Discrete version of a geometric method for detecting chaotic dynamics.

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Abstract

We present a geometric method for detecting chaotic dynamics in discrete dynamical systems.

Keywords: isolating block, Conley index, fixed point index, periodic points, chaos

1 Introduction

In recent years there has been growing interest in the study of chaotic dynamics by means of topological tools. Chaotics dynamics are difficult to study in general and there are few rigorous results about chaotic dynamics in concrete dynamical systems. The advantage of topological approach is that the topological criteria are often easier to verify than the criteria based on smoothness and consequently easier to apply.

Among the first topological criteria for chaos were two criteria presented in [1]. The first one was based on the Conley index, the other on the fixed point index and continuation methods. The Conley index criterion was later applied in [2, 3] to prove chaos in the Lorenz equations. The Conley index criterion was refined by Szymczak [9], who proved a conjecture presented in [1]. The fixed point index criterion was further developed by Zgliczyski [11], who applied it to the Hénon map and Rössler equations [12, 13].

Another topological criterion for chaos was presented in [7]. It is based on the work of Srzednicki [6] who developed the machinery of isolating segments

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to compute the fixed point index of the Poincaré map of a flow directly from the geometric features of an isolating block of the flow. The criterion for chaos based on this machinery applies directly to differential equations with periodic forcing, which is in contrast to the other mentioned criteria: they apply to discrete dynamical systems and may be applied to differential equations only via the study of the Poincaré map. The criterion uses two isolating segments whoose exit sets on a certain Poincaré section are the same but the fixed point indexes computed for the two neighborhoods are different.

Despite the direct applicability to differential equations of the criterion for chaos developed in [7] it is still hard work to construct analytically the necessary isolating segments for concrete differential equations. Therefore it would be very helpful to have a discrete counterpart of the geometric criterion, because this would open the way to computer assisted proofs based on such an analogue. Surprisingly, it is not obvious what a discrete counterpart of the geometric criterion in [7] should be. The aim of this paper is to present a possible analogue of [7] in the discrete case.

Let us remark that the criterion we present should have some connections to the criterion developed in [9]. Originally the authors even intended to use [9] to prove their criterion but so far they failed. Eventually the proof follows relatively closely the proof for differential equations in [7] and should be easy to follow for readers familiar with that proof. In particular let us notice that the reason of introducing an auxiliary map F defined on $3n$ copies of the original space is just to be able to mimic to some extend the proof in [7]. The relation to [9] is not obvious, because the criterion in [9] requires the decomoposition of the isolating neighborhood into at least two connected components. In the example we provide the isolating neighborhoods are connected. Of course the assumptions we make may imply that there are some other isolating neighborhoods inside, which satisfy the criterion in [9], but so far we do not know how to do it. Also, even if we assume that an isolating neighborhood as in [9] exists, it would have to be closer to the invariant set inside, so using it directly in numerical computations would make them much more expensive.

2 Preliminaries

$\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ will denote the sets of real, rational, integer and natural numbers, respectively. For a topological space X and a subset $A \subset X$ the notation $\mathrm{int}_{\mathrm{X}}(A)$, $\mathrm{cl}_{\mathrm{X}}(A)$, $\mathrm{bd}_{\mathrm{X}}(A)$ will be used for the interior, the closure, and the boundary of A in X, respectively. If this causes no misunderstanding, we shall drop the subscript X in the above notation. We say that (A, B) is a pair of subsets of X if $B \subset A \subset X$.

Let X be a locally compact metric space and let $f : X \longrightarrow X$ be a homeomorphism. For a subset $K \subset X$ we define the invariant part of K (with respect to f) by

$$\mathrm{inv_f}(K) = \bigcap_{j=-\infty}^{+\infty} f^j(K).$$

A pair $P = (P_1, P_2)$ of compact subsets of X is called an index pair with respect to f if and only if the following conditions hold

(A) if $x \in P_2$ and $f(x) \in P_1$, then $f(x) \in P_2$

(B) if $x \in P_1$ and $f(x) \notin P_1$, then $x \in P_2$

(C) $\mathrm{inv_f}(\mathrm{cl}(P_1 \setminus P_2)) \subset \mathrm{int}(\mathrm{cl}(P_1 \setminus P_2))$

Let P be an index pair for f and H be the Alexander-Spanier cohomology functor (with rational coefficients). We put

$$f_P : (P_1, P_2) \ni x \longrightarrow f(x) \in (P_1 \cup f(P_2), P_2 \cup f(P_2)),$$

$$i_P : (P_1, P_2) \ni x \hookrightarrow x \in (P_1 \cup f(P_2), P_2 \cup f(P_2)).$$

It follows from the strong excision property that i_P induces an isomorphism in the Alexander-Spanier cohomology.

Definition 1 *The endomorphism $H(f_P) \circ H(i_P)^{-1}$ of $H(P_1, P_2)$ is called the index map associated with the index pair P and is denoted by χ_P.*

The index map was first introduced in [5] (see also [4]).

Let $N \subset X$ be compact. We say that N is an isolating block for f if and only if $f^{-1}(N) \cap N \cap f(N) \subset \mathrm{int}(N)$. One can check that if N is an isolating block, then (N, N^-) is an index pair with $N^- = N \setminus f^{-1}(\mathrm{int}(N))$ (compare [1]). For an isolating block N by χ_N we will denote the index map associated with the index pair (N, N^-).

Assume that X is an ENR i.e. a Euclidean Neighborhood Retract and P is an index pair for f such that χ_P is an isomorphism. Put $S = \mathrm{inv}_f(\mathrm{cl}(P_1 \setminus P_2))$. The following result was proved in [5] (compare also lemma 5.2 in [8])

Theorem 1 *Let $K \subset S$ be the set of fixed points of f contained in S. Then, K is compact and open in the set of fixed points of f, $H(P_1, P_2)$ is of finite type and the fixed point index $\mathrm{ind}(f, K)$ is equal to the Lefschetz number of χ_P.*

3 Main result

Let X be an ENR. Assume that $M \subset N$ are isolating blocks with respect to f such that

(a) $\chi_M = \mathrm{id}_{\mathbb{Q}}$, $\chi_N = -\mathrm{id}_{\mathbb{Q}}$,

(b) $f(N) \cap M \cap f^{-1}(N) \subset \mathrm{int}(M)$,

(c) $f(N \setminus f^{-1}(\mathrm{int}(M))) \cap M \subset N^-$,

(d) all inclusions in the diagram

$$\begin{array}{ccc} (M, M \cap N^-) & \longrightarrow & (N, N^-) \\ \downarrow & & \downarrow \\ (M, M^-) & \longrightarrow & (N, N \setminus f^{-1}(\mathrm{int}(M))) \end{array}$$

induce isomorphisms in the Alexander-Spanier cohomology.

Put $I = \mathrm{inv}_f N = \mathrm{inv}_f(\mathrm{cl}(N \setminus N^-))$. Let $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$ and $\sigma : \Sigma_2 \to \Sigma_2$ be a shift map.

Theorem 2 *There is a continuous, surjective map $g : I \to \Sigma_2$ such that f restricted to I is semiconjugated by g to the shift σ i.e. $g \circ f = \sigma \circ g$. Moreover, for any n-periodic sequence of symbols $c \in \Sigma_2$ its counterimage $g^{-1}(c)$ contains an n-periodic point for f.*

Remark 3 It follows from our proof of Theorem (2) that the condition (a) can be replaced by the condition

(a') $\chi_M = -\mathrm{id}_{\mathbb{Q}}$, $\chi_N = \mathrm{id}_{\mathbb{Q}}$.

Lemma 4 *If $x \in I$, then $x \in \mathrm{int}(M)$ or $x \notin M$.*

Proof: Suppose that $x \in M \setminus \mathrm{int}(M)$. Since $x \in I$, there exists a $y \in I$ such that $x = f(y)$. Consequently, $y \in N \setminus f^{-1}(\mathrm{int}(M))$. From assumption (c) we have

$$x = f(y) \in f(N \setminus f^{-1}(\mathrm{int}(M))) \cap M \subset N^-,$$

a contradiction.

With every point $x \in I$ we associate a symbol $h(x) \in \{0,1\}$ by the rule $h(x) = 0$ if $x \in \mathrm{int}(M)$ and $h(x) = 1$ if $x \notin M$. Define

$$g : I \ni x \longrightarrow \{h(f^n(x))\}_{n \in \mathbb{Z}} \in \Sigma_2. \tag{1}$$

Remark 5 It follows from Lemma 4 that g is continuous. In order to prove the theorem it suffices to prove that periodic sequences are contained in the image of the map g. Indeed, since I is compact and the set of periodic points is dense in Σ_2, g must be surjective.

Let $n \geq 1$ be fixed. We define an auxiliary function

$$F : X \times \mathbb{Z}_{3n} \longrightarrow X \times \mathbb{Z}_{3n}$$

by

$$F(x,i) = (f_i(x), i+1), \tag{2}$$

where the homeomorphism $f_i : X \to X$ is given by

$$f_i(x) = \begin{cases} \text{f(x)}, & \text{if } i \equiv 1 \bmod 3, \\ \text{x}, & \text{otherwise.} \end{cases} \tag{3}$$

One can check that $F^{3n}(x,i) = (f^n(x), i)$. In particular,

$$F^{3n}(X \times \{i\}) \subset X \times \{i\}.$$

For $A \subset X \times \mathbb{Z}_{3n}$ and $i \in \mathbb{Z}_{3n}$ we put

$$A_i = A \cap (X \times \{i\}).$$

Let $c = (c_0, ..., c_{n-1}) \in \Sigma_2$ be an n-periodic sequence. We define the pair $P(c) = (P_1(c), P_2(c)) \subset X \times I_n$ by

$$P_1(c)_{3i+k} = \begin{cases} M \times \{3i+k\}, & \text{if } k = 2 \text{ and } c_i = 0 \ , \\ N \times \{3i+k\}, & \text{otherwise ,} \end{cases} \tag{4}$$

$$P_2(c)_{3i+k} = \begin{cases} (N \setminus f^{-1}(\text{int}(M))) \times \{3i+k\} & \text{if } k = 1 \text{ and } c_i = 0, \\ (M \cap N^-) \times \{3i+k\} & \text{if } k = 2 \text{ and } c_i = 0, \\ N^- \times \{3i+k\} & \text{otherwise,} \end{cases} \tag{5}$$

for $i \in \{0, 1, \ldots, n-1\}$.

Lemma 6 **(1)** *if $x \in N$ and $f(x) \notin M$, then $x \in N \setminus f^{-1}(\text{int}(M))$,*

(2) *if $x \in N \setminus f^{-1}(\text{int}(M))$ and $f(x) \in M$, then $f(x) \in M \cap N^-$.*

Proof: (1) is obvious. It follows from assumption (c) that $f(x) \in M \cap f(N \setminus f^{-1}(\text{int}(M))) \subset N^-$, so $f(x) \in M \cap N^-$. This proves (2).

Put $W = N \cap f^{-1}(N)$.

Lemma 7 *Assume that $x \in f(W) \cap W \cap f^{-1}(W)$. Then*

(W1) $x \in \text{int}(N) \cap f^{-1}(\text{int}(N))$,

(W2) *if $x \in N \cap f^{-1}(M)$ and $f(x) \in M \cap f^{-1}(N)$, then $x \in \text{int}(N) \cap f^{-1}(\text{int}(M))$,*

(W3) *if $x \in (M \cap f^{-1}(N)) \cap f(W)$, then $x \in \text{int}(M) \cap f^{-1}(\text{int}(N))$.*

Proof:
(1) Since $x \in W \cap f^{-1}(W)$, we have $f(x) \in f(N) \cap N \cap f^{-1}(N)$. Since N is an isolating block, we get $f(x) \in \text{int}(N)$. From $x \in f(W)$ it follows that there exists a $y \in W$ such that $f(y) = x \in W$. We now apply this argument again, with x replaced by y and obtain $x = f(y) \in \text{int}(N)$.

(2) Since $x \in N \cap f^{-1}(M)$, we have $f(x) \in f(N) \cap M$, and $f(x) \in f(N) \cap M \cap f^{-1}(N)$. By (b) we get $f(x) \in \text{int}(M)$ and the already proved property (1) shows that $x \in \text{int}(N)$. Consequently $x \in \text{int}(N) \cap f^{-1}(\text{int}(M))$.

(3) In this case $x \in f(N) \cap M \cap f^{-1}(N)$, so again by assumption (b) and property (1) $x \in \text{int}(M) \cap f^{-1}(\text{int}(N))$.

Corollary 8 *For any $n \geq 1$ and an n-periodic sequence $c \in \Sigma_2$ the pair $P(c)$ is an index pair for F.*

Proof: The conditions (A), (B) follow from Lemma 6. In order to prove the condition (C) we have to show that

$$\text{inv}_\text{F}(\text{cl}(P_1(c) \setminus P_2(c))) \subset \text{int}(\text{cl}(P_1(c) \setminus P_2(c))). \tag{6}$$

One can check that for $i \in \{0, 1, \dots, n-1\}$
$(P_1(c) \setminus P_2(c))_{3i+k} =$

$$\begin{cases} (M \cap f^{-1}(\text{int}(N))) \times \{3i+k\}, & \text{if } k = 2 \text{ and } c_i = 0 \ , \\ (N \cap f^{-1}(\text{int}(M))) \times \{3i+k\}, & \text{if } k = 1 \text{ and } c_i = 0 \ , \\ (N \cap f^{-1}(\text{int}(N))) \times \{3i+k\}, & \text{otherwise} \ . \end{cases} \tag{7}$$

It follows that

$$\text{cl}(P_1(c) \setminus P_2(c)) \subset W \times \mathbb{Z}_{3n}. \tag{8}$$

In particular

$$\text{inv}_\text{F}(\text{cl}(P_1(c) \setminus P_2(c))) \subset \text{inv}_\text{F}(W \times \mathbb{Z}_{3n}). \tag{9}$$

Let $(x, 3i+k) \in \text{inv}_\text{F}(\text{cl}(P_1(c) \setminus P_2(c))))$ for some $i \in \{0, 1, \dots, n-1\}$ and $k \in \{0, 1, 2\}$. Since

$$\text{inv}_\text{F}(W \times \mathbb{Z}_{3n}) = \bigcap_{j=-\infty}^{+\infty} F^j(W \times \mathbb{Z}_{3n}),$$

it follows from equations (2) and (3) that

$$x \in f^{-1}(W) \cap W \cap f(W). \tag{10}$$

If $c_i \neq 0$ or $c_i = 0$ and $k = 0$ then by $(W1)$ in Lemma 7 we have that

$$(x, 3i+k) \in (\text{int}(N) \cap f^{-1}(\text{int}(N))) \times \{3i+k\} \subset \text{int}((P_1(c) \setminus P_2(c))_{3i+k}).$$

If $k = 1$ and $c_i = 0$ then by (7) and (3)

$$x \in N \cap f^{-1}(M), \tag{11}$$

$$f(x) \in M \cap f^{-1}(N), \tag{12}$$

so by $(W2)$ in Lemma 7

$$(x, 3i+k) \in \operatorname{int}((P_1(c) \setminus P_2(c))_{3i+k}).$$

If $k = 2$ and $c_i = 0$ then by (7) $x \in M \cap f^{-1}(N)$. On the other hand, since $(x, 3i+2) \in \operatorname{inv}_{\mathrm{F}}(\operatorname{cl}(P_1(c) \setminus P_2(c)))$ there is $(y, 3i+1) \in \operatorname{cl}((P_1(c) \setminus P_2(c))_{3i+1})$ such that $F(y, 3i+1) = (f(y), 3i+2) = (x, 3i+2)$. Again by (7) we obtain that $y \in N \cap f^{-1}(M)$. In particular,

$$x \in M \cap f^{-1}(N) \cap f(W),$$

so by $(W3)$ in Lemma 7

$$(x, 3i+2) \in (\operatorname{int}(M) \cap f^{-1}(\operatorname{int}(N))) \times \{3i+1\} \subset \operatorname{int}((P_1(c) \setminus P_2(c))_{3i+1}),$$

and (6) follows.

Let $\chi_{P(c)}$ be the index map associated with index pair $P(c)$ for n-periodic sequence $c \in \Sigma_2$. Observe that

$$H(P(c)) = \bigoplus_{i=0}^{3n-1} H(P(c)_i),$$

$$\chi_{P(c)} = \bigoplus_{i=0}^{3n-1} \chi_i,$$

where

$$\chi_i = \chi_{P(c)}|_{P(c)_i} : H(P(c)_i) \to H(P(c)_{i-1}).$$

The index map $\chi_{P(c)}$ has the matrix

$$\begin{bmatrix} 0 & \chi_1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \chi_{3n-1} \\ \chi_0 & \dots & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

Lemma 9 *Assume that* 1 *appears in the sequence* c *exactly* k *times. Then for* $i \in \mathbb{Z}_{3n}$

$$(\chi_{P(c)})^{3n}|_{H(P(c)_i)} = (-1)^k \mathrm{id}_{\mathbb{Q}}, \tag{14}$$

so $\chi^{3n}_{P(c)}$ *has the matrix*

$$\begin{bmatrix} (-1)^k & 0 & \dots & 0 \\ 0 & (-1)^k & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & (-1)^k \end{bmatrix} \tag{15}$$

Proof: Put $Y = N \setminus f^{-1}(\mathrm{int}(M))$ and consider the following commutative diagram

$$\begin{tikzcd}
& (M, M^-) \arrow[r, "f_{(M,M^-)}"] \arrow[d] & (M \cup f(M^-), M^- \cup f(M^-)) & (M, M^-) \arrow[l, "i_{(M,M^-)}"'] & \\
& (M, M^-) \arrow[r, "\bar{f}"] \arrow[d] & (M \cup f(M^-), (M \cap N^-) \cup f(M^-)) \arrow[u] \arrow[d] & (M, M \cap N^-) \arrow[l, "\bar{k}"'] \arrow[u] & \\
(N, N^-) \arrow[r, "\bar{f}_{3i}"] & (N, Y) \arrow[r, "\bar{f}_{3i+1}"] & (M \cup f(Y), (M \cap N^-) \cup f(Y)) & (M, M \cap N^-) \arrow[l, "k"'] \arrow[u] \arrow[r, "\bar{f}_{3i+2}"] & (N, N^-)
\end{tikzcd}$$

where $\bar{f}$, $f_{(M,M^-)}$ are induced by f, $\bar{f}_i$ are induced by f_i and all other maps are inclusions. Note that $i_{(M,M^-)}$ and k are excisions and all other inclusions induce isomorphisms in cohomology by the condition (d). Let

$$F_i = F_{P(c)}|_{P(c)_i} : (P_1(c)_i, P_2(c)_i) \to (P_1(c)_{i+1} \cup F(P_2(c)_i), P_2(c)_{i+1} \cup F(P_2(c)_i)),$$

$$i_{P(c)_i} = i_{P(c)}|_{P(c)_i} : (P_1(c), P_2(c))_i \to (P_1(c) \cup F(P_2(c)), P_2(c) \cup F(P_2(c)))_i.$$

If $c_i = 1$ then

$$F_{3i+k} : (N, N^-) \times \{3i+k\} \ni (x, 3i+k) \to (x, 3i+k+1) \in (N, N^-) \times \{3i+k+1\},$$

for $k \in \{0, 2\}$ and

$$F_{3i+1} : (N, N^-) \times \{3i+1\} \to (N \cup f(N^-), N^- \cup f(N^-)) \times \{3i+2\},$$

is induced by f, hence

$$\chi_{3i+3} = \chi_{3i+1} = \mathrm{id}_{\mathbb{Q}},$$
$$\chi_{3i+2} = \chi_N = -\mathrm{id}_{\mathbb{Q}},$$

so

$$\chi_{3i+1} \circ \chi_{3i+2} \circ \chi_{3i+3} = -\mathrm{id}_{\mathbb{Q}}. \tag{16}$$

Assume that $c_i = 0$. Then

$$F_{3i} : (N, N^-) \times \{3i\} \ni (x, 3i) \to (x, 3i+1) \in (N, Y) \times \{3i+1\},$$

is induced by $\bar{f}_{3i}$ and

$$i_{P(c)_{3i+1}} = \mathrm{id}_{H((N,Y)\times\{3i+1\})},$$

so

$$\chi_{3i+1} = H(F_{3i}).$$

Similarly

$$F_{3i+2} : (M, M\cap N^-)\times\{3i+2\} \ni (x, 3i+2) \to (x, 3i+3) \in (N, N^-)\times\{3i+3\},$$

is induced by $\bar{f}_{3i+2}$ and

$$i_{P(c)_{3i+3}} = \mathrm{id}_{H(N,N^-)\times\{3i+3\}},$$

so

$$\chi_{3i+3} = H(F_{3i+2}).$$

One can check that F_{3i+1} is induced by $\bar{f}_{3i+1}$ and $i_{P(c)_{3i+2}}$ is induced by inclusion k so it follows from the diagram and assumptions (a) and (d) that

$$\chi_{3i+1}\circ\chi_{3i+2}\circ\chi_{3i+3} = \mathrm{id}_{\mathbb{Q}}. \tag{17}$$

Since

$$(\chi_{P(c)})^{3n}|_{H(P(c)_0)} = \chi_1\circ\ldots\circ\chi_{3n-1}\circ\chi_0,$$

for $i = 0$ the result follows from (16) and (17).

In a similar way one can check that for $i \in \mathbb{Z}_{3n}$

$$(\chi_{P(c)})^{3n}|_{H(P(c)_i)} = (-1)^k\mathrm{id}_{\mathbb{Q}}.$$

Let $S_c = \mathrm{inv}_F(\mathrm{cl}(P_1(c)\setminus P_2(c)) \subset I\times\mathbb{Z}_{3n}$. By the properties of the Conley index (see theorem (1)) the Lefschetz number of $(\chi_{P(c)})^{3n}$ is well defined and is exactly the fixed point index of F^{3n} in S_c. Let

$$K_c = \{(x,i)\in S_c :\ F^{3n}(x,i) = (x,i)\}. \tag{18}$$

be the set of fixed points of F^{3n} contained in S_c. It follows from Lemma 9 that

$$\mathrm{ind}(F^{3n}, K_c) = (-1)^k 3n. \tag{19}$$

Proof of Theorem 2: Let

$$J_c = \{(x,i)\in S_c : x\in g^{-1}(c),\quad f^n(x) = x\}. \tag{20}$$

Since

$$F^{3n}(x,i) = (f^n(x), i), \tag{21}$$

it suffices to prove that $\mathrm{ind}(F^{3n}, J_c) \neq 0$. This follows from

Lemma 10 *Assume that* 1 *appears in the sequence* c *exactly* $k\leq n$ *times. Then*

$$\mathrm{ind}(F^{3n}, J_c) = (-2)^k 3n.$$

Proof: If $k = 0$ then $c = (0)$, $K_c = J_c$, so by (19)

$$\mathrm{ind}(F^{3n}, J_c) = (\mathrm{ind}(F^{3n}, K_c) = 3n. \tag{22}$$

For $k \geq 1$ we use the induction with respect to k. Let $k = 1$. Since $K_{(0)}$ and J_c form a compact and disjoint covering of K_c,by the additivity property of the fixed point index and (19)

$$-3n = \mathrm{ind}(F^{3n}, K_c) = \mathrm{ind}(F^{3n}, K_{(0)}) + \mathrm{ind}(F^{3n}, J_c),$$

so again by (19)

$$\mathrm{ind}(F^{3n}, J_c) = -6n. \tag{23}$$

Assume now that the lemma holds for $1 \leq k < n$. We will prove it for $k+1$. Denote by Γ the set of all sequences $z = (z_0, ..., z_{n-1})$ such that $c_i = 0$ implies $z_i = 0$ and 1 appears exactly l times in z for some $1 \leq l \leq k$. One can check that $J_c \cup K_{(0)} \cup \bigcup_{z \in \Gamma} J_z$ is a compact and disjoint covering of K_c. Then, again by the additivity of the fixed point index

$$\mathrm{ind}(F^{3n}, K_c) = \mathrm{ind}(F^{3n}, J_c) + \mathrm{ind}(F^{3n}, K_{(0)}) + \sum_{z \in \Gamma} \mathrm{ind}(F_n^{3n}, J_z),$$

hence by the inductive step and (19)

$$\mathrm{ind}(F^{3n}, J_c) = \left((-1)^{k+1} - 1 - \sum_{l=1}^{k} \binom{k+1}{l} (-2)^l \right) 3n.$$

If k is odd then the formula

$$\sum_{l=1}^{k} \binom{k+1}{l} (-2)^l = -2^{k+1} \tag{24}$$

implies the required equation, because 1 appears $k+1$ times in the sequence c, which is an even number. Similarly, if k is even then

$$\sum_{l=1}^{k} \binom{k+1}{l} (-2)^l = -2 + 2^{k+1} \tag{25}$$

and the equation holds.

4 Some generalizations.

In this section we assume the conditions (b), (c), (d) and additionally the conditions

(a_1) $\chi_M = \mathrm{id}_Z$, $\mathrm{Lef}(\chi_N) \neq \chi(Z))$ where $Z = H(N, N^-)$ is a finite dimensional vector space and $\chi(Z)$ is the Euler-Poincaré characteristic of Z,

(a_2) there is a natural number $n_0 \geq 2$ such that

$$\mathrm{Lef}(\chi_N) = \ldots = \mathrm{Lef}(\chi_N^{n_0-1}),$$

$$\chi_N^{n_0} = \mathrm{id}_Z.$$

For $I = \mathrm{inv}_f(N)$ the map $g : I \to \Sigma_2$ is defined by (1). Let $c = (c_0, \ldots, c_{n-1}) \in \Sigma_2$ be an n-periodic sequence of symbols. Let F be the map defined by (2) and (3). Assume that $P(c)$ is the index pair for F associated with the sequence c (see (4) and (5)). Recall that the index map $\chi_{(P(c))}$ has the matrix (13). It follows from the arguments in the proof of (16) and (17) that

$$\chi_{3i+1} \circ \chi_{3i+2} \circ \chi_{3i+3} = \begin{cases} \chi_N, & c_i = 1 \\ \mathrm{id}_Z, & c_i = 0. \end{cases} \tag{26}$$

Recall that the set J_c is the set of fixed points of F^{3n} contained in $\mathrm{inv}_F(\mathrm{cl}(P_1(c) \setminus P_2(c))$ coded by the sequence c (compare (20)).

Lemma 11 *Let the symbol* 1 *appears exactly* $k \in \{0, \ldots, n\}$ *times in the sequence* $c = (c_0, \ldots, c_{n-1}) \in \Sigma_2$. *Then*

$$\mathrm{ind}(F^{3n}, J_c) = \left(\sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \mathrm{Lef}(\chi_N^l) \right) 3n. \tag{27}$$

Proof: For $k = 0$, obviously

$$\mathrm{ind}(F^{3n}, J_{(0)}) = \mathrm{Lef}(\oplus_{i \in \mathbb{Z}_{3n}} \chi_N^0) = \mathrm{Lef}(\oplus_{i \in \mathbb{Z}_{3n}} \mathrm{id}_Z) = 3n\, \chi(Z).$$

For $k \geq 1$ we use the induction with respect to k. Let $k = 1$. Since $K_{(0)}$ and J_c form a compact and disjoint covering of K_c, by (26) and the additivity of the fixed point index

$$\mathrm{Lef}(\chi_N) = \mathrm{ind}(F^{3n}, K_c) = \mathrm{ind}(F^{3n}, K_{(0)}) + \mathrm{ind}(F^{3n}, J_c),$$

hence

$$\mathrm{ind}(F^{3n}, J_c) = (\mathrm{Lef}(\chi_N) - \chi(Z))\, 3n.$$

Assume that the formula holds for $1 \geq k < n$. We will prove it for $k + 1$. Let Γ be the set of all sequences $z = (z_0, \ldots, z_{n-1})$ such that $c_i = 0$ implies $z_i = 0$ and the symbol 1 appears exactly l times in z for some $0 \leq l \leq k$. One can check that $J_c \cup \bigcup_{z \in \Gamma} J_z$ is a compact and disjoint covering of K_c. Then, by the additivity property of the fixed point index

$$\mathrm{ind}(F^{3n}, K_c) = \mathrm{ind}(F^{3n}, J_c) + \sum_{z \in \Gamma} \mathrm{ind}(F^{3n}, J_z).$$

Since by (26)

$$\mathrm{ind}(F^{3n}, K_c) = \mathrm{Lef}(\chi_{P(c)}^{3n}) = 3n\, \mathrm{Lef}(\chi_N^{k+1}),$$

we get from the inductive step

$$\operatorname{ind}(F^{3n}, J_c) = \left(\operatorname{Lef}(\chi_N^{3n}) - \sum_{s=0}^{k} \binom{k+1}{s} \sum_{l=0}^{s} (-1)^{s-l} \binom{s}{l} \operatorname{Lef}(\chi_N^l)\right) 3n =$$

$$\left(\operatorname{Lef}(\chi_N^{k+1}) - \sum_{s=0}^{k} \sum_{l=0}^{s} (-1)^{s-l} \binom{k+1}{s} \binom{s}{l} \operatorname{Lef}(\chi_N^l)\right) 3n =$$

$$\left(\operatorname{Lef}(\chi_N^{k+1}) - \sum_{s=0}^{k} \sum_{l=0}^{s} (-1)^{s-l} \binom{k+1}{l} \binom{k+1-l}{k+1-s} \operatorname{Lef}(\chi_N^l)\right) 3n.$$

Let $s_0 \in \{0, \ldots, k\}$ be fixed. One can easy check that the coefficient of $\operatorname{Lef}(\chi_N^{s_0})$ equals

$$\left[-\sum_{r=s_0}^{k} (-1)^{r-s_0} \binom{k+1-s_0}{k+1-r}\right] \binom{k+1}{s_0} = (-1)^{k+1-s_0} \binom{k+1}{s_0}.$$

Corollary 12

$$\operatorname{ind}(F^{3n}, J_c) = \left(\sum_{n_0|s} (-1)^{k-s} \binom{k}{s}\right) (\chi(Z) - \operatorname{Lef}(\chi_N)). \tag{28}$$

Proof: It follows from assumption (a_2), (27) and

$$\sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} = 0.$$

Theorem 13 *If n_0 is even then g is surjective and for any n-periodic sequence c there exists an $x \in g^{-1}(c)$ such that $f^n(x) = x$.*

Proof: It follows that $\operatorname{ind}(F^{3n}, J_c) \neq 0$, so result follows from density of periodic sequences in Σ_2.

Theorem 14 *If n_0 is odd, then g is surjective. Moreover, for any n-periodic sequence $c \in \Sigma_2$ in which 1 appears k times and k is not odd multiplicity of n_0, then there is $x \in g^{-1}(c)$ such that $f^n(x) = x$.*

Proof: One can check that for n_0 odd

$$\sum_{n_0|s} (-1)^k \binom{k}{s} = 0$$

if and only if k is an odd multiplicity of n_0. Hence the result follows from (28).

Example 15 As an illustration we consider the homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ which maps the unit square $N = ABCD$ as indicated on the picture. Let M denote the thin, red rectangle on the picture. One can easily check that N and M are isolating blocks for f which fulfill the assumptions of Theorem 2.

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