CONLEY INDEX OF POINCARÉ MAPS IN ISOLATING SEGMENTS

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Abstract. We calculate the discrete-time Conley index of the Poincaré map of a time-periodic ordinary differential equation in an isolated invariant set generated by a periodic isolating segment. As an application, we present results on the existence of bounded solutions of some planar equations.

1. Introduction

The index of an isolated invariant set for a continuous-time flow in a locally compact metric space was defined by C. Conley (see [C]). In the case of discrete-time flow, the Conley index theory was initiated by J. Robbin and D. Salomon in the paper [RS]. An improved version of the theory was established by the first author in [Mr1, Mr2] and later developed by A. Szymczak in [Sz]. In [Sr1, Sr2], in order to get results on the existence of periodic solutions of time-periodic non-autonomous equations, the second author introduced the notion of periodic isolating segment (actually, slightly different terminology was used in those papers). Such a segment naturally generates an isolated invariant set for the Poincaré operator. Theorem 1, the main result of the present paper provides the formula on the Conley index of that set. In fact, the theorem can be derived from results of the authors' paper [MRS] written jointly with J. Reineck. However, we present a direct proof, based on the algorithmic approach to the discrete-time Conley index theory given in [Mr4]. Corollary 1 represents the reduction of the formula to the Alexander-Spanier cohomology setting. It is applied here to results on the existence of nonzero bounded solutions of some planar Fourier-Taylor polynomial equations on the plane.

The problem of the existence od bounded solutions of nonlinear ordinary differential equations has received growing attention in the last years. Recent publications on the problem combine analytic and topological or variational arguments, compare for example [FZ, Ma1, Ma2, MT, MW1, MW2, O] for results obtained by guiding functions techniques, applications of the Schauder fixed point theorem or the Leray-Schauder degree, [V] for a variational approach, [IR, Wa1, Wa2, Wa3, Wa4] for applications of the classical homotopy Conley index, and [D, K, Mu] for other methods based on homotopy. Finally, we mention that the notions which appear in the title of the present paper have already been used in the context of bounded solutions; for example, the discrete-time Conley index was applied in the paper [MR] and [Wo] contains an application of isolating segments.

The paper is organized as follows. In Section 2 we provide basic facts concerning non-autonomous equations, in particular we recall the definition of the Poincaré operator. Section 3 presents notions related to the discrete-time Conley index; it includes the definitions of the excisive and normal functors. Section 4 starts with some elementary facts concerning the Ważewski method. We use these facts to introduce the definition of the periodic isolating segment, a concept fundamental in

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the rest of the paper. Theorem 1 together with Corollary 1 are given in Section 5. The next Section 6 is devoted to the proof of Theorem 1. Applications of the main results are given in Section 7.

2. Time-periodic equations and Poincaré maps

We consider a non-autonomous equation

$$\dot{x} = v(t, x), \tag{1}$$

where $v\colon \mathbb{R} \times M \to TM$ is a continuous time-dependent vector-field having the uniqueness property of the corresponding Cauchy problem and M is a Riemannian manifold. Let $t \to \Phi_{(t_0,t)}(x_0)$ denote the solution to the Cauchy problem (1),(2) with the initial condition

$$x(t_0) = x_0. \tag{2}$$

The map Φ is continuous with respect to (t, t_0, x_0) and we call it the *evolutionary operator*. It generates the local flow ϕ on the extended phase space $\mathbb{R} \times M$ given by

$$\phi_\tau(t_0, x_0) := (t_0 + \tau, \Phi_{(t_0,t_0+\tau)}(x_0)).$$

In fact, ϕ is generated by the vector-field $(1, v)$.

Let $T > 0$. In the sequel we assume that v is T-periodic in t. In this case the map

$$\Pi := \Phi_{(0,T)}$$

is called the *Poincaré operator* and will play the main role in the present paper. Its n-th iterate (for $n \in \mathbb{Z}$) is given by

$$\Pi^n = \Phi_{(0,nT)}.$$

3. Isolated invariant sets and Conley index

Let $f\colon U \to X$ be a continuous map, where X is a locally compact metrizable space and U is an open subset of X. A set $S \subset U$ is called *isolated invariant* (for f) if it is compact, invariant (i.e. $f(S) = S$), and there exists V, a neighborhood of S, such that S is the maximal invariant set contained in V. The set V is called an *i*solating neighborhood of S. With such a set S one associates an algebraic object, called *Conley index* (compare [Mr1, RS, Sz]). Following [Mr4] we present briefly its construction.

By a *weak index pair* for the set S we mean a pair (P, Q) of compact subsets of U such that

$$P \cap f(Q) \subset Q, \tag{3}$$

$$P \cap \overline{f(P) \setminus P} \subset Q, \tag{4}$$

$$\overline{P \setminus Q} \text{ is an isolating neighborhood of } S. \tag{5}$$

The conditions (3),(4) were introduced in [Sr4] and their role in algorithmic computation of the Conley index was indicated in [Mr4]. The notion of weak index pair generalizes various notions of index pairs.

Proposition 1 (compare [Sr4], Lemma 6)**.** *(P, Q) is a weak index pair for S if and only if (5) is satisfied and there exists a compact pair $(P', Q') \supset (P, Q)$ such that*

$$f(P) \subset P', f(Q) \subset Q', \text{ and } P \setminus Q = P' \setminus Q'. \tag{6}$$

Following [Mr4, Definition 5.1] we say that (P,Q,P',Q') is a *weak index quadruple* if (P,Q) is a weak index pair, $(P,Q) \subset (P',Q')$ and (6) is satisfied. Various constructive ways of finding the set P',Q' are presented in [Mr4].

In the sequel we deal with the category of compact pairs: their objects are pairs of compact spaces (P,Q) and morphisms are continuous maps $(P,Q) \to (P',Q')$. Let T be a functor from the category of compact pairs to another category. It is called *homotopy invariant* if $T(f) = T(g)$ for homotopic continuous maps $f \simeq g\colon (P,Q) \to (P',Q')$. A continuous map $e\colon (P,Q) \to (P',Q')$ is called *excisive* provided e maps homeomorphically $P \setminus Q$ onto $P' \setminus Q'$. T is called *excisive* if $T(e)$ is an isomorphism for every excisive map e. The functor sending a compact pair (P,Q) onto the quotient space P/Q in the category of pointed spaces is an example of an excisive functor. Due to the strong excision property, the Alexander-Spanier cohomology is an excisive and homotopy invariant functor.

Let $\mathcal{C}$ be a category and let $\mathrm{Endo}(\mathcal{C})$ denote the category of endomorphism; its objects are the endomorphisms in $\mathcal{C}$ and a morphisms from an endomorphism $a\colon A \to A$ to $b\colon B \to B$ is any morphisms $\phi\colon A \to B$ in $\mathcal{C}$ such that $\phi \circ a = b \circ \phi$. In particular, each endomorphism (trivially) defines a morphism in $\mathrm{Endo}(\mathcal{C})$. Let $\mathcal{D}$ be another category. A functor $L\colon \mathrm{Endo}(\mathcal{C}) \to \mathcal{D}$ is called *normal* provided for every endomorphism a (treated as a morphism in $\mathrm{Endo}(\mathcal{C})$) its image $L(a)$ is an automorphism. The universal normal functor is constructed in [Sz]. Particulary important normal functors appear if the target category $\mathcal{D}$ is equal to $\mathrm{Auto}(\mathcal{C})$ being the full subcategory of $\mathrm{Endo}(\mathcal{C})$ whose objects are automorphisms, and the functor $L\colon \mathrm{Endo}(\mathcal{C}) \to \mathrm{Auto}(\mathcal{C})$ is a *normal retractor*, i.e. its restriction to $\mathrm{Auto}(\mathcal{C})$ is equal (up to a natural conjugacy) to the identity functor. In [Mr3], various examples of normal retractors in the category of modules are discussed. They include, in particular, the direct and inverse limit functors, and the Leray functor introduced in [Mr1].

Let us fix an excisive homotopy invariant functor T from the category of compact pairs to a category $\mathcal{C}$ and a normal functor L from $\mathrm{Endo}(\mathcal{C})$ to a category $\mathcal{D}$. For an automorphism $a\colon A \to A$ in $\mathcal{D}$ denote by $[a]$ its conjugacy class. Let S be an isolated invariant set for the map f. Let (P,Q) be a weak index pair for an isolated invariant set S for the map f (a construction of such a pair can be found in [Mr2, Mr4, Sz]). By Proposition 1, there exists a compact pair $(P',Q') \supset (P,Q)$ such that (P,Q,P',Q') is an index quadruple for f. This in particular means that there is an induced map $f\colon (P,Q) \to (P',Q')$ and $T(i)$ is an isomorphism for the inclusion map $i\colon (P,Q) \hookrightarrow (P',Q')$.

The (L,T)-Conley index of S is defined as

$$\mathrm{Conl}_{LT}(f,S) := \begin{cases} [L(T(i)^{-1}T(f))] & \text{if } T \text{ is covariant,} \\ [L(T(f)T(i)^{-1})] & \text{if } T \text{ is contravariant.} \end{cases}$$

One can prove that the definition is correct, in particular it does not depend on the choice of a weak index pair (see [Mr4]).

4. Isolating segments

In this section we recall some concepts related to the Ważewski method as presented in [Sr5]. Assume for a moment that ϕ is a local flow on the space X and $Z \subset X$. The *exit set* of Z (denoted Z^-) is defined as

$$Z^- := \{z \in Z\colon \exists \epsilon_n > 0, \epsilon_n \to 0\colon \phi_{\epsilon_n}(z) \notin Z\}.$$

In the same way the *entrance set* Z^+ of Z is defined; it is equal to the exit set of Z for the reversed local flow $t \to \phi_{-t}$. The Ważewski Lemma asserts that if Z and

Z^- are closed then the *exit-time map*

$$\sigma_Z\colon Z^0 \ni z \to \sup\{t\colon \forall s \in [0,t]\phi_s(z) \in Z\ \} \in [0,\infty)$$

is continuous, where Z^0 consists of those points $z \in Z$ for which there exists a $t > 0$ such that $\phi_t(x) \notin Z$ (see [C, Sr5]). Let us recall recalled here that the lemma immediately implies the Ważewski Theorem, which asserts the existence of a positive semi-trajectory (or a full trajectory in the case both A and A^- are compact) contained in A if A^- is not a strong deformation retract of A.

We return to the situation considered in Section 2; in particular ϕ is now a local flow generated by the vector-field $(1, v)$ and v is T-periodic in the first variable. For a subset Z of $\mathbb{R} \times M$ and $t \in \mathbb{R}$ we put

$$Z_t := \{x \in M\colon (t,x) \in Z\}$$

and by π_1 and π_2 we denote the projections $\mathbb{R}\times M \to \mathbb{R}$ and, respectively, $\mathbb{R}\times M \to M$.

Let W be a compact subset of $\mathbb{R} \times M$. It is called an *isolating segment* over $[0, T]$ for the equation (1) if the exit and entrance sets $W^\pm$ with respect to the local flow ϕ are also compact and

$$\partial W = W^+ \cup W^-,$$

there exist compact subsets W^{--} and W^{++} of ∂W (called, respectively, the *proper exit set* and the *proper entrance set*) such that

$$W^- = (\{T\} \times W_T) \cup W^{--}, \quad W^+ = (\{0\} \times W_0) \cup W^{++},$$

and there exists a homeomorphism

$$h\colon [0,T] \times W_0 \to W$$

satisfying $\pi_1 \circ h = \pi_1$ such that

$$h([0,T] \times W_0^{--}) = W^{--}, \quad h([0,T] \times W_0^{++}) = W^{++}.$$

The isolating segment W is called *periodic* if $W_0 = W_T$ and $W_0^\pm = W_T^\pm$.

Figure 1 shows an example of a periodic isolating segment W over $[0, T]$ for some planar equation. W is equal to the twisted prism with hexagonal base and each

FIGURE 1.

of the sets W^{--} and W^{++} consists of three disjoint ribbons winding around the prism. Actually, if $T = 2\pi$ and the base is sufficiently large, it is a periodic isolating segment for the equation

$$\dot{z} = e^{it}\bar{z}^2 + a(t)z + b(t)\bar{z} + c(t) \tag{7}$$

in the complex plane, where a, b, and c are continuous 2π-periodic functions (compare [Sr1, Sr2, Sr3]).

Let $0 \le s \le t \le T$. The homeomorphism h induces the map

$$m_{(s,t)}\colon W_s \in x \to \pi_2 h(t, \pi_2 h^{-1}(s,x)) \in W_t.$$

The map

$$\mu_W := m_{(0,T)}$$

is called the *monodromy homeomorphism* of the segment W. It depends on the choice of h, however one can easily prove that its homotopy class is h-independent and therefore it is an invariant of the segment. If W and W^{--} are absolute neighborhood retracts then the Lefschetz number of the monodromy homeomorphism $\Lambda(\mu_W)$ is defined. In that case results of [Sr1, Sr2] assert the existence of a fixed point of the Poincaré operator (hence the existence of a periodic solution of the equation) if $\Lambda(\mu_W)$ is nonzero. For example, it is equal to 1 for the segment in Figure 1, hence (7) has a 2π-periodic solution for arbitrary continuous 2π-periodic functions a, b, and c. Actually, $\Lambda(\mu_W)$ is equal to the fixed point index of the Poincaré operator in some set generated by the segment; for applications of that result to planar equations with the Fourier-Taylor polynomial right-hand side we refer to [Sr1, Sr3].

5. Main Theorem

Since now we assume that W is a periodic isolating segment over $[0,T]$ for (1). It naturally generates an isolated invariant set I_W for the Poincaré operator Π given by

$$I_W := \{x \in W_0 \colon \forall t \in [0,T] \Phi_{(0,t)}\Pi^n(x) \in W_t \ \}.$$

The following result provides calculation of its (L,T)-Conley index for some excisive homotopy invariant functor T and a normal functor L.

Theorem 1.

$$\mathrm{Conl}_{LT}(I_W) = [LT(\mu_W)],$$

where the monodromy homeomorphism μ_W is treated as a map $(W_0, W_0^{--}) \to (W_0, W_0^{--})$.

Denote by $\overline{H}$ the Alexander-Spanier cohomology functor over a fixed ring R. Since $\overline{H}(\mu_W)$ is an automorphism, the previous theorem immediately implies

Corollary 1. *The $(L,\overline{H})$-Conley index of I_W does not depend on the choice of a normal retractor L in the category of R-modules and*

$$\mathrm{Conl}_{L\overline{H}}(\Pi, I_W) = [\overline{H}(\mu_W)].$$

In the case of isolating segment W shown in Figure 1, $\overline{H}^1(W_0, W_0^{--})$ is isomorphic to R^2 and $\overline{H}^q(W_0, W_0^{--})$ is trivial for $q \neq 1$. Since μ_Z rotates the hexagon by the angle $2\pi/3$, by Corollary 1 the Conley index of I_W is equal to the conjugacy class of the matrix $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$.

6. Proof of Theorem 1

In order to prove the theorem, without loss of generality we may assume that both L and T are covariant functors. If $x \in W_0$ then $\phi_t(0,x) \notin W$ for $t > T$, hence we can define

$$\Sigma \colon W_0 \ni x \to \sigma_W(0,x) \in [0,T]$$

(see Section 4). By the Ważewski Lemma the map Σ is continuous, hence the following subsets of W_0

$$P := \{x \in W_0 \colon \Sigma(x) = T\}, \tag{8}$$

$$Q' := \{x \in W_0 \colon \phi_{\Sigma(x)}(0,x) \in W^{--}\}, \tag{9}$$

$$Q := P \cap Q' \tag{10}$$

are compact. One easily verifies that (P, Q, W_0, Q') is a weak index quadruple for I_W. We treat the Poincaré operator as the map

$$\Pi \colon (P,Q) \to (W_0, Q')$$

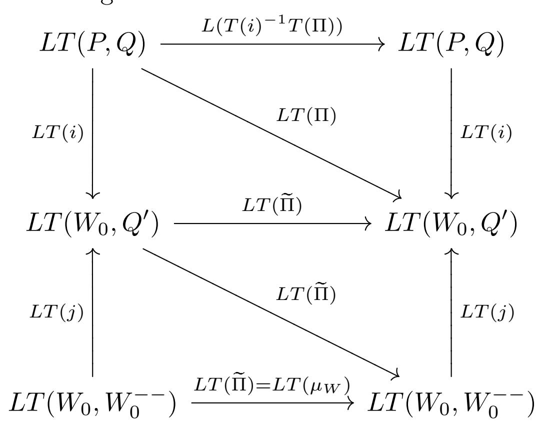
and extend it to the map $\widetilde{\Pi}$ defined in the whole W_0 by the formula

$$\widetilde{\Pi}(x) := \begin{cases} \Pi(x) & \text{if } x \in P, \\ m_{(\Sigma(x),T)}\Phi_{(0,\Sigma(x))}(x) & \text{if } x \in Q'. \end{cases}$$

Note that $\widetilde{\Pi}(Q') = W_0^{--}$ and $\widetilde{\Pi}$ is homotopic to μ_W if it is treated as a map $(W_0, W_0^{--}) \to (W_0, W_0^{--})$; a homotopy is given by

$$W_0 \times [0,1] \ni (x,s) \to \begin{Bmatrix} m_{(\Sigma(x),T)}\Phi_{(0,\Sigma(x))}(x) & \text{if } \Sigma(x) \leq sT \\ m_{(sT,T)}\Phi_{(0,sT)}(x) & \text{if } \Sigma(x) \geq sT \end{Bmatrix} \in W_0.$$

Let $i\colon (P,Q) \hookrightarrow (W_0, Q')$ and $j\colon (W_0, W_0^{--}) \hookrightarrow (W_0, Q')$ be the inclusions. Since i is an excisive map, $T(i)$ is an isomorphism. Thanks to the normality of L, in the following commutative diagram



all horizontal arrows are isomorphisms. $LT(j)$ is also an isomorphism because both the compositions $LT(\widetilde{\Pi}) \circ LT(j)$ and $LT(j) \circ LT(\widetilde{\Pi})$ are isomorphisms (where $\widetilde{\Pi}$ is treated as a map $(W_0, Q') \to (W_0, W_0^{--})$). Thus $L(T(i)^{-1}T(\Pi))$ and $LT(\mu_W)$ are conjugated, hence the result follows. □

7. Applications to bounded solutions

Assume that $M = \mathbb{R}^n$. Each point $x \in I_W$ is an initial point of a bounded solution of the equation (1), hence Theorem 1 can be used as a tool in proving the existence of such solutions. In fact, the Ważewski method (hence also the classical Conley index for continuous flows) is frequently sufficient in determining whether I_W is nonempty; the T-periodic vector-field generates a vector-field on $S^1 \times M$ via the identification S^1 and $\mathbb{R}/T\mathbb{Z}$ and a periodic isolated segment over $[0,T]$ after gluing left and right faces becomes a compact isolating block. Usually it is easy to determine whether its exit set is not its strong deformation retract; in that case there exists a full trajectory contained in it (by the Ważewski Theorem), hence also a bounded solution of the equation.

A more delicate question arises when a bounded solution is known and one would like to determine the existence of another one. In the following two examples of planar equations zero constitutes a bounded solution. In such a situation Conley index with its additivity property becomes helpful in finding a nonzero solution. In the examples we indicate the role of Corollary 1 in the calculation of the required Conley indices.

Let k, p, and q be integers. At first we consider the equation

$$\dot{z} = \frac{1}{p+1} iz + e^{it}\overline{z}^p + e^{ikt}\overline{z}^q \tag{11}$$

for $z \in \mathbb{C}$.

Proposition 2. *Let* $1 \leq p < q$.

(a) *If* $k = 0 \pmod{q+1}$ *then* (11) *has a nonzero* 2π*-periodic solution.*
(b) *If* $k \neq 0 \pmod{q+1}$ *then* (11) *has a nonzero bounded solution.*

Proof. Part (a) is a particular case of more general theorems in [Sr1, Sr2, Sr3], hence we restrict to the proof of (b). It follows from the results of the just cited papers that the equation has two periodic isolating segments over $[0, 2\pi]$ surrounding the 0-axis. The smaller one looks like the one in Figure 1 having the hexagon replaced by the regular $2(p+1)$-polygon (hence three ribbons representing the proper exit set are replaced by $p+1$ ribbons). Its monodromy map is equal to the rotation by the angle $2\pi/(p+1)$, hence its first cohomology has the matrix

$$A := \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & 0 & \dots & 0 & -1 \\ 0 & 1 & 0 & \dots & 0 & -1 \\ 0 & 0 & 1 & \dots & 0 & -1 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \end{bmatrix} \tag{12}$$

where the number of rows and columns is equal to p. The larger segment is equal to the union of k-copies of similar segments (with p replaced by q) connected by the side faces. Consequently, the first cohomology of its monodromy is equal to A^k, where A is the $q \times q$ matrix given in (12). Since $p \neq q$, the cohomologies of the mondoromy homeomorphisms of the segments cannot be conjugated, hence Corollary 1 implies that the invariant set corresponding to the larger segment contains points different from zero, which results in the existence of a nonzero bounded solution. □

In the next example the matrices generated by the monodromy maps have the same dimension and we use determinants in order to distinguish their conjugacy classes. Let r be another integer. We consider the equation

$$\dot{z} = \frac{1}{p+1} iz + e^{it}\overline{z}^p + e^{ikt} z^q \overline{z}^r \tag{13}$$

Proposition 3. *Let $p \geq 1$ be odd, $p < q + r$, and let $r - q = p$.*

(a) *If $k = 0 \bmod p+1$ then* (13) *has a 2π-periodic nonzero solution.*
(b) *If $k \neq 0 \bmod p+1$ and k is even then* (13) *has a bounded nonzero solution.*

Proof. As before, we skip the proof of (a), since it is a consequence of [Sr1, Sr2, Sr3]. For a proof of (b) we again consider two periodic isolating segments which existence follows from the cited papers. The smaller one is exactly the same as in the proof of Proposition 2; in particular it is constructed using a regular $2(p+1)$-polygon. The larger one is build as the union of k-copies of some similar segment, but contrary to the previous proof the number of ribbons forming the proper exit sets of both segments is equal to $p+1$. The first cohomologies of their monodromy homeomorphisms are represented by A and, respectively, A^k, where A is the $p \times p$-matrix given in (12). Since the determinant of A is equal to $(-1)^p$, p is odd and k is even, there is no conjugacy and Corollary 1 implies the result. □

It is not clear whether the bounded solutions in points (b) of Propositions 2 and 3 are periodic; since the corresponding Lefschetz numbers are all equal to 1, the methods of [Sr1, Sr2] do not recognize other periodic solutions from the zero one.

References

[C] C. Conley, Isolated invariant sets and the Morse index, CBMS Regional Conference Series in Mathematics 38, American Mathematical Society, Providence, RI, 1978.

[D] Z. Dzedzej, Remarks on bounded solutions for some nonautonomous ODE, Univ. Iagel. Acta Math. No. 41 (2003), 157–162, erratum: 42 (2004), 199.

[FZ] A. Fonda, F. Zanolin, Bounded solutions of nonlinear second order ordinary differential equations, Discrete Contin. Dyn. Systems 4 (1998) 91–98.

[IR] M. Izydorek, S. Rybicki, Bifurcations of bounded solutions of 1-parameter ODEs, J. Differential Equations 130 (1996), 267–276.

[K] W. Karpińska, On bounded solutions of nonlinear differential equations at resonance, Nonlinear Anal. – Theory Meth. Appl. 51 (2002), 723–733.

[Ma1] J. Mawhin, Bounded solutions of nonlinear ordinary differential equations, in: F. Zanolin (ed.), Non linear analysis and boundary value problems for ordinary differential equations, CISM Courses and Lectures 371, Springer, Vienna 1996, 121–147.

[Ma2] J. Mawhin, Remarques sur les solutions bornées ou presque périodiques de l'équation du pendule forcé, Ann. Sci. Math. Québec 22 (1998), 213–224.

[MT] J. Mawhin, H. B. Thompson, Periodic or bounded solutions of Carathèodory systems of ordinary differential equations, J. Dynam. Differential Equations 15 (2003), 327–334.

[MW1] J. Mawhin, J. R. Ward, Jr., Bounded solutions of some second order nonlinear differential equations, J. London Math. Soc. (2) 58 (1998), 733–747.

[MW2] J. Mawhin, J. R. Ward, Jr., Guiding-like functions for periodic or bounded solutions of ordinary differential equations, Discrete Contin. Dyn. Syst. 8 (2002), 39–54.

[Mr1] M. Mrozek, Leray functor and cohomological Conley index for discrete dynamical systems, Trans. Amer. Math. Soc. 318 (1990), 149–178.

[Mr2] M. Mrozek, Shape index and other indices of Conley type for continuous maps in locally compact metric spaces, Fund. Math. 145 (1994), 15–37.

[Mr3] M. Mrozek, Normal functors and retractors in categories of endomorphisms, Univ. Jagiell. Acta. Math. 29 (1992), 181–198.

[Mr4] M. Mrozek, Index pairs algorithms, Foundations Comput. Math., accepted.

[MRS] M. Mrozek, J. Reineck, R. Srzednicki, The Conley index over the circle, J. Dynamics Differential Equations 12 (2000), 385–409.

[MR] M. Mrozek, K. P. Rybakowski, Nontrivial full bounded solutions of time-periodic semilinear PDEs, Proc. Roy. Soc. Edinburgh 117A (1991), 305–315.

[Mu] E. Mukhamadiev, Bounded solutions and homotopic invariants of systems of nonlinear differential equations, Dokl. Akad. Nauk 351 (1996), 596–598.

[O] R. Ortega, A boundedness result of Landesman-Lazer type, Differential Integral Equations 8 (1995), 729–734.

[RS] J. Robbin, D. Salamon, Dynamical systems, shape theory and the Conley index, Ergodic Theory Dynamical Systems 8* (1988), 375–393.

[Sr1] R. Srzednicki, A geometric method for the periodic p roblem in ordinary differential equations, Séminaire d'Analyse Moderne, no. 22, Université de Sherbrooke, 1992.

[Sr2] R. Srzednicki, Periodic and bounded solutions in blocks for time-periodic non-autonomous ordinary differential equations, Nonlinear Anal. – Theory Meth. Appl. 22 (1994), 707–737.

[Sr3] R. Srzednicki, On periodic solutions of planar polynomial differential equations with periodic coefficients, J. Differential Equations 114 (1994), 77–100.

[Sr4] R. Srzednicki, Generalized Lefschetz Theorem and a fixed point index formula, Topology Appl. 81 (1997), 207–224.

[Sr5] R. Srzednicki, Ważewski Method and Conley Index, in: A. Cañada, P. Drábek, A. Fonda (eds.), Handbook of Differential Equations: Ordinary Differential Equations, vol. 1, Elsevier B.V., Amsterdam 2004, 591–684.

[Sz] A. Szymczak, The Conley index for the discrete semidynamical systems, Topology Appl. 66 (1995), 215–240.

[V] G. Verzini, Bounded solutions to superlinear ODEs: a variational approach, Nonlinearity 16 (2003), 2013–2028.

[Wa1] J. R. Ward, Jr., A topological method for bounded solutions of non-autonomous ordinary differential equations, Trans. Amer. Math. Soc. 333 (1992), 709–720.

[Wa2] J. R. Ward, Jr., Global continuation for bounded solutions of ordinary differential equations, Topol. Methods Nonlinear Anal. 2 (1993), 75–90.

[Wa3] J. R. Ward, Jr., Homotopy and bounded solutions of ordinary differential equations, J. Differential Equations 107 (1994), 428–445.

[Wa4] J. R. Ward, Jr., Bifurcation of bounded solutions of ordinary differential equations, J. Korean Math. Soc. 37 (2000), 707–720.

[Wo] K. Wójcik, Remark on bounded solutions of certain nth order differential equations, Univ. Iagel. Acta Math. 38 (2000), 307–313.

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