AN ALGORITHMIC APPROACH TO THE CONLEY INDEX THEORY

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Abstract. We introduce a class of representable sets which is closed under the operations of set theoretical union, intersection, difference and topological interior and closure. We use this class to construct an algorithm which verifies if for a given dynamical system a given set is an isolating neighborhood. In case of a positive answer the algorithm constructs an index pair.

Keywords: Conley index, isolated invariant set, index pairs, algorithms, grids, discretisation

1. Introduction

The computer assisted, Conley index based proof of chaos in the Lorenz equations [4] showed the importance of the Conley index theory in rigorous numerics of dynamical systems. The proof utilized a chaos criterion developed in [3]. The role of the computer was to verify whether the assumptions of that criterion are satisfied for some region in the phase space of the Lorenz system. Since the assumptions are based on the Conley index, it was necessary to verify that the Conley indexes are defined in the appropriately chosen domains and compute the indexes. In order to translate this problem to one treatable by algorithmic means, the Conley index theory for multivalued dynamical systems ([2]) was used. The latter theory, via its combinatorial and inheritable character, forms a bridge between the continuous dynamics of a differential equation and the finite dynamics of a computer (see [5]). A similar topological approach was used in the computer assisted proofs of chaos presented in [1] and [7].

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A natural question arises to what extent the techniques developed to prove chaos in the Lorenz system may be applied to other problems in differential equations. One of the subquestions is: given a continuous dynamical system and a compact set is it possible to verify in finite time if the set is an isolating neighborhood? And if the answer is yes, is it possible to construct an index pair in finite time and find the associated Conley index? A combinatorial approach to this question is presented in [6].

The aim of the present paper is to set up a general framework in topology which lets us give positive answers to the above questions as well as some similar questions which may be raised in the future. To achieve this task we first define the class of representable sets, which generalizes the class considered in [4] and [5]. The class is closed under the operations of set theoretical union, intersection, difference and topological interior and closure. Since it is also finite, each of these five operations as well as any operation defined in terms of them, may be easily performed by means of an algorithm. In the next step we introduce multivalued representable maps. Their main feature is the fact that they preserve representable sets under the image and both weak and strong preimage and consequently these operations also may be performed by an algorithm. Next we show that the invariant part of a representable set under a multivalued dynamical system with a representable generator is a representable set which leads to the conclusion that the question if a given representable set is an isolating neighborhood may be determined algorithmically. Finally we show that a representable isolating neighborhood always admits a representable index pair and we give an explicit formula for it. The formula may be easily adapted to an algorithm computing the index pair.

In the sequel we will use the following notation. For any set X the notation $\mathcal{P}(X)$ will stand for the family of all subsets of X. If X is a metric space then $\operatorname{dist}(x, y)$ will denote the distance form x to y. If $A \subset X$, then we denote the diameter, the boundary, the interior and the closure of A respectively by $\operatorname{diam} A$, $\operatorname{bd} A, \operatorname{int} A$ and $\operatorname{cl} A$. If $x \in X$ and $r > 0$ then $B(x, r)$ will denote the closed ball of center x and radius r. Similarly, if $A \subset X$ then we put $B(A, r) := \{y \in X \mid \operatorname{dist}(y, A) \leq r\}$. The notation $f : X \multimap Y$ will stand for a partial map from X to Y, i.e. a map whose domain is not necessarily the whole space X.

2. Preliminaries

We begin with introducing some convenient notation and summarizing elementary features related to this notation.

Let X be a fixed locally compact metric space. For $\mathcal{A} \subset \mathcal{P}(X)$ and $A \subset X$ put

$$
\begin{aligned}
\operatorname{diam} \mathcal{A} &:= \sup \{ \operatorname{diam} a \mid a \in \mathcal{A} \} \\
\mathcal{A}(A) &:= \{ a \in \mathcal{A} \mid a \cap A \neq \emptyset \} \\
\mathcal{A}(x) &:= \mathcal{A}(\{ x \}) \\
|\mathcal{A}| &:= \bigcup \mathcal{A}
\end{aligned}
$$

For $\mathcal{C} \subset \mathcal{A}$ put

$$\langle \mathcal{C} \rangle := \langle \mathcal{C} \rangle_{\mathcal{A}} := \{ x \in X \mid \mathcal{A}(x) = \mathcal{C} \}$$

and for $a \in \mathcal{A}$ put

$$\langle a \rangle := \langle \{ a \} \rangle.$$

We will say that a subfamily $\mathcal{C} \subset \mathcal{A}$ is *proper* if $\langle \mathcal{C} \rangle \neq \emptyset$.

The following proposition is straightforward.

Proposition 2.1.

$$\langle\emptyset\rangle = \emptyset \tag{1}$$

$$A \subset B \;\Rightarrow\; \mathcal{A}(A) \subset \mathcal{A}(B) \tag{2}$$

$$\mathcal{A}(\bigcup_i A_i) = \bigcup_i \mathcal{A}(A_i) \tag{3}$$

$$\mathcal{A} \subset \mathcal{B} \;\Rightarrow\; |\mathcal{A}| \subset |\mathcal{B}| \tag{4}$$

$$|\bigcup_i \mathcal{A}_i| = \bigcup_i |\mathcal{A}_i| \tag{5}$$

$$A \cap |\mathcal{A}| \subset |\mathcal{A}(A)| \tag{6}$$

$$\langle\mathcal{A}\rangle \subset \bigcap \mathcal{A} \subset |\mathcal{A}| \tag{7}$$

$$x \in \langle\mathcal{A}(x)\rangle \tag{8}$$

Lemma 2.2. *If $\mathcal{C} \subset \mathcal{A}$ is proper then $\mathcal{A}(\langle\mathcal{C}\rangle) = \mathcal{C}$.*

Proof: Take $a \in \mathcal{A}(\langle\mathcal{C}\rangle)$. Then $a \cap \langle\mathcal{C}\rangle \neq \emptyset$. Let $z \in a \cap \langle\mathcal{C}\rangle$. Then $\mathcal{A}(z) = \mathcal{C}$. Since $a \in \mathcal{A}(z)$, we get $a \in \mathcal{C}$. To prove the opposite inclusion take $c \in \mathcal{C}$ and $x \in \langle\mathcal{C}\rangle$. Since $\mathcal{A}(x) = \mathcal{C}$, we have $c \in \mathcal{A}(x) \subset \mathcal{A}(\langle\mathcal{C}\rangle)$. QED

Corollary 2.3. *If $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{A}$ are proper then*

$$\langle\mathcal{C}_1\rangle = \langle\mathcal{C}_2\rangle \Leftrightarrow \mathcal{C}_1 = \mathcal{C}_2$$

Corollary 2.4. *If $\langle\mathcal{C}_1\rangle \cap \langle\mathcal{C}_2\rangle \neq \emptyset$ then $\mathcal{C}_1 = \mathcal{C}_2$.*

Proof: Let $x \in \langle\mathcal{C}_1\rangle \cap \langle\mathcal{C}_2\rangle$. Then $\mathcal{C}_1 = \mathcal{A}(x) = \mathcal{C}_2$. QED

Before we define the class of representable sets it is convenient to introduce the notion of a grid. Grids will serve as tools in constructing representable sets.

Definition 2.5. The family $\mathcal{A} \subset \mathcal{P}(X)$ will be called *a grid* in X if

$$\text{for every } a \in \mathcal{A} \text{ the set } a \text{ is compact and } \{a\} \text{ is proper} \tag{9}$$

$$\text{for every compact } K \subset X \text{ we have } 1 \leq \#\mathcal{A}(K) < \infty \tag{10}$$

$$\text{for every proper } \mathcal{C} \subset \mathcal{A} \text{ we have } \operatorname{cl}\langle\mathcal{C}\rangle = \bigcap \mathcal{C} \tag{11}$$

The simplest example of a grid is the standard γ-grid in $\mathbb{R}$ defined by

$$\mathcal{G}_\gamma^d := \mathcal{G}_\gamma := \{ [n_1\gamma, (n_1+1)\gamma] \times \ldots \times [n_d\gamma, (n_d+1)\gamma] \mid (n_1, \ldots, n_d) \in \mathbb{Z}^d \}$$

It is straightforward to check that every γ-grid is indeed a grid. Another example of a grid is provided by the following proposition.

Proposition 2.6. *If T is a triangulation of $\mathbb{R}$ then the family of all simplexes in T of dimension d is a grid in $\mathbb{R}$.*

We leave its proof to the reader.

From (9) and the local compactness of X we easily conclude the following proposition.

Proposition 2.7. *Every proper subfamily of a grid is finite.*

Here are the basic properties of a grid.

Proposition 2.8. *Assume $\mathcal{A}$ is a grid. Then*

$$\forall x \in X\ \exists V \text{ a neighborhood of } x, \text{ such that} \tag{12}$$

$$y \in V \ \Rightarrow\ \mathcal{A}(y) \subset \mathcal{A}(x) \tag{13}$$

$$\forall a \in \mathcal{A}\ \operatorname{cl}\langle a \rangle = a \tag{14}$$

$$\forall a \in \mathcal{A}\ \langle a \rangle = \operatorname{int} a \tag{15}$$

$$\forall a \in \mathcal{A}\ a = \operatorname{cl}\operatorname{int} a \tag{16}$$

$$X = |\mathcal{A}| \tag{17}$$

$$\forall a_1, a_2 \in \mathcal{A}\ \ a_1 \neq a_2 \ \Rightarrow\ \operatorname{int} a_1 \cap \operatorname{int} a_2 \neq \emptyset \tag{18}$$

$$A \subset \langle \mathcal{A}(A) \rangle \tag{19}$$

Proof: To prove (13) assume it is not true. Then there exists an $x \in X$ and a sequence $x_n \to x$ such that $\mathcal{A}(x_n) \not\subset \mathcal{A}(x)$. Without loss of generality we may assume that $\{ x_n \} \subset V$ for a compact neighborhood of x. Then $\mathcal{A}(x_n) \subset \mathcal{A}(V)$. By (10) $\mathcal{A}(V)$ is finite. Hence, by passing to an appropriate sequence we may assume that $\mathcal{A}(x_n) = \text{const} =: \mathcal{B}$. Let $a \in \mathcal{B} \setminus \mathcal{A}(x)$. Then $x_n \in a$ and $x \notin a$. However, since $x_n \to x$ we get $x \in a$, a contradiction.

Observe that by (11)

$$\operatorname{cl}\langle a\rangle = \bigcap\{\, a \,\} = a$$

which proves (14). To prove (15) consider first the inclusion $\langle a\rangle \subset \operatorname{int} a$. Let $x \in \langle a\rangle$. Applying (13) to x we can take V a neighborhood of x such that $y \in V \;\Rightarrow\; \mathcal{A}(y) \subset \mathcal{A}(x) = \{\, a \,\}$. By (8) it follows that $V \subset \langle a\rangle \subset a$, therefore $x \in \operatorname{int} a$. To prove the opposite inclusion assume it is not true. Then there exists an $x \in \operatorname{int} a \setminus \langle a\rangle$. Let $V \subset a$ be a neighborhood of x. Since $x \notin \langle a\rangle$, there exists a $b \in \mathcal{A}$, $b \neq a$ be such that $x \in b = \operatorname{cl}\langle b\rangle$. Since $b = \operatorname{cl}\langle b\rangle$, we can find a sequence $\{\, x_n \,\} \subset \langle b\rangle$ such that $x_n \to x$. Then $x_n \in V \subset a$ for almost all n and consequently $x_n \notin \langle b\rangle$, a contradiction.

Property (16) is a straightforward consequence of properties (14) and (15). Property (17) is obvious. To prove (18) observe that by Corollary 2.4

$$\operatorname{int} a_1 \cap \operatorname{int} a_2 = \langle a_1\rangle \cap \langle a_2\rangle = \emptyset$$

The last property is straightforward. QED

3. Representable Sets

We are now ready to introduce the class of representable sets. Let $\mathcal{A}$ be a fixed grid. By an *elementary representable set* over $\mathcal{A}$ we mean a set of the form $\langle\mathcal{C}\rangle$, where $\mathcal{C}$ is a proper subfamily of $\mathcal{A}$. The set A will be called *representable* over $\mathcal{A}$ if it is a finite union of elementary representable sets. It will be called *strongly representable* over $\mathcal{A}$, if it is a finite sum of a subfamily of $\mathcal{A}$. We will denote the family of elementary representable sets (representable sets, strongly representable sets) over $\mathcal{A}$ respectively by $\mathrm{ER}(\mathcal{A}), \mathrm{R}(\mathcal{A}), \mathrm{SR}(\mathcal{A})$.

Let us begin our study of representable sets by the following two simple observations.

Proposition 3.1.

$$X = \bigcup_{E\in\mathrm{ER}(\mathcal{A})} E$$

Proposition 3.2. *If $K \subset X$ is compact then*

$$\#\{\, E \in \mathrm{ER}\,\mathcal{A} \mid E \subset K \,\} < \infty.$$

The following characterization of representable sets in terms of elementary representable sets will be useful in the sequel.

Theorem 3.3. *The set A is representable if and only if the following two conditions are satisfied*

$$\mathrm{cl}\, A \text{ is compact} \tag{20}$$

$$\forall E \in \mathrm{ER}(\mathcal{A}) \;\; E \cap A \neq \emptyset \;\Rightarrow\; E \subset A. \tag{21}$$

Proof: Properties (20) and (21) follow immediately from Proposition 2.8 . On the other hand, if properties (20) and (21) are satisfied then by Proposition 3.1

$$A = A \cap X = \bigcup_{E \in \mathrm{ER}(\mathcal{A})} A \cap E = \bigcup_{E \in \mathrm{ER}(\mathcal{A}), E \subset A} E, \tag{22}$$

which shows that A is a representable set. Since $\mathrm{cl}\, A$ is compact, it follows from Proposition 3.2 that the last sum in (22) is finite. Hence A is a representable set. QED

Let us now summarize the basic properties of representable sets.

Proposition 3.4.

$$A_1, A_2 \in \mathrm{ER}(\mathcal{A}) \;\Rightarrow\; A_1 = A_2 \text{ or } A_1 \cap A_2 = \emptyset \tag{23}$$

$$A \in \mathrm{R}(\mathcal{A}) \;\Rightarrow\; A = \bigcup \{\, E \in \mathrm{ER}(\mathcal{A}) \mid E \subset A \,\} \tag{24}$$

$$A, B \in \mathrm{R}(\mathcal{A}) \;\Rightarrow\; A \cup B, A \cap B, A \setminus B \in \mathrm{R}(\mathcal{A}) \tag{25}$$

$$A, B \in \mathrm{SR}(\mathcal{A}) \;\Rightarrow\; A \cup B \in \mathrm{SR}(\mathcal{A}) \tag{26}$$

$$K \text{ compact } \;\Rightarrow\; \mathrm{card}\,\{\, A \in \mathrm{R}(\mathcal{A}) \mid A \subset K \,\} < \infty \tag{27}$$

Proof: Property (23) follows immediately from Corollary 2.4 and property (24) from Theorem 3.3.

To prove (25) take $A = A_1 \cup \cdots \cup A_n$ and $B = B_1 \cup \ldots B_m$ such that $A_i, B_j \in \mathrm{ER}(\mathcal{A})$. Then obviously $A \cup B \in \mathrm{R}(\mathcal{A})$. Since

$$A \cap B = \bigcup_{i,j} A_i \cap B_j$$

and by (23) $A_i \cap B_j \in \mathrm{ER}(\mathcal{A})$ we get $A \cap B \in \mathrm{R}(\mathcal{A})$. Similarly, since

$$A \setminus B = (A_1 \setminus B) \cup \cdot \cup (A_n \setminus B)$$

and $A_i \setminus B \in \mathrm{ER}(\mathcal{A})$ we get $A \setminus B \in \mathrm{R}(\mathcal{A})$. This proves (25). Property (26) is obvious and the last property follows form Proposition 3.2. QED

The following lemma gives a characterization of closures of elementary representable sets.

Lemma 3.5. *For any proper $\mathcal{C} \subset \mathcal{A}$*

$$\mathrm{cl}\,\langle\mathcal{C}\rangle = \bigcup_{\mathcal{C}\subset\mathcal{C}'\subset\mathcal{A}, \mathcal{C}' \text{ proper}} \langle\mathcal{C}'\rangle \tag{28}$$

Proof: Assume $x \in \mathrm{cl}\,\langle\mathcal{C}\rangle$. Then $x = \lim x_n$ for a sequence $\{x_n\}$ such that $\mathcal{A}(x_n) = \mathcal{C}$. Let $c \in \mathcal{C}$. Then $x_n \in c$ for all $n \in \mathbb{N}$. It follows that $c \in \mathcal{A}(x)$, i.e. $\mathcal{C}' := \mathcal{A}(x) \supset \mathcal{C}$ and we get $x \in \langle\mathcal{C}'\rangle$. This proves that the left hand side of (28) is contained in the right hand side.

To prove the opposite inclusion take an $x \in \langle\mathcal{C}'\rangle$ for some proper $\mathcal{C}'$ satisfying $\mathcal{A} \supset \mathcal{C}' \supset \mathcal{C}$. Then, by (11) $x \in \bigcap\mathcal{C}' \subset \bigcap\mathcal{C} = \mathrm{cl}\,\langle\mathcal{C}\rangle$. This completes the proof. QED

Theorem 3.6. *If A is a representable set then so is its closure.*

Proof: Assume first that $A = \langle\mathcal{C}\rangle$ for a proper $\mathcal{C} \subset \mathcal{A}$. In view of (28) all what we need to show is that there are only finitely many proper $\mathcal{C}'$ such that $\mathcal{C} \subset \mathcal{C}' \subset \mathcal{A}$. To prove this observe that if for such a $\mathcal{C}'$, $x \in \langle\mathcal{C}'\rangle$ and $c \in \mathcal{C}'$ then, by (28), $x \in c \cap \mathrm{cl}\, A$, i.e. $c \in \mathcal{A}(\mathrm{cl}\, A)$ and consequently $\mathcal{C}' \subset \mathcal{A}(\mathrm{cl}\, A)$. However, since $\mathrm{cl}\,\mathcal{A} = \bigcap\mathcal{C}$ is compact by (9), we see from (11) that the family $\mathcal{A}(\mathrm{cl}\, A)$ is finite and consequently the union in (28) is also finite. Thus the assertion is proved if A is an elementary representable set. If A is any representable set then $A = \bigcup_{i=1,\dots n} A_i$, where A_i are elementary representable sets and $\mathrm{cl}\, A = \bigcup_{i=1,\dots n} \mathrm{cl}\, A_i$ is representable as a finite union of representable sets. QED

Corollary 3.7. *Elements of $\mathcal{A}$ are representable over $\mathcal{A}$. Strongly representable sets over $\mathcal{A}$ are representable over $\mathcal{A}$.*

Lemma 3.8. *For any $A \subset X$*

$$A \subset \operatorname{int} |\mathcal{A}(A)|$$

Proof: The proof will proceed in several steps.

Step1. We make an extra assumption that $A = \langle \mathcal{C} \rangle$ for some proper $\mathcal{C} \subset \mathcal{A}$. Since by Lemma 2.2 $\mathcal{A}(\langle \mathcal{C} \rangle) = \mathcal{C}$, it is sufficient to show that $\langle \mathcal{C} \rangle \subset \operatorname{int} |\mathcal{C}|$. Assume the contrary. Then there exists an $x \in \langle \mathcal{C} \rangle \setminus \operatorname{int} |\mathcal{C}|$. Choose a sequence $x_n \to x$ such that $x_n \notin |\mathcal{C}|$. Then $\mathcal{C} \cap \mathcal{A}(x_n) = \emptyset$. By (14) we can select V, a neighborhood of x such that $\mathcal{A}(y) \subset \mathcal{A}(x)$ for all $y \in V$. In particular we get $\mathcal{A}(x_n) \subset \mathcal{A}(x) = \mathcal{C}$ for almost all $n \in \mathbb{N}$, which implies $\mathcal{C} \cap \mathcal{A}(x_n) = \mathcal{A}(x_n) \neq \emptyset$, a contradiction.

Step2. A is a representable set. Then $A = A_1 \cup A_2 \cup \ldots A_n$, where A_i are elementary representable sets. We get from the previous step

$$A = \bigcup_{i=1,\ldots n} A_i \subset \bigcup_{i=1,\ldots n} \operatorname{int} |\mathcal{A}(A_i)| \subset \\ \operatorname{int} \bigcup_{i=1,\ldots n} |\mathcal{A}(A_i)| = \operatorname{int} |\mathcal{A}(\bigcup_{i=1,\ldots n} A_i)| = \operatorname{int} |\mathcal{A}(A)|$$

which implies the assertion.

Step3. A is a relatively compact set. Consider the family

$$\mathcal{B} := \{\, B \in \mathrm{R}(\mathcal{A}) \mid A \subset B \subset |\mathcal{A}(A)| \,\}$$

Since by (10) $\operatorname{card} \mathcal{A}(A) \leq \operatorname{card} \mathcal{A}(\operatorname{cl} A) < \infty$, the set $|\mathcal{A}(A)|$ is compact and consequently also representable. It follows that $|\mathcal{A}(A)| \in \mathcal{B}$, i.e. the family $\mathcal{B} \neq \emptyset$. Put $B := \bigcap \mathcal{B}$. Obviously $A \subset B$ and B is representable. We will show that $\mathcal{A}(A) = \mathcal{A}(B)$. Obviously $\mathcal{A}(A) \subset \mathcal{A}(B)$. Assume the opposite inclusion is not true. Then there exists an $a \in \mathcal{A}(B) \setminus \mathcal{A}(A)$, which means $a \cap B \neq \emptyset$ and $a \cap A = \emptyset$. Put $B' := B \setminus a$. B' is a representable subset of B and $B' \neq B$. However, since $A \subset B' \subset |\mathcal{A}(A)|$, we get $B' \in \mathcal{B}$ and consequently $B \subset B'$, a contradiction. This proves that $\mathcal{A}(A) = \mathcal{A}(B)$. Hence we get

$$A \subset B \subset \operatorname{int} |\mathcal{A}(B)| = \operatorname{int} |\mathcal{A}(A)|$$

Step4. A is an arbitrary subset of X. Then $A = \bigcup_{i=1,\dots\infty} A_i$, where A_i are relatively compact. It follows from the previous step that

$$A \subset \bigcup_{i=1,\dots\infty} \operatorname{int} |\mathcal{A}(A_i)| \subset \operatorname{int} \bigcup_{i=1,\dots,\infty} |\mathcal{A}(A_i)| = \operatorname{int} |\mathcal{A}(A)|$$

This completes the proof. QED

Theorem 3.9. *If A is a representable set then* $\operatorname{int} A$ *is a representable set.*

Proof: Put $N := |\mathcal{A}(A)|$. Then N is a representable compact set and by Lemma 3.8 $A \subset \operatorname{int} N$. It follows that

$$\operatorname{int} A = \operatorname{int}_N A = N \setminus \operatorname{cl}_N(N \setminus A) = N \setminus \operatorname{cl}(N \setminus A)$$

and the latter set is representable by (25) and Theorem 3.6 . QED

We complete this section with the following corollary, which is an immediate consequence of property (25), Theorems 3.6 and 3.9 and the fact that $A \subset B$ is equivalent to $A \cap B = A$.

Corollary 3.10. *If A, B are representable sets then the set $A \cup B, A \cap B, A \setminus B, \operatorname{cl} A, \operatorname{int} A$ may be constructed in finite time. Moreover, the inclusion $A \subset B$ may also be verified in finite time.*

4. Neighborhoods

Let $\mathcal{G}$ be a fixed grid in X and let $A \subset X$. Let us define the following two sets

$$o_{\mathcal{G}}(A) := \bigcup \{\, E \in \operatorname{ER}(\mathcal{G}) \mid \operatorname{cl} E \cap A \neq \emptyset \,\}$$
$$n_{\mathcal{G}}(A) := \bigcup \{\, \gamma \in \mathcal{G} \mid \gamma \cap A \neq \emptyset \,\}$$

The following two propositions are straightforward.

Proposition 4.1.

$$A \subset B \;\Rightarrow\; o_{\mathcal{G}}(A) \subset o_{\mathcal{G}}(B),\; n_{\mathcal{G}}(A) \subset n_{\mathcal{G}}(B)$$
$$A = \bigcup_{\iota \in I} A_\iota \;\Rightarrow\; o_{\mathcal{G}}(A) = \bigcup_{\iota \in I} o_{\mathcal{G}}(A_\iota),\; n_{\mathcal{G}}(A) = \bigcup_{\iota \in I} n_{\mathcal{G}}(A_\iota)$$
$$o_{\mathcal{G}}(A) \subset n_{\mathcal{G}}(A)$$

Proposition 4.2. *If A is representable then both $o_{\mathcal{G}}(A)$ and $n_{\mathcal{G}}(A)$ may be constructed in finite time.*

Next we will prove two useful criteria for a representable set to be closed or open.

Theorem 4.3. *Assume A is a representable set. Then A is closed iff for every elementary representable set E*

$$E \subset A \ \Rightarrow \ \mathrm{cl}\, E \subset A. \tag{29}$$

Similarly, A is open iff for every elementary representable set E

$$E \cap A = \emptyset \ \Rightarrow \ \mathrm{cl}\, E \cap A = \emptyset. \tag{30}$$

Proof: If A is closed then obviously $\mathrm{cl}\, E \subset A$ for any $E \subset A$, i.e. in particular for any elementary representable subset $E \subset A$. To prove that this condition is sufficient take the decomposition $A = E_1 \cup \ldots \cup E_n$ of A into elementary representable sets. We have

$$\mathrm{cl}\, A = \mathrm{cl}\, E_1 \cup \ldots \cup \mathrm{cl}\, E_n \subset A,$$

which proves that A is closed.

Assume in turn that A is open. Then obviously $\mathrm{cl}\, E \cap A = \emptyset$ for any set E such that $E \cap A = \emptyset$, in particular for any elementary representable set E disjoint from A. To show that this condition is sufficient put $N := |\mathcal{G}(A)|$. We will prove that condition (30) implies that $N \setminus A$ is closed.

For this end take an elementary representable set $E \subset N \setminus A$. Then $E \cap A = \emptyset$ and consequently $A \cap \mathrm{cl}\, E = \emptyset$, i.e. $\mathrm{cl}\, E \subset N \setminus A$. Thus $N \setminus A$ is closed. Since by Lemma 3.8 $A \subset \mathrm{int}\, N$, we conclude that A is open. QED

We want now to characterize better the two neighborhoods $o_{\mathcal{G}}(A)$ and $n_{\mathcal{G}}(A)$.

Lemma 4.4. *For any $A \subset X$ the set $o_{\mathcal{G}}(A)$ is open and the set $n_{\mathcal{G}}(A)$ is closed*

Proof: The fact that $n_{\mathcal{G}}(A)$ is closed follows from (9) and (10). To show that $o_{\mathcal{G}}(A)$ is open it is enough to verify the criterion (30) from Theorem 4.3. Let E be an elementary representable set such that $E \cap o_{\mathcal{G}}(A) = \emptyset$. We need to show that $\operatorname{cl} E \cap o_{\mathcal{G}}(A) = \emptyset$. Assume the contrary. Let $\operatorname{cl} E = E_1 \cup \ldots \cup E_n$, where $E_i \in \mathrm{ER}(\mathcal{G})$. Then $E_i \cap o_{\mathcal{G}}(A) \neq \emptyset$ for some $i \in \{1, 2, \ldots n\}$. It follows from (21) that $E_i \subset o_{\mathcal{G}}(A)$ which implies $\operatorname{cl} E_i \cap A \neq \emptyset$ and we get $\operatorname{cl} E \cap A \neq \emptyset$ and consequently $E \subset o_{\mathcal{G}}(A)$, a contradiction. QED

Theorem 4.5. *The set $o_{\mathcal{G}}(A)$ is the smallest open representable set containing A, i.e.*

$$U \in \mathrm{R}(\mathcal{G}),\ U \text{ open},\ U \supset A \ \Rightarrow\ U \supset o_{\mathcal{G}}(A).$$

Proof: Let U be an open representable set which contains A. Let $x \in o_{\mathcal{G}}(A)$. Then $x \in E$ for some elementary representable set E such that $\operatorname{cl} E \cap A \neq \emptyset$. It follows that $\operatorname{cl} E \cap U \neq \emptyset$ and $E \cap U \neq \emptyset$, because U is open. Thus $E \subset U$ by (21), which means that $x \in U$. QED

The following lemma will be useful in the sequel.

Lemma 4.6. *Assume A is a compact representable set. Then there exists an open neighborhood V of A such that for any representable set B*

$$B \subset V \ \Rightarrow\ B \subset A.$$

Proof: Put

$$\mathcal{E} := \{\, E \mid E \in \mathrm{ER}(\mathcal{G}),\ E \subset n_{\mathcal{G}}(A),\ E \not\subset A \,\}$$

and for $E \in \mathcal{E}$ define

$$\alpha_E : \operatorname{cl} E \ni x \to \operatorname{dist}(x, E) \in \mathbb{R}$$
$$c_E := \sup_{x \in \operatorname{cl} E} \alpha_E(x)$$
$$c := \min\{\, c_E \mid E \in \mathcal{E} \,\}$$

For each $E \in \mathcal{E}$ select $x_E \in \operatorname{cl} E$ such that $c_E = \alpha_E(x_E)$. Put $U := B(A, c) \cap o_{\mathcal{G}}(A)$ and take V an open neighborhood of A such that $\operatorname{cl} V \subset U$. Let B be a representable subset of V. We want to show that $B \subset A$. Assume the contrary. Let $B = E_1 \cup \ldots \cup E_n$ be the decomposition of B into elementary representable sets. Then $E_i \not\subset A$ for some $i \in \{ 1, 2, \ldots n \}$. However $E_i \subset B \subset V \subset U \subset n_{\mathcal{G}}(A)$, which means that $E_i \in \mathcal{E}$. It follows that $\alpha_{E_i}(x_{E_i}) \geq c$, which means that $x_{E_i} \in \operatorname{cl} B \setminus U$. But this contradicts $\operatorname{cl} B \subset \operatorname{cl} V \subset U$. QED

5. Subgrids.

Obviously one fixed grid cannot be sufficient to treat all problems. If the diameter of a grid is too large for a particular problem, then one needs to subdivide it in some way. For this purpose we introduce subgrids.

First we need a definition. We will say that the family $\mathcal{G}$ is *centered* if

$$\{ \gamma_1, \ldots, \gamma_n \} \subset \mathcal{G},\ \gamma_i \cap \gamma_j \neq \emptyset \ \Rightarrow\ \gamma_1 \cap \ldots \cap \gamma_n \neq \emptyset. \tag{31}$$

The reader will easily check the following proposition.

Proposition 5.1. *Any standard grid in $\mathbb{R}$ is centered.*

Assume that $\mathcal{G}, \mathcal{G}'$ are two centered grids in X. We will say that $\mathcal{G}'$ is a *subgrid* of $\mathcal{G}$ if the following two conditions are satisfied

$$\forall \tau \in \mathcal{G}' \exists \sigma \in \mathcal{G}\ \ \tau \subset \sigma \tag{32}$$

$$\forall \tau \in \mathcal{G}' \forall E_1, E_2 \in \mathrm{ER}(\mathcal{G})\ \ \tau \cap \operatorname{cl} E_i \neq \emptyset \ \Rightarrow\ \tau \cap \operatorname{cl} E_1 \cap \operatorname{cl} E_2 \neq \emptyset. \tag{33}$$

One easily verifies the following proposition.

Proposition 5.2. *For every $\eta > 0$ the grid $\mathcal{G}_{\eta/2}$ is a subgrid of $\mathcal{G}_\eta$.*

The following proposition is a technical one which will be needed in the sequel.

Proposition 5.3. *Assume $\mathcal{G}'$ is a subgrid of $\mathcal{G}$ and A_1, A_2 are two closed sets representable over $\mathcal{G}$. If $\tau \in \mathcal{G}'$ then*

$$\tau \cap A \neq \emptyset \neq \tau \cap B \ \Rightarrow\ \tau \cap A \cap B \neq \emptyset$$

Proof: Let $A = A_1 \cup A_2 \cup \dots A_k$, $B = B_1 \cup B_2 \dots \cup B_l$ be decompositions of A, B into elementary representable sets over $\mathcal{G}$. Assume $\tau \cap A \cap B = \emptyset$. Then for any $i \in \{1, 2, \dots k\}$, $j \in \{1, 2, \dots l\}$ we have $\tau \cap \operatorname{cl} A_i \cap \operatorname{cl} B_j = \emptyset$, i.e., by (33), $\tau \cap \operatorname{cl} A_i = \emptyset$ or $\tau \cap \operatorname{cl} B_j = \emptyset$. However $\tau \cap A \neq \emptyset$ implies that there exists an $i \in \{1, 2, \dots k\}$ such that $\tau \cap A_i \neq \emptyset$ and similarly we conclude that also $\tau \cap B_j \neq \emptyset$ for some $j \in \{1, 2, \dots l\}$, a contradiction. QED

The next proposition summarizes main properties of subgrids.

Proposition 5.4. *Assume $\mathcal{G}'$ is a subgrid of $\mathcal{G}$. Then*

$$\forall \tau \in \mathcal{G}' \exists! \sigma =: \sigma_\tau \in \mathcal{G} \;\; \tau \subset \sigma \tag{34}$$

$$\forall \sigma \in \mathcal{G} \; \sigma = \bigcup \{ \tau \in \mathcal{G}' \mid \tau \subset \sigma \} \tag{35}$$

$$E \in \mathrm{ER}(\mathcal{G}),\; E' \in \mathrm{ER}(\mathcal{G}'),\; E \cap E' \neq \emptyset \;\Rightarrow\; E' \subset E \tag{36}$$

$$\mathrm{R}(\mathcal{G}) \subset \mathrm{R}(\mathcal{G}') \tag{37}$$

Proof: To prove (34) take a $\tau \in \mathcal{G}'$ and assume $\tau \subset \sigma_1$, $\tau \subset \sigma_2$ for some $\sigma_1, \sigma_2 \in \mathcal{G}$ such that $\sigma_1 \neq \sigma_2$. Then

$$\operatorname{int} \tau \subset \operatorname{int}(\sigma_1 \cap \sigma_2) = \operatorname{int} \sigma_1 \cap \operatorname{int} \sigma_2 = \emptyset,$$

and by (16) we get $\tau = \operatorname{cl} \operatorname{int} \tau = \emptyset$, a contradiction.

Let $\sigma \in \mathcal{G}$. All what we need to show (33) is $\sigma \subset \bigcup \{ \tau \in \mathcal{G}' \mid \tau \subset \sigma \}$. Thus take $x \in \sigma$. By (16) $x = \lim x_n$, where $x_n \in \operatorname{int} \sigma$. Let $x_n \in \tau_n \in \mathcal{G}'$. Taking a subsequence, if necessary, we may assume that $\tau_n = \tau$ for some $\tau \in \mathcal{G}'$. It follows that $\sigma_\tau \cap \sigma \neq \emptyset$ and by (16) $\operatorname{int} \sigma_\tau \cap \operatorname{int} \sigma \neq \emptyset$. Thus (18) yields $\sigma_\tau = \sigma$, i.e. $x \in \operatorname{cl} \tau = \tau \subset \sigma$, which proves (33).

To prove (36) take $E = \langle \mathcal{E} \rangle$, $E' = \langle \mathcal{E}' \rangle$, where $\mathcal{E} = \{ \gamma_1, \gamma_2, \dots \gamma_k \} \subset \mathcal{G}$ and $\mathcal{E}' = \{ \gamma'_1, \gamma'_2, \dots \gamma'_l \} \subset \mathcal{G}'$. Let $x \in E \cap E'$. Then $\mathcal{G}(x) = \mathcal{E}$, $\mathcal{G}'(x) = \mathcal{E}'$. Put $\rho_i := \sigma_{\gamma'_i}$. We will show that

$$\mathcal{E} = \{ \rho_1, \rho_2, \dots, \rho_l \} \tag{38}$$

To show that the right-hand side is contained in the left-hand side take $\rho_i \in \{ \rho_1, \dots, \rho_n \}$. Since $\rho_i \supset \sigma_{\gamma'_i} \ni x$, we see that $\rho_i = \mathcal{G}(x) = \mathcal{E}$. To see the opposite inclusion take $\gamma \in \mathcal{E}$. Then $x \in \gamma$ and by (35) there exists $\gamma' \in \mathcal{G}'$ such that $x \in \gamma' \subset \gamma$. It follows that $\gamma' \in \mathcal{G}'(x) =$

$\{\gamma'_1, \gamma'_2, \dots \gamma'_l\}$, i.e. $\gamma' = \gamma'_i \subset \rho_i$ for some $i \in \{1, 2, \dots l\}$. We get now from (34) that $\gamma = \rho_i$, which implies $\gamma \in \{\rho_1, \dots, \rho_n\}$. This proves (38) and we get

$$E' = \gamma'_1 \cap \gamma'_2 \cap \dots \cap \gamma'_l \subset \rho_1 \cap \rho_2 \cap \dots \cap \rho_l = E$$

and (36) is proved.

To prove (37) it is sufficient to show that for $E \in \mathrm{ER}(\mathcal{G})$ we have

$$E = \bigcup \{ E' \in \mathrm{ER}(\mathcal{G}') \mid E' \subset E \},$$

which is an immediate consequence of (36). This finishes the proof. QED

We are now able to prove the following two theorems.

Theorem 5.5. *Assume $\mathcal{G}'$ is a subgrid of $\mathcal{G}$ and $A \in \mathrm{R}(\mathcal{G})$ is closed. Then*

$$n_{\mathcal{G}'}(A) \subset o_{\mathcal{G}}(A).$$

Proof: Assume the contrary. Then there exists an $x \in n_{\mathcal{G}'}(A) \setminus o_{\mathcal{G}}(A)$. Choose $\tau \in \mathcal{G}'$ such that $x \in \tau$ and $\tau \cap A \neq \emptyset$. Let $E \in \mathrm{ER}(\mathcal{G}')$ be such that $x \in E$ and $\mathrm{cl}\, E \cap A = \emptyset$. Let $A = A_1 \cup A_2 \cup \dots \cup A_n$ be the decomposition of A into elementary sets in $\mathcal{G}$. Then $\tau \cap \mathrm{cl}\, A_i \neq \emptyset$ for some $i \in \{1, 2, \dots n\}$. Since also $\tau \cap \mathrm{cl}\, E \neq \emptyset$ we get from (33) that $\emptyset \neq \tau \cap \mathrm{cl}\, E_i \cap \mathrm{cl}\, E \subset A \cap \mathrm{cl}\, E$, a contradiction. QED

Theorem 5.6. *Assume $\mathcal{G}'$ is a subgrid of $\mathcal{G}$. If $A, B \in \mathrm{R}(\mathcal{G})$ are closed then*

$$n_{\mathcal{G}'}(A) \cap o_{\mathcal{G}'}(B) \subset o_{\mathcal{G}}(A \cap B).$$

Proof: Take $x \in n_{\mathcal{G}'}(A) \cap o_{\mathcal{G}'}(B)$. Let $\tau \in \mathcal{G}'$ be such that $x \in \tau$ and $\tau \cap A \neq \emptyset$. Let $\{\nu_1, \nu_2, \dots, \nu_k\} \subset \mathcal{G}'$ be such that $F := \langle \{\nu_1, \nu_2, \dots, \nu_k\} \rangle \in \mathrm{ER}(\mathcal{G}')$ satisfies $x \in F$ and $\mathrm{cl}\, F \cap B \neq \emptyset$. Then $\tau \in \{\nu_1, \dots, \nu_k\}$ and $\mathrm{cl}\, F = \nu_1 \cap \dots \cap \nu_k \subset \tau$. Let $\mathcal{G}(x) =: \{\mu_1, \mu_2, \dots, \mu_l\}$ and $E := \langle \mathcal{G}(x) \rangle$. Since $\tau \cap B \supset \mathrm{cl}\, F \cap B \neq \emptyset$ and $\tau \cap A \neq \emptyset$, we get from Proposition 5.3 that $\tau \cap A \cap B \neq \emptyset$, in particular $A \cap B \neq \emptyset$.

Let us proceed first with the case when both A and B are closures of elementary sets over $\mathcal{G}$. Then $A = \sigma_1 \cap \sigma_2 \cap \ldots \cap \sigma_p$ and $B = \rho_1 \cap \rho_2 \cap \ldots \rho_q$, where $\sigma_1, \ldots, \sigma_p, \rho_1, \ldots, \rho_q \in \mathcal{G}$. We will show that

$$\operatorname{cl} E \cap A \cap B \neq \emptyset \tag{39}$$

We have $\sigma_i \cap \rho_j \supset A \cap B \neq \emptyset$ and $\mu_i \cap \rho_j \supset \operatorname{cl} E \cap B \neq \emptyset$. We will show that also $\mu_i \cap \sigma_j \neq \emptyset$. Indeed, $\tau \cap \mu_i \supset \tau \cap \operatorname{cl} E \supset \tau \cap \operatorname{cl} F = \operatorname{cl} F \neq \emptyset$ and $\tau \cap \sigma_j \supset \tau \cap A \neq \emptyset$, thus, by Proposition 5.3 $\mu_i \cap \sigma_j \supset \tau \cap \mu_i \cap \sigma_j \neq \emptyset$. Property (39) follows now from (31). and since $x \in E$ we get $x \in o_{\mathcal{G}}(A \cap B)$.

It remains to consider the case of arbitrary closed representable sets A, B. Then $A = \operatorname{cl} A = \operatorname{cl} A_1 \cup \ldots \operatorname{cl} A_k$ and $B + \operatorname{cl} B = \operatorname{cl} B_1 \cup \ldots \cup \operatorname{cl} B_l$, where $A_1, \ldots A_k, B_1, \ldots B_l \in ER(\mathcal{G})$ and we get by Proposition 4.1 and the just proved case

$$n_{\mathcal{G}'}(A) \cap o_{\mathcal{G}'}(B) = \bigcup_{i=1,\ldots k} n_{\mathcal{G}'}(\operatorname{cl} A_i) \cap \bigcup_{j=1,\ldots,l} o_{\mathcal{G}'}(\operatorname{cl} B_j) =$$
$$\bigcup_{i=1,\ldots,k, j=1,\ldots,l} n_{\mathcal{G}'}(\operatorname{cl} A_i) \cap o_{\mathcal{G}'}(\operatorname{cl} B_j) \subset$$
$$\bigcup_{i=1,\ldots,k, j=1,\ldots l} o_{\mathcal{G}}(\operatorname{cl} A_i \cap \operatorname{cl} B_j) = o_{\mathcal{G}}(\bigcup_{i=1,\ldots,k, j=1,\ldots l} (\operatorname{cl} A_i \cap \operatorname{cl} B_j) = o_{\mathcal{G}}(A \cap B).$$

QED

6. Multivalued representable maps

We turn now our attention to multivalued dynamics. We begin with defining representable maps and studying their properties.

By a *multivalued map* $F : X \rightrightarrows Y$ we mean a map $F : X \to \mathcal{P}Y$. The *domain* of a multivalued map $F : X \rightrightarrows Y$ is the set

$$\operatorname{dom} F := \{ x \in X \mid F(x) \neq \emptyset \}$$

and the image of F is defined as the set

$$\operatorname{im} F := \bigcup_{x \in X} F(x).$$

For $A \subset X, B \subset Y$ we define *the image, strong preimage* and *weak preimage* of A, B under F as the sets

$$F(A) := \bigcup_{x \in A} F(x)$$
$$F^{o-1}(B) := \{\, x \in \operatorname{dom} F \mid F(x) \subset B \,\}$$
$$F^{*-1}(B) := \{\, x \in X \mid F(x) \cap B \neq \emptyset \,\}$$

The inverse of F is defined as a multivalued map

$$F^{-1} : Y \ni y \to \{\, x \in X \mid y \in F(x) \,\} \subset X$$

The reader should note that the image under the inverse map $F^{-1}(B)$ coincides with the weak (not strong) preimage of B under F.

One can easily check the following proposition.

Proposition 6.1. *For any multivalued map F*

$$(F^{-1})^{-1} = F$$

If $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ are two multivalued maps then we define their composition as a multivalued map

$$G \circ F : X \ni x \to G(F(x)) \subset Z$$

Now let X, X' be two locally compact metric spaces and let $\mathcal{G}, \mathcal{G}'$ be grids respectively in X, X'. We will say that a multivalued map $F : X \rightrightarrows X'$ is *representable* over $\mathcal{G}, \mathcal{G}'$ if the following three conditions are satisfied

$$\operatorname{card} \mathcal{G}(\operatorname{dom} F) < \infty \tag{40}$$
$$\forall x \in X \;\; F(x) \in \mathrm{R}(\mathcal{G}) \tag{41}$$
$$\forall E \in \mathrm{ER}(\mathcal{G}) \;\; F_{|E} = \mathrm{const} \tag{42}$$

We have the following theorem characterizing the properties of representable multivalued maps.

Theorem 6.2. *If $F : X \rightrightarrows X'$ is representable, $A \in \mathrm{R}(\mathcal{G})$, $A' \in \mathrm{R}(\mathcal{G}')$ then*

$$\operatorname{dom} F,\ F^{o-1}(A'),\ F^{*-1}(A') \in \mathrm{R}(\mathcal{G}) \tag{43}$$

$$F(A) \in \mathrm{R}(\mathcal{G}'). \tag{44}$$

Proof: Observe that $\operatorname{dom} F = \bigcup_{x \in \operatorname{dom} F} \langle \mathcal{G}(x) \rangle$ and since $x \in \operatorname{dom} F \Rightarrow \mathcal{G}(x) \subset \mathcal{G}(\operatorname{dom} F)$, the union is finite, which proves that $\operatorname{dom} F \in \mathrm{R}(\mathcal{G})$. An analogous argument applies to $F^{*-1}(A')$ and $F^{o-1}(A')$.

Finally $F(A) = \bigcup_{x \in A} F(x)$ and again one easily verifies that the union is finite. QED

As an immediate consequence we obtain the following corollary.

Corollary 6.3. *If $F : X \rightrightarrows X'$ is representable, $A \in \mathrm{R}(\mathcal{G})$, $A' \in \mathrm{R}(\mathcal{G}')$ then the sets $\operatorname{dom} F,\ F^{o-1}(A'),\ F^{*-1}(A') \in \mathrm{R}(\mathcal{G}), F(A) \in \mathrm{R}(\mathcal{G}')$ may be constructed in finite time.*

The following theorem shows that representable uppersemicontinuous maps behave in a quite simple way.

Theorem 6.4. *Let $F : X \rightrightarrows X'$ be representable and upper semicontinuous. If A is a compact representable set then*

$$F(o_{\mathcal{G}}(A)) = F(A)$$

Proof: Obviously it is sufficient to show that $F(o_{\mathcal{G}}(A)) \subset F(A)$. By Lemma 4.6 we can find V, an open subset of $F(A)$ such that for any representable set $B \subset V$ we have $B \subset A$. Since F is upper semicontinuous, there exists an open neighborhood U of A such that $F(U) \subset V$. We will show that

$$F(o_{\mathcal{G}}(A)) \subset F(U).$$

Indeed, if $y \in F(o_{\mathcal{G}}(A))$ then there exists an elementary representable set E and $x \in E$ such that $A \cap \operatorname{cl} E \neq \emptyset$ and $y \in F(x)$. Then also $U \cap \operatorname{cl} E \neq \emptyset$, hence we can find an $x' \in U \cap E$. It follows that $y \in F(x) = F(x') \subset F(U)$ and the required inclusion is proved, so that we conclude that $F(o_{\mathcal{G}}(A)) \subset V$. However $F(o_{\mathcal{G}}(A))$ is representable as an image of an representable set, therefore we obtain $F(o_{\mathcal{G}}(A)) \subset A$. QED

Representability of multivalued maps carries over to their inverses, as the following theorem shows.

Theorem 6.5. *If $F : X \rightrightarrows X'$ is representable then so is F^{-1}.*

Proof: We have

$$\operatorname{dom} F^{-1} = \{ x' \in X' \mid \exists x \in X \; x' \in F(x) \} = F(\operatorname{dom} F) = \operatorname{im} F$$

thus $\operatorname{dom} F^{-1}$ is representable as the image of a representable set under a representable map. In particular we get

$$\mathcal{G}'(\operatorname{dom} F^{-1}) = \mathcal{G}'(\operatorname{im} F) = \bigcup_{x' \in \operatorname{im} F} \mathcal{G}'(x'),$$

which is a finite family, because $\mathcal{G}'(x')$ is constant on elementary sets and $\operatorname{im} F$ is a finite union of elementary representable sets. This shows (40) In order to prove (41) first observe that

$$F^{-1}(x') = \{ x \in X \mid x' \in F(x) \} = \bigcup_{x \in F^{-1}(x')} \langle \mathcal{G}(x) \rangle \subset \operatorname{dom} F \tag{45}$$

Indeed, let $y \in \langle \mathcal{G}(x) \rangle$ for some $x \in F^{-1}(x')$. Then $x' \in F(x) = F(y)$, i.e. $y \in F^{-1}(x')$. The opposite inclusion is obvious. Hence (41) follows from (45)

It remains to show (42) . Let $E' \in \mathrm{ER}(\mathcal{G}')$ and $x, y \in E'$.. We have to show that

$$F^{-1}(x') = F^{-1}(y').$$

Due to the symmetry of the above equation it is enough to just prove that the left-hand side is contained in the right-hand side. Let $x \in F^{-1}(x')$. Then $x' \in F(x)$, i.e. $E \subset F(x)$. This shows that $y' \in F(x)$ and $x \in F^{-1}(y')$. Thus (42) is proved. QED

Let $N \subset X$ be a compact set and $F : X \rightrightarrows X$ be a representable map. Let $F_N : X \ni x \to N \cap F(x) \subset X$. The following proposition is straightforward.

Proposition 6.6. *If F is representable and N is representable then F_N is representable.*

QED

7. Representations of continuous maps

The next thing we want to show is that every Lipschitz continuous map may be approximated with an arbitrary accuracy by a multivalued representable map.

Assume X, Y are two locally compact metric spaces with given grids $\mathcal{G}$ and $\mathcal{H}$. Let $f : X \multimap Y$ be a continuous map. We say that a multivalued representable map $F : X \rightrightarrows Y$ is a *representation of f over $\mathcal{G}, \mathcal{H}$* if $\operatorname{dom} f \subset \operatorname{dom} F$ and $x \in \operatorname{dom} f$ implies $f(x) \in F(x)$. It is convenient to define also a combinatorial representation of f as a multivalued map $\mathcal{F} : \mathcal{G} \rightrightarrows \mathcal{H}$ such that the following conditions are satisfied

$$\# \operatorname{dom} \mathcal{F} < \infty \tag{46}$$

$$\forall g \in \mathcal{G} \; \#\mathcal{F}(g) < \infty \tag{47}$$

$$\operatorname{dom} f \subset |\operatorname{dom} \mathcal{F}| \tag{48}$$

$$x \in \operatorname{dom} f, g \in \operatorname{dom} \mathcal{F}, x \in g \;\Rightarrow\; f(x) \in |\mathcal{F}(g)| \tag{49}$$

With a combinatorial multivalued map $\mathcal{F} : \mathcal{G} \rightrightarrows \mathcal{H}$ we associate the following two multivalued maps:

$$\lfloor \mathcal{F} \rfloor : |\mathcal{G}| \ni x \to \bigcap_{g \in \mathcal{G}(x)} |\mathcal{F}(G)| \tag{50}$$

$$\lceil \mathcal{F} \rceil : |\mathcal{G}| \ni x \to \bigcup_{g \in \mathcal{G}(x)} |\mathcal{F}(G)| \tag{51}$$

Theorem 7.1. *If $\mathcal{F} : \mathcal{G} \rightrightarrows \mathcal{H}$ is a combinatorial representation of f, then $\lfloor \mathcal{F} \rfloor$ and $\lceil \mathcal{F} \rceil$ are representations of f. Moreover, $\lfloor \mathcal{F} \rfloor$ is lower semicontinuous and $\lceil \mathcal{F} \rceil$ is upper semicontinuous.*

Proof: We will show first that $\lceil \mathcal{F} \rceil$ is representable. One easily verifies that $\operatorname{dom} \lceil \mathcal{F} \rceil = |\operatorname{dom} \mathcal{F}|$, hence $\operatorname{dom} \lceil \mathcal{F} \rceil$ is strongly representable. If $x \in \operatorname{dom} \lceil \mathcal{F} \rceil$ then $\lceil \mathcal{F} \rceil(x) = \bigcup_{g \in \mathcal{G}(x)} |\mathcal{F}(g)|$, which shows that $\lceil \mathcal{F} \rceil(x)$ is representable. If $\mathcal{E} \in R(\mathcal{G})$ and $x, y \in \langle \mathcal{E} \rangle$, then $\mathcal{G}(x) = \mathcal{E} = \mathcal{G}(y)$ and consequently $\lceil \mathcal{F} \rceil(x) = \lceil \mathcal{F} \rceil(y)$. This shows that $\lceil \mathcal{F} \rceil$ is representable. Obviously $f(x) \in \lceil \mathcal{F} \rceil(x)$.

In order to show that $\lceil\mathcal{F}\rceil$ is upper semicontinuous, take $x \in \operatorname{dom}\lceil\mathcal{F}\rceil$ and an open set U in Y such that $\lceil\mathcal{F}\rceil(x) \subset U$. By (13) we can select a neighborhood V of x such that $y \in V \;\Rightarrow\; \mathcal{G}(y) \subset \mathcal{G}(x)$. Let $y \in V$. Then

$$\lceil\mathcal{F}\rceil(y) = \bigcup_{g\in\mathcal{G}(y)} |\mathcal{F}(g)| \subset \bigcup_{g\in\mathcal{G}(x)} |\mathcal{F}(g)| = \lceil\mathcal{F}\rceil(x) \subset U.$$

The proof in case of $\lfloor\mathcal{F}\rfloor$ is similar. QED

Let $F_n : X \rightrightarrows Y$ be a sequence of multivalued maps and $f : X \to Y$ be a single valued map such that $\operatorname{dom} f \subset \operatorname{dom} F_n$. We say that F_n converges uniformly to f and write $F_n \to f$ if

$$\forall\epsilon > 0 \exists N \in \mathbb{N} \forall x \in \operatorname{dom} f \;\; F_n(x) \subset B(f(x), \epsilon) \tag{52}$$

Theorem 7.2. *Assume $\mathcal{G}_n, \mathcal{H}_n$ are sequences of grids respectively in X and Y such that* $\operatorname{diam}\mathcal{G}_n \to 0$, $\operatorname{diam}\mathcal{H}_n \to 0$. *Let $f : X \not\to Y$ be a Lipschitz function such that* $\operatorname{dom} f$ *is relatively compact. Then there exist sequences $F_n, G_n : x \rightrightarrows Y$ such that*

$$F_n, G_n \textit{ are representations of } f \tag{53}$$

$$F_n \textit{ is lower semicontinuous and } G_n \textit{ is upper semicontinuous,} \tag{54}$$

$$F_n \to f, G_n \to f \tag{55}$$

Proof: First fix an $n \in \mathbb{N}$ and put $\mathcal{G} := \mathcal{G}_n, \mathcal{H} := \mathcal{H}_n$, $\alpha := \operatorname{diam}\mathcal{G}, \beta := \operatorname{diam}\mathcal{H}$. Let L be a Lipschitz constant for f. Since by (10) $\#\mathcal{G}(\operatorname{dom} f) < \infty$, we can select a finite set of points $\{x_g\}_{g\in\mathcal{G}(\operatorname{dom} f)}$ such that $x_g \in g$. Put

$$\mathcal{F} : \mathcal{G} \ni g \to \begin{cases} \mathcal{H}(B(f(x_g), L\alpha)) & \text{if } g \in \mathcal{G}(\operatorname{dom} f) \\ \emptyset & \text{otherwise} \end{cases}$$

One easily verifies that $\mathcal{F}$ satisfies properties (46),(47) and (48). To prove (49) observe that if $x \in \operatorname{dom} f \cap g$ then $\operatorname{dist}(x, x_g) \le \alpha$ and $\operatorname{dist}(f(x), f(x_g)) \le L\alpha$. Thus $\mathcal{F}$ is a combinatorial representation of f. Let $\mathcal{F}_n$ denote the above construction with respect to $\mathcal{G}_n, \mathcal{H}_n$. Put

$F_n := \lfloor \mathcal{F}_n \rfloor$ and $G_n := \lceil \mathcal{G}_n \rceil$. Then by Theorem 7.1 F_n, G_n are representations of f respectively lower and upper semicontinuous and obviously $F_n(x) \subset G_n(x)$.

Obviously, in order to prove property (55) it is sufficient to show that

$$G_n(x) \subset B(f(x), 2L\alpha_n + \beta_n). \tag{56}$$

To see this, let us take a $y \in G_n(x)$. Then $y \in \lceil \mathcal{F}_n \rceil(g)$ for some $g \in \mathcal{G}(x)$. Thus there exist an $h \in \mathcal{H}$ and $\bar{y} \in h \cap B(y_g, L\alpha_n)$ such that $y \in h$ and we get

$$\operatorname{dist}(y, f(x_g)) \le \operatorname{dist}(y, \bar{y}) + \operatorname{dist}(\bar{y}, f(x_g)) \le \beta_n + L\alpha_n$$

and

$$\operatorname{dist}(y, f(x)) \le \operatorname{dist}(y, f(x_g)) + \operatorname{dist}(f(x_g), f(x)) \le \beta_n + L\alpha_n + L\alpha_n = 2L\alpha_n + \beta_n$$

which proves (56). QED

8. Representable index pairs.

Finally we turn our attention to the Conley index theory. We first recall some definitions.

Let $F : X \rightrightarrows X$ be a multivalued map and $N \subset X$ be compact. Define the positive (negative) *invariant set* of N by

$$\operatorname{Inv}^+(N, F) := \Big\{ x \in N \mid \exists \sigma : \mathbb{Z}^+ \to N \text{s.t. } \sigma(0) = x \text{ and } \sigma(n+1) \in F(\sigma(n)) \Big\}.$$

and

$$\operatorname{Inv}^-(N, F) := \Big\{ x \in N \mid \exists \sigma : \mathbb{Z}^- \to N \text{s.t. } \sigma(0) = x \text{ and } \sigma(n+1) \in F(\sigma(n)) \Big\}.$$

Define the invariant part of N, denoted $\operatorname{Inv} N$, as the intersection of the positive and negative invariant parts. The diameter of F over N is defined as

$$\operatorname{diam}_N F := \sup\{\operatorname{diam} F(x) \mid x \in N\}.$$

The following proposition is straightforward

Proposition 8.1. *If F and N are representable then* $\operatorname{diam}_N F$ *may be computed in finite time.*

The set N is an *isolating neighborhood* for F if

$$B_{\mathrm{diam}_N F}(\mathrm{Inv}\, N) \subset \mathrm{int}\, N. \tag{57}$$

The computation of the Conley index is reduced to the computation of certain homology if one is able to construct for a given isolating neighborhood an *index pair*, i.e. a pair $P = (P_1, P_2)$ of compact subsets $P_2 \subset P_1 \subset N$ such that

$$F(P_i) \cap N \subset P_i,\ i = 1, 2, \tag{58}$$

$$F(P_1 \backslash P_2) \subset N, \tag{59}$$

$$\mathrm{Inv}\, N \subset \mathrm{int}(P_1 \backslash P_2) \tag{60}$$

The following results show that such a computation is indeed possible.

Theorem 8.2. *If F and N are representable then there exist numbers $p, q \in \mathbb{N}$ such that*

$$\mathrm{Inv}^-(N, F) = \bigcap_{i=0,\infty} F_N^i(N) = \bigcap_{i=0,p} F_N^i(N) \tag{61}$$

$$\mathrm{Inv}^+(N, F) = \bigcap_{i=0,\infty} F_N^{*-i}(N) = \bigcap_{i=0,q} F_N^{*-i}(N) \tag{62}$$

Proof: Put $A_n := \bigcap_{i=0,n} F_N^i(N)$ and $B_n := \bigcap_{i=0,n} F_N^{*-i}(N)$. Observe that both $\{A_n\}_{n=1,\ldots\infty}$ and $\{B_n\}_{n=1,\ldots\infty}$ are descending sequences of representable subset of N. Since there is only a finite number of representable subsets of N, both sequences must be constant for all but a finite number of elements. In other words $\bigcap_{i=0,\infty} F_N^i(N) = \bigcap_{i=0,p} F_N^i(N)$ and $\bigcap_{i=0,\infty} F_N^{*-i}(N) = \bigcap_{i=0,q} F_N^{*-i}(N)$.

In order to show the first equality in (61) let us take an $x \in \mathrm{Inv}^-(N, F)$. Let $\sigma : \mathbb{Z}^- \to N$ be a solution of F through x in N. We will show by induction that

$$\sigma(-i + k) \in F_N^k(N) \text{ for } k = 0, 1, \ldots i \text{ and } i \in \mathbb{Z} \tag{63}$$

Indeed, if $k = 0$, then $\sigma(-i) \in N = F_N^0(N)$. If (63) is satisfied for some $k < i$ then

$$\sigma(-i + k + 1) \in N \cap F(\sigma(-i + k)) \subset N \cap F(F_n^k(N)) = F_N^{k+1}(N),$$

which proves (63). Thus we have $x = \sigma(0) = \sigma(-i+i) \in F_N^i(N)$ and one inclusion is proved. To prove the opposite inclusion let us take an $x \in \bigcap_{i=0,\infty} F_N^i(N)$ and fix an $i \in \mathbb{Z}^+$. We will define recursively a solution $\sigma_i : \{-i, -(i-1), \dots, 0\} \to N$ through x such that

$$\sigma_i(j) \in F_N^{i+j}(N) \text{ for } j \in \{-i, -(i-1), \dots 0\}. \tag{64}$$

To this end let us put $\sigma_i(0) := x$ and assume that $\sigma_i(j)$ is already defined for a $j > -i$ in such a way that $\sigma_i(j) \in F_N^{i+j}(N) = F_N(F_N^{i+j-1}(N))$. This means that $\sigma_i(j) \in F_N(w)$ for some $w \in F_N^{i+j-1}(N)$. Putting $\sigma_i(j-1) := w$ we obtain a solution through x, which obviously satisfies (64).

Now, the standard diagonal-limiting process provides the required left solution through x in N. This proves (61).

To prove the first equality in (62) let us take first an $x \in \mathrm{Inv}^+(N)$. Let $\sigma : \mathbb{Z}^+ \to N$ be a positive solution through x in N. Let $i \in \mathbb{Z}^+$ be fixed. We will show by induction that

$$\sigma(k) \in F_N^{*(-i+k)}(N) \text{ for } k = 0, 1, \dots i. \tag{65}$$

Indeed, if $k = i$ then the property is obvious. Assume it is satisfied for some $k > 0$. Since σ is a solution, we have $\sigma(k) \in F_N(\sigma(k-1))$, which implies that $F_N(\sigma(k-1)) \cap F_N^{*(-i+k)}(N) \neq \emptyset$. It follows that $\sigma(k-1) \in F_N^{*-1}(F_N^{*(-i+k)}(N)) = F_N^{*(-i+k-1)}(N)$ and (65) is proved. In particular we get from (65) that $x = \sigma(0) \in F_N^{*-i}(N)$ for all $i \in \mathbb{Z}^+$ and one inclusion is proved. To prove the opposite inclusion, let us take an $x \in \bigcap_{i=0,\infty} F_N^{*-i}(N)$ and fix an $i \in \mathbb{Z}^+$. We will define recursively a solution $\sigma_i : \{0, 1, \dots, i\} \to N$ through x such that

$$\sigma_i(j) \in F_N^{*(-i+j)}(N) \text{ for } j = 0, 1, \dots, i \tag{66}$$

To this end let us put $\sigma_i(0) := x$ and assume that $\sigma_i(j)$ is already defined for some $j < i$ in such a way that $\sigma_i(j) \in F_N^{*(-i+j)}(N) = F_N^{*-1}(F_N^{*(-i+j+1)}(N)$. This means that $F_N(\sigma_i(j)) \cap F_N^{*(-i+j+1)}(N) \neq \emptyset$. Choose a $z \in F_N(\sigma_i(j)) \cap F_N^{*(-i+j+1)}(N)$ and put $\sigma_i(j+1) := z$. One easily verifies that σ_i is a solution satisfying (66). Again, a standard diagonal-limiting process provides the required right solution through x in N. The theorem is proved. QED

Corollary 8.3. *If* N *is representable and* F *is representable then* $\operatorname{Inv}^- N, \operatorname{Inv}^+ N$ *and* $\operatorname{Inv} N$ *are representable.*

Theorem 8.4. *Assume* $F : X \rightrightarrows X$ *is a representable usc multivalued map and* N *is a representable isolating neighborhood with respect to a grid* $\mathcal{G}$ *in* X. *Let* $\mathcal{H}$ *be a subgrid of* $\mathcal{G}$. *Put* $P_1 := n_{\mathcal{H}}(\operatorname{Inv}^- N) \cap N$, $P_2 := P_1 \setminus o_{\mathcal{H}}(\operatorname{Inv}^+ N)$. *Then* P_1, P_2 *are representable with respect to* $\mathcal{H}$ *and* $P := (P_1, P_2)$ *is an index pair for* F *in* N.

Proof: The fact that P_1, P_2 are representable sets follows immediately from Theorem 8.2. To prove that P is an index pair, let us consider first property (58). We have from Theorem 5.5 and from Theorem 6.4

$$\begin{aligned} & F(P_1) \cap N = N \cap F(n_{\mathcal{H}}(\operatorname{Inv}^- N) \cap N) \subset \\ & N \cap F(o_{\mathcal{G}}(\operatorname{Inv}^- N) \cap N) \subset N \cap F(o_{\mathcal{G}}(\operatorname{Inv}^- N)) \subset N \cap F(\operatorname{Inv}^- N). \end{aligned} \tag{67}$$

One can now easily check that $N \cap F(\operatorname{Inv}^- N) \subset \operatorname{Inv}^- N$ which implies $F(P_1) \cap N \subset \operatorname{Inv}^- N \subset P_1$ and proves (58).

Now take $x \in P_2$ and $y \in N \cap F(x) = F_N(x)$. Since $P_2 \subset P_1$, we get $y \in P_1$. To show that $y \in P_2$, assume the contrary. Then $y \in o_{\mathcal{H}}(\operatorname{Inv}^+ N)$. Thus we can find an elementary representable set $E \in \operatorname{ER}(\mathcal{H})$ such that $y \in E$ and $\operatorname{Inv}^+ N \cap \operatorname{cl} E \neq \emptyset$. In particular $E \cap F_N(x) \neq \emptyset$ and since $F_N(x)$ is representable and closed, we get from (21) that $\operatorname{cl} E \subset F_N(x)$. It follows that $F_N(x) \cap \operatorname{Inv}^+ N \neq \emptyset$ and consequently $x \in \operatorname{Inv}^+ N \subset P_1 \setminus P_2$, a contradiction. Thus also P_2 satisfies (58).

In order to prove (59) let us take an $x \in P_1 \setminus P_2$ and $y \in F(x)$. Then, by Theorem 5.6 $x \in o_{\mathcal{H}}(\operatorname{Inv}^+ N) \cap n_{\mathcal{H}}(\operatorname{Inv}^- N) \subset o_{\mathcal{G}}(\operatorname{Inv} N)$. Let $E \in \operatorname{ER}(\mathcal{G})$ be such that $x \in E$ and $\operatorname{cl} E \cap \operatorname{Inv} N \neq \emptyset$. Let $w \in \operatorname{cl} E \cap \operatorname{Inv} N$. Then $x \in o_{\mathcal{G}}(w)$ and by Theorem 6.4 $F(x) \subset F(w)$. Since also $w \in \operatorname{Inv} N$, we can find a $u \in F(w) \cap \operatorname{Inv} N$. Now, since

$y, u \in F(w)$ and $u \in \operatorname{Inv} N$, we get from (57) that $y \in N$. This shows (59) .

It remains to show (60). Observe that

$$\operatorname{Inv}^- N \subset o_{\mathcal{H}}(\operatorname{Inv}^- N) \subset \operatorname{int} n_{\mathcal{H}}(\operatorname{Inv}^- N)$$

Since

$$P_1 \setminus P_2 = o_{\mathcal{H}}(\operatorname{Inv}^+ N) \cap n_{\mathcal{H}}(\operatorname{Inv}^- N) \cap N$$

we get

$$\operatorname{int}(P_1 \setminus P_2) = o_{\mathcal{H}}(\operatorname{Inv}^+ N) \cap \operatorname{int} n_{\mathcal{H}}(\operatorname{Inv}^- N) \cap \operatorname{int} N \supset$$
$$\operatorname{Inv}^+ N \cap \operatorname{Inv}^- N \cap \operatorname{int} N = \operatorname{Inv} N \cap \operatorname{int} N = \operatorname{Inv} N$$

which shows (60) .

The proof is completed. QED

For a set $A \subset X$ we define a representable ball of radius r around A by

$$B_{\mathcal{G}}(A) := \bigcup \{\, \tau \in \mathcal{G} \mid \operatorname{dist}(\tau, A) \leq r \,\}.$$

The following proposition is straightforward.

Proposition 8.5. *The set $B_{\mathcal{G}}(A)$ is representable. Moreover, if A is representable then $B_{\mathcal{G}}(A)$ may be computed in finite time.*

We finish this paper with the following algorithm based on Theorem 8.4

Algorithm 8.6. :

```
function find_index_pair(N, F)
begin
    A := N; B := N;
    repeat
        A' := A; B' := B;
        A := F_N(A); B := F_N^{*-1}(B);
    until (A = A' and B = B');
    C := A ∩ B;
    r := diam_N F;
    if (B_G(C, r) ⊂ int N) then
    begin
        P1 := n_H(A) ∩ N;
        P2 := P1 \ o_H(B);
        return (P1, P2);
    end
    else return "Failure";
```

end ;

Theorem 8.7. *If Algorithm 8.6 is called with N, a representable compact set and F, a representable multivalued dynamical system on input, then it always stops. If it does not output "Failure", then N is an isolating neighborhood for F and the algorithm outputs an index pair (P_1, P_2) for F in N.*

Proof: Due to Corollary 3.10, Proposition 4.2, Corollary 6.3, Proposition 8.1 and Proposition 8.5 every individual operation in the above algorithm may be performed in finite time. The only loop in Algorithm 8.6 is always completed due to Theorem 8.2. Therefore the conclusion follows from Theorem 8.4. QED

The implementation of the above algorithm is under construction. Its theoretical and practical performance will be discussed elsewhere.

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