Homology of representable sets*

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Abstract

We generalize the notion of cubical homology to the class of locally compact representable sets in order to propose a new convenient method of reducing the complexity of a set while computing its homology.

1 Introduction

In this note we present a new reduction method for preprocessing homology computations. The method is based on the notion of the homology of representable sets modelled on the single space homology theory (see [6, 11]). In this theory there is no need of relative homology for building the long exact sequence. We make use of this in a purely combinatorial and self-contained setting. Such an approach enables us to perform reduction process deeper than in the classical case of the cubical homology.

The need for very efficient homology algorithms arises from many applications, in particular from applications to rigorous numerics of dynamical systems, where one can easily encounter cubical sets consisting of millions of cubes. The existing methods are not good enough to find homology of such sets in reasonable time. The classical algorithm for homology computations is based on Smith diagonalization of the matrix of the boundary homomorphism [13, Section 1.11]. The computational complexity of the best available Smith diagonalization algorithm is $O(n^{3.376\cdots})$ ([14]). Various alternatives for

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the classical approach have also been developed [7, 1, 5, 3, 15]. Delfinado and Edelsbrunner [2] present an algorithm for Betti numbers which runs in near linear time, however the applicability of this algorithm is restricted to dimension three. Some improvements in the Smith algorithm may be obtained by applying probabilistic methods [3, 4]. Unfortunately, such methods cannot be used in rigorous numerics.

Smith diagonalization is a purely algebraic method. To speed up algorithmic homology computations one may consider methods specific to computational topology. The methods of chain complex reduction originated in [9] and then developed in [10, 8, 12] constitute such an approach. They consist in iterating the process of replacing the chain complex by a smaller one with the same homology and computing the homology only when no more reductions are possible. This way one postpones the process of computing the homology of the chain complex until the complex is acceptably small.

In this paper we show how one can benefit from the elementary reductions in homology computations by extending the cubical homology introduced in [8] to locally compact representable sets. This larger class of sets enables us to perform reductions significantly deeper than in the realm of cubical homology. This results in much faster homology algorithms for cubical sets.

This paper concentrates on the theoretical foundations of the new method. Details of the algorithm, numerical experiments and generalizations are in progress and will be published elsewhere.

2 Homology of representable sets

In what follows an essential use will be made of the notion of the cubical homology introduced in [8].

If I is an interval then the associated *elementary cell* is (cf. [8], Definition 2.13)

$$\overset{\circ}{I} = \begin{cases} (l, l+1) & \text{if } I = [l, l+1], \\ [l] & \text{if } I = [l, l]. \end{cases}$$

For a general elementary cube $Q = I_1 \times I_2 \times \ldots \times I_d \subset \mathbb{R}^d$ we define the associated *elementary cell* as

$$\overset{\circ}{Q} = \overset{\circ}{I}_1 \times \overset{\circ}{I}_2 \times \ldots \times \overset{\circ}{I}_d \,.$$

We let the dimension of $\mathring{Q}$ to be the number of nondegenerate components in Q. Elementary cells in $\mathbb{R}^2$ are shown in Figure 1.

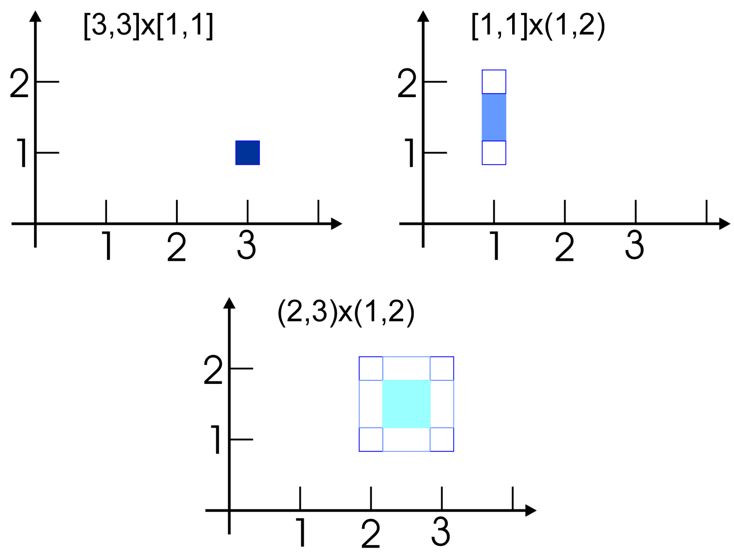

Figure 1: Cells in $\mathbb{R}^2$: 0-dimensional, 1-dimensional and 2-dimensional.

Definition 1 (cf. [8], Definition 6.1) A set $X \subset \mathbb{R}^d$ will be called *representable* if it is a finite union of elementary cells (see Figure 2). Let us denote the set of all k-dimensional cells of X by

$$\mathring{\mathcal{K}}_k(X) := \{\mathring{Q} \subset X : \dim \mathring{Q} = k\}$$

and the set of all cells of X by

$$\mathring{\mathcal{K}}(X) := \bigcup_{k=0}^{d} \mathring{\mathcal{K}}_k(X)$$

It is obvious that every cubical set is representable but the converse is not true.

We are going to endow a representable set X with an algebraic structure identifying any elementary cell $\mathring{Q} \in \mathring{\mathcal{K}}(X)$ with an algebraic object $\hat{Q}$. Let

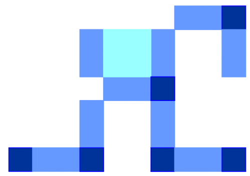

Figure 2: A representable set in $\mathbb{R}^2$.

$\hat{\mathcal{K}}(X) := \{\hat{Q} : \overset{\circ}{Q} \in \overset{\circ}{\mathcal{K}}(X)\}$. A finite sum

$$c = \sum_{i=1}^{m} \alpha_i \hat{Q}_i, \quad \text{for } \hat{Q}_i \in \hat{\mathcal{K}}_k(X), \alpha_i \in \mathbb{Z}$$

will be called a *k-chain of a representable set X*. By $C_k(X)$ we will denote the set of all k-chains of X. Let us observe that $C_k(X)$ is an abelian free group generated by $\hat{\mathcal{K}}_k(X)$, with respect to the ordinary chain addition as a group operation. Going towards the homology we connect the algebra and the topology of X defining the boundary of a chain. Observe that at least on the level of algebra it doesn't matter if a given chain is generated by elementary cells or elementary cubes. Thus, one may treat any chain of a representable set as a cubical one. Keeping this in mind we define the boundary $\overset{\circ}{\partial}{}_k^X(c)$ of a chain $c \in C_k(X)$ as a cubical boundary $\partial_k(c)$ projected onto $\overset{\circ}{\mathcal{K}}_{k-1}(X)$. For instance, if $X = (1,2]$ we let $\overset{\circ}{\partial}{}_1^X(\widehat{(1,2)}) = \widehat{[2]}$.

Definition 2 Given $k \in \mathbb{Z}$, we define a *cell boundary operator*

$$\overset{\circ}{\partial}{}_k^X : C_k(X) \to C_{k-1}(X)$$

by

$$\overset{\circ}{\partial}{}_k^X(c) := \sum_{\overset{\circ}{Q} \in \overset{\circ}{\mathcal{K}}_{k-1}(X)} \langle \partial_k(c), \hat{Q} \rangle \hat{Q} \tag{1}$$

where ∂_k stands for the ordinary cubical boundary operator and $\langle \cdot, \cdot \rangle$ denotes the scalar product of chains, i.e. $\langle c_1, c_2 \rangle := \sum_{i=1}^{m} \alpha_i \beta_i$ for $c_1 = \sum_{i=1}^{m} \alpha_i \hat{Q}_i$ and $c_2 = \sum_{i=1}^{m} \beta_i \hat{Q}_i$.

If X is a cubical set then $\overset{\circ}{\partial}^X \equiv \partial^X$. Thus, we simplify the notation and omit upperscript "$\circ$". Similarly, if the set X will be clear from the context we will write ∂ instead of ∂^X.

By the linearity of both the cubical boundary operator and the scalar product of chains, we infer that

Proposition 1 *$\partial_k : C_k(X) \to C_{k-1}(X)$ is linear.*

One of the most required property of the boundary operator is $\partial \circ \partial \equiv 0$. Observe that in general it is not true.

Example 1 Let us consider a representable set $X = [2,3] \times [1,2] \setminus (2,3) \times [1]$. Then

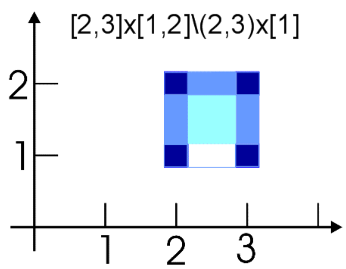

Figure 3: $X = [2,3] \times [1,2] \setminus (2,3) \times [1]$.

$$\partial(\partial((2,3) \widehat{\times} (1,2))) = [2] \widehat{\times} [1] - [3] \widehat{\times} [1].$$

We write $Q \prec R$ if a cube Q is a proper face of a cube R, i.e. $Q \subset R$ and $Q \neq R$.

Lemma 1 *A representable set $X \subset \mathbb{R}^d$ is locally compact if and only if for arbitrary cubes $Q \prec R \prec S$ with $\overset{\circ}{Q}, \overset{\circ}{S} \in \overset{\circ}{\mathcal{K}}(X)$ we have $\overset{\circ}{R} \in \overset{\circ}{\mathcal{K}}(X)$.*

Proof. Assume that X is locally compact and consider arbitrary cubes $Q \prec R \prec S$ with $\overset{\circ}{Q}, \overset{\circ}{S} \in \overset{\circ}{\mathcal{K}}(X)$. We need to prove that $\overset{\circ}{R} \in \overset{\circ}{\mathcal{K}}(X)$. Suppose the contrary and consider $z \in \overset{\circ}{Q}$. There exists a sequence $(z_n)_{n \in \mathbb{N}}$ in $\overset{\circ}{R}$ such that $z = \lim_{n \to \infty} z_n$ as $z \in \overset{\circ}{Q} \subset R = \operatorname{cl} \overset{\circ}{R}$. Now, since for each $n \in \mathbb{N}$ $z_n \in \overset{\circ}{R} \subset S = \operatorname{cl} \overset{\circ}{S}$, there exists a sequence $(z_{n_m})_{m \in \mathbb{N}}$ in $\overset{\circ}{S}$ convergent to z_n. Thus, in an arbitrary neighborhood of $z \in X$ there exists a limit of points in X which does not belong to X. This means that X is not locally compact and brings a contradiction.

To prove the converse implication assume that for arbitrary cubes $Q \prec R \prec S$ with $\overset{\circ}{Q}, \overset{\circ}{S} \in \overset{\circ}{\mathcal{K}}(X)$ we have $\overset{\circ}{R} \in \overset{\circ}{\mathcal{K}}(X)$. If $\mathrm{cl}X \setminus X$ is closed then X is locally compact as a difference of compact sets. Thus, assume the contrary and consider $\overset{\circ}{R} \subset \mathrm{cl}X \setminus X$ with $R \cap X \neq \emptyset$. One can find $Q \subset R$ such that $\overset{\circ}{Q} \subset X$. On the other hand there exists $\overset{\circ}{S} \subset X$ with $R \subset S$, as $\overset{\circ}{R} \subset \mathrm{cl}X$. But this results in $\overset{\circ}{R} \subset X$ which contradicts our assumption. □

Theorem 1 *If $X \subset \mathbb{R}^d$ is representable and locally compact then $\partial \circ \partial \equiv 0$.*

Proof. Since ∂ is linear, it suffices to prove that $\partial_{k-1}(\partial_k(\hat{S})) = 0$ for any given elementary cell $\overset{\circ}{S} \in \overset{\circ}{\mathcal{K}}(X)$, $k \in \mathbb{Z}$. Let us observe that

$$\begin{aligned}
\partial^S_{k-1}(\partial^S_k(\hat{S})) &= \partial^S_{k-1}\left(\sum_{R \in \mathcal{K}_{k-1}(S)} \langle \partial^S_k \hat{S}, \hat{R} \rangle \hat{R}\right) \\
&= \sum_{R \in \mathcal{K}_{k-1}(S)} \langle \partial^S_k \hat{S}, \hat{R} \rangle \sum_{Q \in \mathcal{K}_{k-2}(S)} \langle \partial^S_{k-1} \hat{R}, \hat{Q} \rangle \hat{Q} \\
&= \sum_{Q \in \mathcal{K}_{k-2}(S)} \left(\sum_{R \in \mathcal{K}_{k-1}(S)} \langle \partial^S_k \hat{S}, \hat{R} \rangle \langle \partial^S_{k-1} \hat{R}, \hat{Q} \rangle\right) \hat{Q} \\
&= \sum_{Q \in \mathcal{K}_{k-2}(S)} \left(\sum_{R \in \mathcal{K}_{k-1}(S): Q \prec R \prec S} \langle \partial^S_k \hat{S}, \hat{R} \rangle \langle \partial^S_{k-1} \hat{R}, \hat{Q} \rangle\right) \hat{Q}.
\end{aligned}$$

This means that for an arbitrary elementary cube Q with $\dim Q = k-2$ we have

$$\sum_{R \in \mathcal{K}_{k-1}(S): Q \prec R \prec S} \langle \partial^S_k \hat{S}, \hat{R} \rangle \langle \partial^S_{k-1} \hat{R}, \hat{Q} \rangle = 0 \tag{2}$$

as $\partial^S \circ \partial^S \equiv 0$ in a cubical set S. On the other hand proceeding in a similar with respect to ∂^X we have

$$\partial^X_{k-1}(\partial^X_k(\hat{S})) = \sum_{\overset{\circ}{Q} \in \overset{\circ}{\mathcal{K}}_{k-2}(X)} \left(\sum_{\overset{\circ}{R} \in \overset{\circ}{\mathcal{K}}_{k-1}(X): Q \prec R \prec S} \langle \partial^S_k \hat{S}, \hat{R} \rangle \langle \partial^S_{k-1} \hat{R}, \hat{Q} \rangle\right) \hat{Q}.$$

Now it is enough to show that

$$\sum_{\overset{\circ}{R}\in\overset{\circ}{\mathcal{K}}_{k-1}(X):Q\prec R\prec S} \langle\partial_k^S \hat{S}, \hat{R}\rangle\langle\partial_{k-1}^S \hat{R}, \hat{Q}\rangle = 0 \tag{3}$$

for arbitrarily fixed $\overset{\circ}{Q}\in\overset{\circ}{\mathcal{K}}_{k-2}(X)$. Assume the contrary. Then comparing (2) and (3) we infer that there exists $\overset{\circ}{R}\notin\overset{\circ}{\mathcal{K}}_{k-1}(X)$ such that $Q \prec R \prec S$, which contradicts the local compactness of X and completes the proof. □

Now we are in a position to define the homology of X.

Definition 3 Let $X \subset \mathbb{R}^d$ be representable and locally compact. The *k-th cell homology group* of X is a quotient group

$$H_k(X) := Z_k(X)/B_k(X),$$

where $Z_k(X) := \ker \partial_k$ is the set of all *k-cycles of X* and $B_k(X) := \mathrm{im}\partial_{k+1}$ is the set of all *k-boundaries of X*. Observe that both $Z_k(X)$ and $B_k(X)$ are subgroups of $C_k(X)$. The collection of all homology groups of X is called *homology of X* and is denoted by

$$H(X) := \{H_k(X)\}_{k\in\mathbb{Z}}.$$

3 Exact sequence

Theorem 2 *If $X, A \subset \mathbb{R}^d$ are representable and locally compact and A is closed in X then $X \setminus A$ is representable and locally compact. Moreover, there exists a long exact sequence*

$$\ldots \longrightarrow H_k(A) \longrightarrow H_k(X) \longrightarrow H_k(X\setminus A) \longrightarrow H_{k-1}(A) \longrightarrow \ldots. \tag{4}$$

Proof. Under the assumptions of the theorem $X \setminus A$ is locally compact. By Proposition 6.3 in [8] it is representable. Consider the following sequence

$$0 \longrightarrow C(A) \xrightarrow{i} C(X) \xrightarrow{j} C(X\setminus A) \longrightarrow 0, \tag{5}$$

where $i : C(A) \to C(X)$ is an inclusion and $j : C(X) \to C(X \setminus A)$ is a linear extension of

$$j(\hat{Q}) := \begin{cases} \hat{Q} & \text{if } \hat{Q} \in \hat{\mathcal{K}}(X\setminus A) \\ 0 & \text{if } \hat{Q} \in \hat{\mathcal{K}}(A) \end{cases}$$

Since X, A and $X \setminus A$ are representable and locally compact $\{C(A), \partial^A\}$, $\{C(X), \partial\}$ and $\{C(X \setminus A), \partial^{X\setminus A}\}$ are chain complexes. It is easy to prove that i is a chain map and that the sequence (5) is exact at $C(A)$, $C(X)$ and $C(X \setminus A)$. We will show that j is a chain map. Indeed, if $\hat{Q} \in \hat{\mathcal{K}}(A)$ then $\partial_k^{X\setminus A}(j_k(\hat{Q})) = \partial_k(0) = 0$. On the other hand, since A is closed we have $\partial_k(\hat{Q}) = \sum_{P \prec Q} \alpha_P \hat{P}$ for a suitable α_P and $j_{k-1}(\partial_k(\hat{Q})) = j_{k-1}(\sum_{P\prec Q} \alpha_P \hat{P}) = \sum_{P\prec Q} \alpha_P j_{k-1}(\hat{P}) = 0$. Now let $\hat{Q} \in \hat{\mathcal{K}}(X \setminus A)$. We have

$$\begin{aligned} \partial_k(j_k(\hat{Q})) &= \partial_k(\hat{Q}) \\ &= \sum_{\overset{\circ}{P} \in \overset{\circ}{\mathcal{K}}_{k-1}(X\setminus A)} \langle \partial_k^Q(\hat{Q}), \hat{P}\rangle \hat{P} \end{aligned}$$

and

$$\begin{aligned} j_{k-1}(\partial_k(\hat{Q})) &= j_{k-1}\left(\sum_{\overset{\circ}{P} \in \overset{\circ}{\mathcal{K}}_{k-1}(X)} \langle \partial_k^Q(\hat{Q}), \hat{P}\rangle \hat{P} \right) \\ &= \sum_{\overset{\circ}{P} \in \overset{\circ}{\mathcal{K}}_{k-1}(X)} \langle \partial_k^Q(\hat{Q}), \hat{P}\rangle j_{k-1}(\hat{P}) \\ &= \sum_{\overset{\circ}{P} \in \overset{\circ}{\mathcal{K}}_{k-1}(X\setminus A)} \langle \partial_k^Q(\hat{Q}), \hat{P}\rangle \hat{P}. \end{aligned}$$

Now the exactness of (4) follows from a basic homological algebra. □

4 Reduction method

Definition 4 A representable set A will be called a 0-*space* if $H_k(A) = 0$ for each $k \in \mathbb{Z}$.

The existence of a long exact sequence results in the following Theorem.

Theorem 3 *Let $X \subset \mathbb{R}^d$ be a representable and locally compact set and let $A \subset X$ be a* 0*-space. If A is either closed or open in X then*

$$H(X) \cong H(X \setminus A).$$

Proof. If A is closed in X then by Theorem 2 the following sequence

$$\ldots \longrightarrow 0 \longrightarrow H_k(X) \xrightarrow{\varphi_*} H_k(X \setminus A) \longrightarrow 0 \longrightarrow \ldots$$

is exact. By the exactness $\varphi_* : H_k(X) \to H_k(X \setminus A)$ is an isomorphism. If A is open in X then one can apply Theorem 2 with respect to $X \setminus A$ and proceed as above to finish the proof. $\square$

The above theorem provides us with a convenient method to reduce the complexity of a set while computing its homology.

Example 2 Let us find the homology of $X = (1,3) \times [1,2]$. Step by step we eliminate closed or open 0-space (see Figure 4). Finally we get

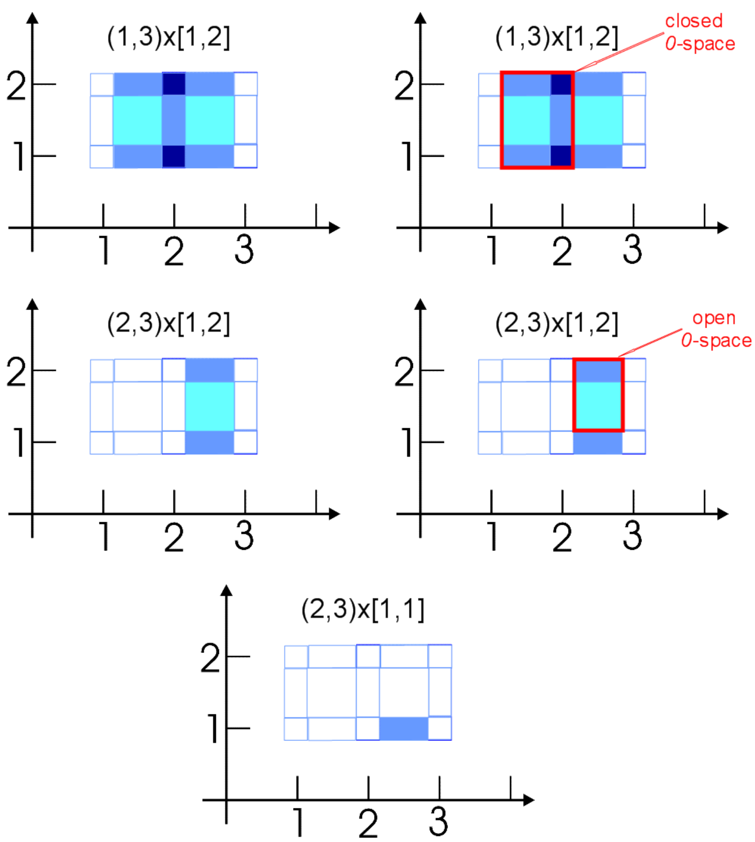

Figure 4: Reduction via 0-spaces.

$$H_k(X) \cong \begin{cases} \mathbb{Z} & \text{for } k = 1 \\ 0 & \text{for } k \neq 1. \end{cases}$$

As it was mentioned in the introduction our approach enables us to perform reduction process deeper than in the classical case of the cubical homology, which results in much faster homology algorithms for cubical sets. In order to visualize this let us consider cubical set Γ^1 shown in Figure 5. First observe that no classical reduction is possible as each vertex belongs to

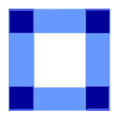

Figure 5: Γ^1

the boundary of exactly two edges, so there are no free faces. However, when we treat the cubical complex as the reduced complex, i.e. we assume that the empty set is an additional simplex of dimension -1 which is the boundary of any vertex, then the reduction via 0-spaces is possible. The resulting sequence of reductions is presented in Figure 6. Now the outcome is the

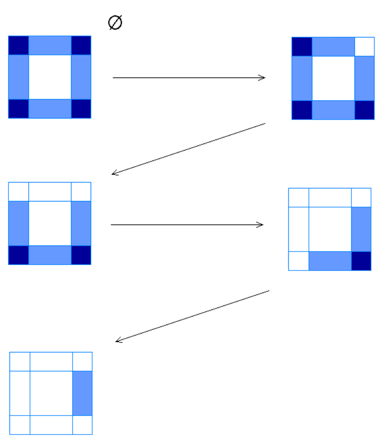

Figure 6: Reduction of Γ^1 via 0-spaces.

complex consisting of exactly one edge whose boundary is zero. Therefore this generator is also a homology generator.

5 Relative homology

As we saw in Section 3 there is no need of relative homology in building the exact sequence. In this section we show how the homology of representable sets may be compared with relative homology.

Let $X, A \subset X$ be a pair of representable and locally compact sets. Associated to these sets are groups $C(X)$ and $C(A)$ with bases $\hat{\mathcal{K}}(X)$ and $\hat{\mathcal{K}}(A)$, respectively. Since $\hat{\mathcal{K}}(A) \subset \hat{\mathcal{K}}(X)$, the quotient group $C_k(X)/C_k(A)$ is a free abelian group. Thus we have the following definitions.

Definition 5 Let X and $A \subset X$ be representable and locally compact sets. The *relative chains of X modulo A* are the elements of the free abelian groups

$$C_k(X, A) := C_k(X)/C_k(A).$$

The *relative chain complex of X modulo A* is given by

$$\{C_k(X, A), \partial_k^{(X,A)}\}$$

where $\partial_k^{(X,A)} : C_k(X, A) \to C_{k-1}(X, A)$ is the boundary map induced by the standard boundary map on $C_k(X)$. One can verify that the induced boundary map is well defined and $\partial_{k-1}^{(X,A)} \partial_k^{(X,A)} = 0$. We define *relative k-cycles* ,

$$Z_k(X, A) := \ker \partial_k^{(X,A)},$$

the *relative k-boundaries* ,

$$B_k(X, A) := \mathrm{im} \partial_{k+1}^{(X,A)}$$

and finally the *relative homology groups*

$$H_k(X, A) := Z_k(X, A)/B_k(X, A).$$

What we are going to do now is to prove that the cell homology is actually the relative homology.

Theorem 4 *Let $X, A \subset X$ be a pair of representable and locally compact sets such that $X \setminus A$ is locally compact. Then*

$$H(X, A) \cong H(X \setminus A).$$

Proof. Consider the map $f_k : C_k(X, A) \to C_k(X \setminus A)$ given by

$$f_k([c]) := \sum_{\overset{\circ}{Q} \in \overset{\circ}{\mathcal{K}}_k(X \setminus A)} \langle c, \hat{Q} \rangle \hat{Q} \quad \text{for } [c] \in C_k(X, A).$$

One can show that f is well defined and is an isomorphism. Moreover

$$f_{k-1} \partial_k^{(X,A)} = \partial^{(X \setminus A)} f_k,$$

thus f is a chain isomorphism, so it induces an isomorphism on the level of homology. □

Observe that if a set X is representable and locally compact then $\mathrm{cl} X \setminus X$ is locally compact and, by Proposition 6.3 in [8], representable. Thus, as an immediate consequence of Theorem 4 we have the following corollary.

Corollary 1 *If X is a representable and locally compact set then $H(X) \cong H(\mathrm{cl} X, \mathrm{cl} X \setminus X)$.*

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