 we conclude that $\operatorname{im} \varphi \subseteq M_r$. This completes the proof that the collection M is a Morse decomposition of X with respect to F.

The Conley indices of $\mathcal{M}_r$ and M_r coincide by Theorem 5.4. The fact that the Conley-Morse graphs coincide as well follows from Theorems 9.7 and 9.5(a) via an argument similar to the argument for condition (b) of Definition 9.4 and is left to the reader. □

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