Finding Eigenvalues of Self-maps with the Kronecker Canonical Form

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Abstract Recent research has examined how to study the topological features of a continuous self-map by means of the persistence of the eigenspaces, for given eigenvalues, of the endomorphism induced in homology over a field. This raised the question of how to select dynamically significant eigenvalues. The present paper aims to answer this question, giving an algorithm that computes the persistence of eigenspaces for every eigenvalue simultaneously, also expressing said eigenspaces as direct sums of "finite" and "singular" subspaces.

Keywords Computational topology · Persistent homology · Self-maps · Matrix pencils · Kronecker canonical form

1 Introduction

The theory of persistent homology [2, 6] has proved in the past two decades to be a very useful tool in several branches of applied mathematics and computer science. In [1], a novel application of persistence to the computational analysis of dynami-

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cal systems is introduced. Building upon the concept of *towers* in a given category (a tower in the category of modules or vector spaces is equivalent to a *persistence module* as defined in [6]), the authors define the tower of eigenspaces for an endomorphism of a tower of (finite dimensional) vector spaces. When these vector spaces are obtained as the homology over a field $\mathbb{F}$ of a filtration representing the underlying topological space, and the endomorphism is the map induced in homology by a self-map of said topological space, the eigenvectors are homology classes invariant under the self-map and provide a first step towards understanding the persistence of this map.

When the self-map is expanding, there is no guarantee that the image of a homology class by the endomorphism is in the filtration at the same step or even at any step. To overcome this difficulty, the authors of [1] adapted persistent homology to the study of a self-map by using two towers of vector spaces, which are equivalent to persistence modules indexed over integer numbers: (Y_i, η_i), a tower of homology spaces obtained from a filtration of the underlying topological space, and (X_i, ξ_i), a tower of homology spaces obtained by restricting domains such that maps induced by the self-map are simplicial. The morphisms $\varphi_i \colon X_i \to Y_i$, $\psi_i \colon X_i \to Y_i$ are obtained, respectively, from the self-map and from the inclusion map. In [1], the *eigenspace for pairs* $E_t(\varphi, \psi)$ was constructed by defining, for every $t \in \mathbb{F}$,

$$\overline{E}_t(\varphi, \psi) = \ker(\varphi - t\psi)$$

and then quotienting out the common kernel of φ and ψ, that is,

$$E_t(\varphi, \psi) = \overline{E}_t(\varphi, \psi)/(\ker \varphi \cap \ker \psi). \tag{1}$$

Nevertheless, despite quotienting out the common kernel of φ and ψ, it may happen that $E_t(\varphi, \psi)$ is non-trivial for every $t \in \mathbb{F}$, a phenomenon that was termed "abundance of eigenvalues" in [1]. This difficulty in finding the eigenvalues for the pair (φ, ψ), and in identifying them as dynamically significant, leads to the question whether there exists a way to compute the eigenspace towers for a pair of morphisms, for all eigenvalues simultaneously. The present article aims to answer this question in the affirmative, providing an algorithm to extract eigenvectors for every eigenvalue all at once. In addition, using the theory of the Kronecker canonical form for matrix pencils (a generalization of the Jordan form to polynomial matrices of the form $tB - A$), the eigenspace for every eigenvalue can be expressed as the direct sum of a "finite" and a "singular" part, the latter of which being associated with the abundance of eigenvalues phenomenon. We believe that the dynamically significant eigenvectors are contained in the former, finite part.

In Sect. 2, we reintroduce the concept of the Kronecker canonical form along with invariant polynomials of polynomial matrices, which while belonging to classical theories in linear algebra, appear not to be part of the common mathematical knowledge. Section 3 is dedicated to the algorithm to extract eigenvectors, as well as generalized eigenvectors, for all eigenvalues simultaneously. Section 4 shows numerical examples.

2 Kronecker Canonical Form

By the term *linear matrix pencil*, or simply *matrix pencil*, we refer to the polynomial matrix $tB - A$, where $A, B \in M^{m\times n}(\mathbb{F})$ and $\mathbb{F}$ is a fixed field. Fix a value $\widehat{t} \in \mathbb{F}$; if the equation

$$(\widehat{t}B - A)\, x = 0$$

possesses a nonzero solution $x \in \mathbb{F}^n$, then x is said to be an *eigenvector* for the *eigenvalue* $\widehat{t}$. In addition, if there is a finite sequence $x_1, x_2, \ldots, x_k \in \mathbb{F}^n$ of nonzero vectors such that the system

$$\begin{aligned}(\widehat{t}B - A)\, x_1 &= 0,\\ (\widehat{t}B - A)\, x_2 &= Bx_1,\\ &\vdots\\ (\widehat{t}B - A)\, x_k &= Bx_{k-1}\end{aligned}$$

has a solution, then this sequence is called a *sequence of generalized eigenvectors* for the eigenvalue $\widehat{t}$. Let $tB_1 - A_1$ and $tB_2 - A_2$ be two $m \times n$ pencils; if there exist invertible matrices $Q \in M^{m\times m}(\mathbb{F})$, $R \in M^{n\times n}(\mathbb{F})$ such that $Q^{-1}(tB_1 - A_1)R = tB_2 - A_2$, then the pencils are said to be *similar*.

In order to study the eigenstructure of the pencil $tB - A$, that is find its eigenvalues and the dimension of its eigenspaces and generalized eigenspaces, and hence to extract (generalized) eigenvectors, we recall the classical concepts of invariant polynomials and of Kronecker indices of matrix pencils.

We first start by considering a particular type of pencil. Call the *rank* of a pencil, $\operatorname{rank}(tB - A)$, the largest integer k such that there exist non-vanishing $k \times k$ minors of $tB - A$. If a pencil $tB - A$ is square ($B, A \in M^{n\times n}(\mathbb{F})$) and has rank n, it is said to be *regular*. If it is non-square, or if it is $n \times n$ square but its rank is strictly lower than n, it is said to be *singular*. We will additionally say that a pencil has *full row rank* (respectively *full column rank*) if its rank equals its number of rows (respectively its number of columns).

Let us recall the well-known rational canonical form and primary rational canonical form of a square matrix.

Definition 1 For $p(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{k-1}t^{k-1} + t^k$ a monic polynomial, the $k \times k$ matrix

$$C(p) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0\\ 1 & 0 & \cdots & 0 & -c_1\\ 0 & 1 & \cdots & 0 & -c_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & -c_{k-1}\end{bmatrix}$$

is called the *companion matrix* of d.

Theorem 2 [4, Theorem 11.17] *Let T be a square matrix, then T is similar to a unique square matrix*

$$\operatorname{diag}\{C(d_1), C(d_2), \ldots, C(d_s)\} \tag{2}$$

where $C(d_i)$ is the companion matrix of a non-constant monic polynomial d_i and $d_1|d_2|\ldots|d_s$.

Theorem 3 [4, Theorem 11.20] *Let T be a square matrix, then T is similar to a square matrix*

$$\operatorname{diag}\{C(p_1), C(p_2), \ldots, C(p_r)\} \tag{3}$$

where each $p_i = q_i^{s_i}$ is a power of a monic prime polynomial q_i, and $C(p_i)$ its companion matrix. This matrix is uniquely determined up to the order of the blocks $C(p_i)$ on the diagonal.

We refer to the form (2) as the *rational canonical form* of T, and to the form (3) as the *primary rational canonical form* of T.

Proposition 4 *Every regular pencil $tB - A$ over $\mathbb{F}$ is similar to a pencil in the form*

$$\operatorname{diag}\{tN - I_{r_1}, tI_{r_2} - C\} \tag{4}$$

where N is the direct sum of nilpotent companion matrices, C is a square matrix in rational canonical form, and I_{r_1} and I_{r_2} are identity matrices of the given size.

Proof This proof proceeds similarly to the proof of [3, Chapter 12, Theorem 3]. If $tB - A$ is regular, then there exists $\widehat{t} \in \mathbb{F}$ such that $\widehat{t}B - A$ has full rank. Call $\widetilde{A}$ the matrix $-(\widehat{t}B - A)$, then

$$\begin{aligned} tB - A &= (t - \widehat{t})\, B - \widetilde{A} \\ \Rightarrow \widetilde{A}^{-1}\,(tB - A) &= (t - \widehat{t})\, \widetilde{A}^{-1}B - I. \end{aligned}$$

We can write the primary rational canonical form of $\widetilde{A}^{-1}B$ by ordering the blocks such that the block corresponding to t^{r_1} for $r_1 > 0$, if it exists, is in the top left. The pencil $tB - A$ is thus similar (in the sense for pencils given above) to

$$(t - \widehat{t})\, \operatorname{diag}\{C_0, C_1\} - I = \operatorname{diag}\left\{t\, C_0 - (I_{r_1} + \widehat{t}\, C_0), (t\, C_1 - (I_{r_2} + \widehat{t}\, C_1)\right\}$$

where C_0 is the companion matrix of $p(t) = t^{r_1}$. Since $I_{r_1} + \widehat{t}\, C_0$ is invertible, and so is C_1, we can left-multiply the above pencil by $\operatorname{diag}\left\{(I_{r_1} + \widehat{t}\, C_0)^{-1}, C_1^{-1}\right\}$, yielding

$$\operatorname{diag}\left\{t\,(I_{r_1} + \widehat{t}\, C_0)^{-1}\, C_0 - I_{r_1}, t\, I_{r_2} - C_1^{-1}\,(I_{r_2} + \widehat{t}\, C_1)\right\}.$$

The result is obtained by putting the matrices $(I_{r_1} + \widehat{t}\, C_0)^{-1}\, C_0$ and $C_1^{-1}\,(I_{r_2} + \widehat{t}\, C_1)$ into their rational canonical forms, respectively N and C. ■

Matrix N in (4) is a block diagonal matrix, whose blocks are nilpotent companion matrices N_i, $i = 1, \ldots, l$. Each such matrix is the companion of the polynomial t^{k_i}, $k_i \geq 1$, and so N has only 0 as eigenvalue. We say that $tB - A$ possesses l *infinite elementary divisors*, whose orders are $k_1, k_2, \ldots, k_l$.

We also encountered in (4) a matrix C in rational canonical form, that is

$$C = \operatorname{diag}\{C(d_1), C(d_2), \ldots, C(d_s)\}$$

with $d_1|d_2|\ldots|d_s$. These polynomials are referred to as the *invariant polynomials* of the pencil $tB - A$. We will refer to the eigenstructure of C as the *finite* eigenstructure of the pencil. From Proposition 4 and the fact that $tN - I_{r_1}$ has no eigenvalue, we see that t is an eigenvalue of a regular pencil if and only if it is a root of one of its invariant polynomials, with the dimension of its eigenspace being the number of such invariant polynomials. In [3, Chapter 6], the classical algorithm to put a polynomial matrix into Smith normal form is shown to yield a diagonal matrix in canonical form, whose first diagonal elements are ones followed by the invariant polynomials of the matrix, with zero rows at the bottom and zero columns at the right. A regular pencil is of full rank and can, therefore, not have zero rows or columns, so the classical Smith normal form algorithm provides invertible matrices $Q(t)$, $R(t)$ such that

$$Q(t)^{-1}\,(tB - A)\,R(t) = \operatorname{diag}\{1, \ldots, 1, d_1, \ldots, d_s\}.$$

We easily see that if $R(t) = [y_1(t)\, y_2(t) \ldots y_{n-s}(t)\, x_1(t)\, x_2(t) \ldots x_s(t)]$, and if $\widehat{t}$ is a root of polynomial d_i, then

$$(\widehat{t}B - A)\, x_i(\widehat{t}) = 0.$$

Since $R(t)$ is invertible, its columns are linearly independent for every value t. Therefore, if $\widehat{t}$ is a root of more than one invariant polynomial, we can find the same number of linearly independent eigenvectors.

Now, consider a general $m \times n$ pencil $tB - A$. We can study solutions of

$$\forall t \in \mathbb{F} \quad (tB - A)\, x(t) = 0, \tag{5}$$

where $x : \mathbb{F} \to \mathbb{F}^n$ is the variable. If there exists a linear dependence over $\mathbb{F}[t]$ between the columns of $tB - A$, then there exists a polynomial solution of Eq. (5) which we call a *polynomial eigenvector* for the pencil. Write such a solution as

$$x(t) = x_0 + t\, x_1 + t^2\, x_2 + \cdots + t^{\varepsilon}\, x_{\varepsilon},\ \ \varepsilon \geq 0 \tag{6}$$

with x_i, $i = 0, \ldots, \varepsilon$ vectors in $\mathbb{F}^n$, and $x_\varepsilon \neq 0$, where ε is the degree of the polynomial eigenvector. Without loss of generality, we can assume that $x_0 \neq 0$. Indeed, suppose that $x_0 = x_1 = \cdots = x_{k-1} = 0$ and $x_k \neq 0$ for $k \leq \varepsilon$ in Eq. (6). Then we can factor out t^k, leaving

$$t^k\,(tB - A)\,(x_k + t\, x_{k+1} + t^2\, x_{k+1} + \cdots + t^{\varepsilon-k}\, x_\varepsilon) = 0,$$

that is, $x_k + t\, x_{k+1} + t^2\, x_{k+1} + \cdots + t^{\varepsilon-k}\, x_\varepsilon$ is a new polynomial eigenvector with nonzero constant term. Therefore, if $x(t)$ is a polynomial eigenvector of $tB - A$ with nonzero constant term, then for every $\widehat{t} \in \mathbb{F}$, $x(\widehat{t})$ is an eigenvector of $tB - A$ for eigenvalue $\widehat{t}$.

Theorem 5 [3, Chapter 12, Theorem 4] *Suppose that ε is the smallest positive integer such that the pencil $tB - A$ possesses a polynomial solution (6) of degree $\varepsilon > 0$. Then the pencil is similar to*

$$\begin{bmatrix} L_\varepsilon & 0 \\ 0 & t\widehat{B} - \widehat{A} \end{bmatrix}$$

where

$$L_\varepsilon = \begin{bmatrix} t & -1 & & \\ & \ddots & \ddots & \\ & & t & -1 \end{bmatrix} \tag{7}$$

is a bidiagonal pencil of dimension $\varepsilon \times (\varepsilon + 1)$, known as a column Kronecker block *of* index ε, *and $t\widehat{B} - \widehat{A}$ has no polynomial eigenvector analogous to (6) of degree less than ε.*

Theorem 5 is also valid in the case where $\varepsilon = 0$, in which case a "0×1" block L_0 means a column of zeros to the left of $t\widehat{B} - \widehat{A}$.

Proposition 6 *A vector $x_0 \in \ker A \cap \ker B$ if and only if $x(t) = x_0$ is a polynomial eigenvector of $tB - A$ of degree* 0.

Proof If $x_0 \in \ker A \cap \ker B$, then obviously $(tB - A)\, x_0 = 0$. Now suppose that $(tB - A)\, x_0 = 0$, then for every $t \in \mathbb{F}$, $A\, x_0 = t\, B\, x_0$. Since $A\, x_0$ and $B\, x_0$ are elements of $\mathbb{F}$, then this can only be true if $A\, x_0 = B\, x_0 = 0$. ■

The last theorem in this section concerns a decomposition of the pencil $tB - A$:

Theorem 7 *Any $m \times n$ pencil $tB - A$ over $\mathbb{F}$ is similar to the pencil*

$$\mathrm{diag}\{L_{\varepsilon_1}, \ldots, L_{\varepsilon_p}, L_{\eta_1}^T, \ldots, L_{\eta_q}^T, t\overline{B} - \overline{A}\}$$

where $t\overline{B} - \overline{A}$ is a regular and therefore square pencil.

Proof Repeatedly Applying Theorem 5, we may successively extract from $tB - A$ Kronecker blocks of nonincreasing index until we end up with the following decomposition: $tB - A$ is similar to

$$\mathrm{diag}\{L_{\varepsilon_1}, L_{\varepsilon_2}, \ldots, L_{\varepsilon_p}, t\widehat{B} - \widehat{A}\}$$

where the columns of $t\widehat{B} - \widehat{A}$ are linearly independent and the blocks L_{ε_i} may be ordered in a way that $0 \le \varepsilon_1 \le \varepsilon_2 \le \cdots \le \varepsilon_p$. At this point, $t\widehat{B} - \widehat{A}$ may still have a linearly dependent set of rows, in which case it would possess *left polynomial eigenvectors* $y(t)$ such that $y(t)\,(t\widehat{B}-\widehat{A})=0$. This is obviously equivalent to $(t\widehat{B}^T-\widehat{A}^T)\,y^T(t)=0$, and therefore Theorem 5 can now be applied to this transposed subpencil, yielding *row Kronecker blocks* $L^T_{\eta_j}$, $j=1,\dots,q$.

Since we already know the decomposition $t\overline{B}-\overline{A}$ of (4), this completes the presentation of the *Kronecker canonical form* of a pencil. We will more precisely call this form the *rational Kronecker canonical form* since it includes a matrix in rational canonical form; the classical Kronecker canonical form is a generalization of the Jordan form and therefore is only guaranteed to exist when working with an algebraically closed field.

Let us now show an example.

Example 8 Consider the following pencil over $\mathbb{Q}$:

$$tB-A=\begin{bmatrix} -t & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ t-1 & 0 & t-1 & t-1 & 0 & -t & 1 & 0\\ -1 & 0 & 0 & t & 0 & 0 & 0 & 0\\ 0 & -t-1 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & t+1 & t+1 & 0 & 0 & 0\\ 0 & 0 & -1 & -t-1 & -t-1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

We can show that this pencil has column Kronecker indices $\varepsilon_1=\varepsilon_2=1$, row Kronecker index $\eta_1=0$, one infinite elementary divisor of order 1 and invariant polynomials $t+1$ and t^2-1. Therefore, the rational Kronecker canonical form of $tB-A$ is

$$\begin{bmatrix} t & -1 & & & & & & \\ & & t & -1 & & & & \\ & & & & & & & \\ & & & & -1 & & & \\ & & & & & t+1 & & \\ & & & & & & t & -1\\ & & & & & & -1 & t \end{bmatrix}.$$

We leave to the reader to verify that the following transition matrices put $tB-A$ into this canonical form:

$$Q^{-1}=\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & -1 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

3 Algorithm

Van Dooren [5] introduced an algorithm to transform a pencil $tB - A$ into a form from which its column Kronecker indices can be computed, and the associated polynomial eigenvectors are easily extracted. This is done by successively column- and row-reducing subpencils of $tB - A$. In the following algorithm, indices denote step number except for zero and identity matrices, where they denote dimension.

Algorithm 9

Input: $tB - A$
$j := 1;\ m_1 := m;\ n_1 := n;$
$A_{1,1} := A;\ B_{1,1} := B;\ Q^{-1} := I_m;\ R := I_n;$
while (true)
  if $B_{j,j}$ has n_j linearly independent columns
    $l := j - 1;$
    return Q^{-1}, R;
  $\left[\, B_{j+1} \middle| 0_{m_j \times s_j} \,\right] := B_{j,j}\ R_j;$
  Let R_j be obtained through column reduction algorithm on $B_{j,j}$
  $\left[\, A_{j+1} \middle| A_j \,\right] := A_{j,j}\ R_j;$
  for $i = 1$ **to** $j - 1$ **do**
    (* Update other blocks in column j *)
    $\left[\, B_{j+1,i} \middle| B_{j,i} \,\right] := B_{j,i}\ R_j;$
    $\left[\, A_{j+1,i} \middle| A_{j,i} \,\right] := A_{j,i}\ R_j;$
  (* Update transition matrix R *)
  $R := R \left[\begin{array}{c|c} R_j & 0_{n_j \times (n-n_j)} \\ \hline 0_{(n-n_j) \times n_j} & I_{n-n_j} \end{array}\right];$
  $\left[\begin{array}{c} 0_{(m_j - r_j) \times s_j} \\ \hline A_{j,j} \end{array}\right] := Q_j^{-1}\ A_j;$
  Let Q_j^{-1} be obtained through row reduction algorithm on A_j
    and permutation so zero rows are on top
  $\left[\begin{array}{c} A_{j+1,j+1} \\ \hline A_{j+1,j} \end{array}\right] := Q_j^{-1}\ A_{j+1};\ \left[\begin{array}{c} B_{j+1,j+1} \\ \hline B_{j+1,j} \end{array}\right] := Q_j^{-1}\ B_{j+1};$
  (* Update transition matrix Q^{-1} *)

$$Q^{-1} := \left[\begin{array}{c|c} Q_j^{-1} & 0_{m_j\times(m-m_j)} \\ \hline 0_{(m-m_j)\times m_j} & I_{m-m_j} \end{array}\right] Q^{-1};$$
$$m_{j+1} := m_j - r_j;\; n_{j+1} := n_j - s_j;$$
$$j := j+1;$$

Theorem 10 *Algorithm 9 stops when $B_{l+1,l+1}$ has full column rank. At this point, the output are the matrices Q^{-1} and R such that $Q^{-1}(tB-A)R$ is the following block lower triangular matrix:*

$$\begin{array}{cc} \left[\begin{array}{c|c|c|c|c} tB_{l+1,l+1}-A_{l+1,l+1} & 0 & \cdots & 0 & 0 \\ \hline tB_{l+1,l}-A_{l+1,l} & -A_{l,l} & \cdots & 0 & 0 \\ \hline \vdots & \ddots & \ddots & \vdots & \vdots \\ \hline tB_{l+1,2}-A_{l+1,2} & tB_{l,2}-A_{l,2} & \cdots & -A_{2,2} & 0 \\ \hline tB_{l+1,1}-A_{l+1,1} & tB_{l,1}-A_{l,1} & \cdots & tB_{2,1}-A_{2,1} & -A_{1,1} \end{array}\right] & \begin{array}{c} m_{l+1} \\ r_l \\ \vdots \\ r_2 \\ r_1 \end{array} \\ \begin{array}{ccccc} n_{l+1} & s_l & \cdots & s_2 & s_1 \end{array} & \end{array} \tag{8}$$

where the $A_{j,j}$'s have full row rank r_j for $j = 1, \ldots, l$, and the $B_{j,j-1}$'s have full column rank s_j for $j = 2, \ldots, l$. Some of the r_j's can equal 0.

Proof Form (8) is a direct consequence of the algorithm. Indeed, the initial form of the pencil is

$$tB_{1,1} - A_{1,1},$$

and at step j, the left block of columns,

$$\left[\begin{array}{c} tB_{j,j}-A_{j,j} \\ \hline tB_{j,j-1}-A_{j,j-1} \\ \hline \vdots \\ \hline tB_{j,1}-A_{j,1} \end{array}\right]$$

is the only part of the pencil to change, being transformed by multiplying on the right by R_j and on the left by

$$\begin{bmatrix} Q_j^{-1} & \\ & I_{m-m_j} \end{bmatrix}$$

into

$$\left[\begin{array}{c|c} tB_{j+1,j+1}-A_{j+1,j+1} & 0_{(m_j-r_j)\times s_j} \\ \hline tB_{j+1,j}-A_{j+1,j} & -A_{j,j} \\ \hline tB_{j+1,j-1}-A_{j+1,j-1} & tB_{j,j-1}-A_{j,j-1} \\ \hline \vdots & \vdots \\ \hline tB_{j+1,1}-A_{j+1,1} & tB_{j,1}-A_{j,1} \end{array}\right].$$

The $A_{j,j}$ blocks, $j = 1, \ldots, l$, have full row rank r_j, being obtained from the nonzero rows of a row-reduced matrix. In addition, at every step j, the block B_{j+1} created has full column rank, being obtained from the nonzero columns of a column-reduced matrix. Multiplying it on the left by Q_j^{-1} yields

$$\left[\begin{array}{c} B_{j+1,j+1} \\ \hline B_{j+1,j} \end{array}\right]. \tag{9}$$

If $B_{j+1,j+1}$ has full column rank, then the algorithm stops and this block becomes the upper left block $B_{l+1,l+1}$. Otherwise, the block (9) is multiplied on the right by R_{j+1}, yielding

$$\left[\begin{array}{c|c} B_{j+2} & 0 \\ \hline B_{j+2,j} & B_{j+1,j} \end{array}\right] = Q_j^{-1} \, B_{j+1} \, R_{j+1}.$$

Since B_{j+1} has full column rank, then $B_{j+1,j}$ also does. This is true for $j = 1, \ldots, l-1$, proving the theorem. ■

From the row and column ranks r_j and s_j, we then compute (putting $s_{l+1} := 0$)

$$e_j := s_j - r_j \geq 0 \text{ for } j = 1, \ldots, l;$$
$$d_j := r_j - s_{j+1} \geq 0 \text{ for } j = 1, \ldots, l.$$

As shown in [5, Proposition 4.3], the indices d_j and e_j fully determine the infinite elementary divisors and the column Kronecker indices, respectively. More precisely, they tell us that $tB - A$ has d_j infinite elementary divisors of degree j, $j = 1, \ldots, l$, and e_j column Kronecker blocks L_{j-1} of size $(j-1) \times j$, $j = 1, \ldots, l$. The pencil $tB_{l+1,l+1} - A_{l+1,l+1}$ additionally contains the finite structure of the original pencil.

In [5] a dual algorithm is also described. It extracts the infinite elementary divisors and row Kronecker indices of $tB - A$. Here, let us recall that if B is an identity matrix, that is for the classical eigenproblem for a square matrix A, there exists a natural isomorphism between the left and right eigenspaces, and generalized eigenspaces, of A. Indeed, the left generalized eigenspace of A (equivalently the generalized eigenspace of A^T) for every given eigenvalue is the dual space of its (right) generalized eigenspace. This natural isomorphism breaks down in the case of matrix pencils since column and row Kronecker indices are completely independent of each other, but it is possible to retain it by quotienting out vectors from the column (respectively row) Kronecker structure from the eigenspace (respectively left eigenspace).

When the pencil has been put into form (8), we can further use the fact that $B_{l+1,l+1}$ has full column rank, as do the blocks $B_{i,i-1}$ for $i = 2, \ldots, l$, and that $A_{i,i}$ has full row rank for $i = 1, \ldots, l$, to zero out the majority of subdiagonal blocks in the following way.

Algorithm 11

Input: Q^{-1}, R, $Q^{-1}\,(tB - A)\,R$ from Algorithm 9
for $i = 1$ **to** l

(* Zero out block $B_{l+1,l+1-i}$ *)
Find X such that $B_{l+1,l+1-i} = X\, B_{l+1,l+1}$;
$B_{l+1,l+1-i} := 0$;
$A_{l+1,l+1-i} := A_{l+1,l+1-i} - X\, A_{l+1,l+1}$;

$$Q^{-1} := \begin{bmatrix} I_{m_{l+1}} & & & \\ & \ddots & & \\ -X & & I_{r_{l+1-i}} & \\ & & & \ddots \\ & & & & I_{r_1} \end{bmatrix} Q^{-1};$$

for $j = 1$ **to** $i - 2$
(* Zero out block $B_{l+1-j,l+1-i}$ *)
Find Z such that $B_{l+1-j,l+1-i} = Z\, B_{l+1-j,l-j}$;
$B_{l+1-j,l+1-i} := 0$;
$A_{l-j,l+1-i} := A_{l-j,l+1-i} - Z\, A_{l-j,l-j}$;

$$Q^{-1} := \begin{bmatrix} I_{m_{l+1}} & & & & & \\ & \ddots & & & & \\ & & I_{r_{l-j}} & & & \\ & & & \ddots & & \\ & & -Z & & I_{r_{l+1-i}} & \\ & & & & & \ddots \\ & & & & & & I_{r_1} \end{bmatrix} Q^{-1};$$

for $j = 1$ **to** i
(* Zero out block $A_{l+1-i+j,l+1-i}$ *)
Find Y such that $A_{l+1-i+j,l+1-i} = A_{l+1-i,l+1-i}\, Y$;
$A_{l+1-i+j,l+1-i} := 0$;
for $k = 1$ **to** $l - i$
$A_{l+1-i+j,k} := A_{l+1-i+j,k} - A_{l+1-i,l+1-i-k}\, Y$;
$B_{l+1-i+j,k} := B_{l+1-i+j,k} - B_{l+1-i,l+1-i-k}\, Y$;

$$R := R \begin{bmatrix} I_{n_{l+1}} & & & & & \\ & \ddots & & & & \\ & & I_{s_{l+1-i+j}} & & & \\ & & & \ddots & & \\ & & -Y & & I_{s_{l+1-i}} & \\ & & & & & \ddots \\ & & & & & & I_{s_1} \end{bmatrix};$$

At this point, having reused the names of the blocks, $Q^{-1}\,(tB - A)\,R$ equals

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$$
\begin{array}{c}
\left[\begin{array}{c|c|c|c|c|c}
tB_{l+1,l+1}-A_{l+1,l+1} & 0 & 0 & \cdots & 0 & 0 \\ \hline
0 & -A_{l,l} & 0 & \cdots & 0 & 0 \\ \hline
0 & tB_{l,l-1} & -A_{l-1,l-1} & \cdots & 0 & 0 \\ \hline
0 & 0 & tB_{l-1,l-2} & \ddots & \vdots & \vdots \\ \hline
\vdots & \vdots & \vdots & \ddots & -A_{2,2} & 0 \\ \hline
0 & 0 & 0 & \cdots & tB_{2,1} & -A_{1,1}
\end{array}\right]
\begin{array}{l} m_{l+1} \\ r_l \\ r_{l-1} \\ \vdots \\ r_2 \\ r_1 \end{array} \\
\begin{array}{cccccc} n_{l+1} & s_l & s_{l-1} & \cdots & s_2 & s_1 \end{array}
\end{array} . \tag{10}
$$

Note that the blocks $A_{i,i}$ have $s_i - r_i = e_i$ zero columns, which is also the number of Kronecker blocks of index $i-1$, each of which corresponds to a polynomial eigenvector of degree $i-1$. Therefore, using the blocks $A_{i,i}$ to zero out the blocks $tB_{i+1,i}$, $i = 1$ going up to $l-1$ in this order, will expose zero columns in the pencil.

Algorithm 12

Input: R, $Q^{-1}(tB - A)R$ from Algorithm 11
for $i = 1$ **to** $l-1$
 Find Y such that $B_{i+1,i} = A_{i,i}Y$;
 $B_{i+1,i} := 0$;

$$
R := R \begin{bmatrix} I_{n_{l+1}} & & & & \\ & \ddots & & & \\ & & I_{s_{i+1}} & & \\ & & tY & I_{s_i} & \\ & & & & \ddots \\ & & & & & I_{s_1} \end{bmatrix};
$$

Note that R is now a matrix over $\mathbb{F}[t]$, and for every zero column of $Q^{-1}(tB - A)R$, we find a column $x(t) \in \mathbb{F}[t]^n$ of R which is a polynomial eigenvector of $tB - A$. In addition, Algorithm 12 ensures that the degrees of such columns of R are equal to the column Kronecker indices of $tB - A$.

Furthermore, since the block $tB_{l+1,l+1} - A_{l+1,l+1}$ contains the whole finite structure of the pencil, we can at this point (also updating R) put it into Smith normal form, whose non-constant diagonal elements will be the invariant polynomials of $tB - A$. Here as well, we expose a column in the pencil that is zero except for one entry, an invariant polynomial of $tB - A$. When evaluated at a root t_0 of this polynomial, the corresponding column of R is an eigenvector for eigenvalue t_0.

When working on $\mathbb{Q}$, the rational roots of a polynomial with integer (or rational) coefficients can be obtained with the following well-known theorem:

Theorem 13 (Rational Root Theorem) *Let*

$$
a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0 \tag{11}
$$

be a polynomial equation with integer coefficients, and suppose that $a_n \neq 0$, $a_0 \neq 0$. *Then every rational root* p/q *of (11), where* p, q *are relatively prime, has the property that* $p|a_0$ *and* $q|a_n$.

Applying the previous theorem to the invariant polynomials of $tB - A$ over $\mathbb{Q}$ (multiplying by an integer if necessary) allows one to find every rational eigenvalue.

Note that in case left eigenvectors are required, the dual algorithm of [5] can be used instead of Algorithm 9, followed by a dual "row" version of Algorithms 11 and 12, keeping track of the left transition matrix Q^T.

Example 14 Consider again the pencil of Example 8. Applying Algorithm 9 yields

$$\begin{array}{cc}
\left[\begin{array}{ccc|cc|ccc}
t+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-t-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t & -t & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -t & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & t-1 & t-1 & 0 & -1 & 1 & 0 \\
-1 & -t & t-1 & t & t+1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right]
&
\begin{array}{c} \\ \\ m_3 \\ r_2 \\ \\ r_1 \\ \\ \end{array}
\\
\begin{array}{ccc} \quad n_3 \quad & \qquad s_2 \qquad & \quad s_1 \end{array} &
\end{array},$$

so we can verify the presence of $s_2 - r_2 = 2$ column Kronecker blocks of index 1, and $r_1 - s_2 = 1$ infinite elementary divisor of order 1. Applying Algorithms 11, 12 and the Smith normal form algorithm, we obtain

$$tB - A \sim \left[\begin{array}{ccc|cc|ccc}
1 & 0 & 0 & & & & & \\
0 & t+1 & 0 & & & & & \\
0 & 0 & t^2-1 & & & & & \\
0 & 0 & 0 & & & & & \\
\hline
 & & & 0 & 0 & -1 & 0 & 0 \\
 & & & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & -1
\end{array}\right]$$

$$R = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -t & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & -t-1 & 0 & -t-1 & 1 & 1 & 0 \\
1 & 0 & t & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & -t-1 & 1 & 1 & 0 \\
-1 & 0 & -t & -t & -t-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

verifying that $t+1$ and t^2-1 are invariant polynomials of this pencil. We conclude that

$$x_1(t) = [0, 0, 0, 0, 0, -1, -t, 0]^T; \quad x_2(t) = [0, -1, -t-1, 0, 1, -t-1, -t-1, 0]^T$$

are polynomial eigenvectors,

$$x_3 = [0, 1, 0, 0, 0, 0, 0, 0]^T$$

is an eigenvector of $tB - A$ for eigenvalue -1, and

$$x_4(t) = [1, -t, -t - 1, t, 0, -1, -t, 0]^T$$

is a vector that can be evaluated at ± 1 to yield an eigenvector for each of these two eigenvalues.

We note that, as can been seen in Example 14, our algorithm allows us to identify, for every eigenvector for a given eigenvalue, whether it originates from the singular structure of the pencil or not. Since we believe that the singular structure is not associated with topologically significant eigenvectors, this identification is useful in applications.

Let us now discuss the computation of generalized eigenvectors of a pencil. Algorithm 12 provides polynomial eigenvectors of degree equal to the column Kronecker indices. A pencil whose Kronecker structure has one index ε possesses a sequence of $\varepsilon + 1$ generalized eigenvectors. To see this, consider the Kronecker block L_ε in (7). It can easily be checked that for every field value t, the sequence

$$\begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^{\varepsilon-1} \\ t^\varepsilon \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2\,t \\ \vdots \\ -(\varepsilon - 1)\,t^{\varepsilon-2} \\ -\varepsilon\,t^{\varepsilon-1} \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ (-1)^{\varepsilon-1} \\ (-1)^{\varepsilon-1}\,\varepsilon\,t \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ (-1)^\varepsilon \end{bmatrix}$$

is a sequence of $\varepsilon + 1$ linearly independent generalized eigenvectors of L_ε. We also notice that the above sequence of polynomial vectors has been obtained by formal differentiation over the ring $\mathbb{F}[t]$ of the first vector. This is a ring homomorphism denoted $\frac{\mathrm{d}}{\mathrm{d}t}$ (or with prime notation) with the property that

$$\frac{\mathrm{d}}{\mathrm{d}t} t^k = k\,t^{k-1} \text{ for } k \in \mathbb{N}, \text{ and } \frac{\mathrm{d}}{\mathrm{d}t} c = 0, c \in \mathbb{F}.$$

If we denote $x(t) = [1, t, t^2, \ldots, t^{\varepsilon-1}, t^\varepsilon]^T$, then the above sequence is

$$x(t), -x'(t), \frac{1}{2}\,x''(t), \ldots, \frac{(-1)^{\varepsilon-1}}{(\varepsilon - 1)!}\,x^{(\varepsilon-1)}(t), \frac{(-1)^\varepsilon}{\varepsilon!}\,x^\varepsilon(t).$$

This property generalizes to other pencils. Suppose that $tB - A$ possesses a polynomial eigenvector $x(t)$ of degree ε, as obtained for example by Algorithm 12. Then $x(t)$ satisfies Eq. (5). Formal differentiation verifies the chain rule, and so we can apply it repeatedly to both sides of this equation:

$$
\begin{aligned}
(tB - A)x'(t) &= -Bx(t), \\
(tB - A)x''(t) &= -2Bx'(t), \\
&\vdots \\
(tB - A)x^{(\varepsilon)}(t) &= -\varepsilon B x^{(\varepsilon-1)}(t).
\end{aligned}
$$

From this it can easily be seen that $((-1)^k/k!\,x^{(k)}(t))$, $k = 0, \ldots, \varepsilon$ is a sequence of $\varepsilon + 1$ linearly independent generalized eigenvectors for $tB - A$. The $(\varepsilon + 1)$st derivative of $x(t)$ is the zero vector and therefore not linearly independent. The same procedure can also be applied to the columns of R in the output of Algorithm 12, say $x(t)$, that correspond to invariant polynomials of $tB - A$ in the sense that

$$Q^{-1}\,(tB - A)\,x(t) = d(t)\,e$$

for d an invariant polynomial and e a vector of the canonical basis of $\mathbb{F}^m$. Indeed, if t_0 is a root of d, we can write $d(t) = (t - t_0)^{k+1}\,r_0(t)$ for a certain $k \geq 0$, where r does not have t_0 as a root. Then

$$(tB - A)\,x(t) = (t - t_0)^{k+1}\,r_0(t)\,Q\,e.$$

Proposition 15 *For $i \leq k$, applying formal differentiation i times to both sides of the previous equation yields*

$$(tB - A)x^{(i)}(t) = -iBx^{(i-1)}(t) + (t - t_0)^{k+1-i} r_i(t) Q\,e \tag{12}$$

where $r_i(t)$ is another polynomial such that $r_i(t_0) \neq 0$.

Proof Suppose, for $0 \leq i \leq k - 1$, that (12) holds. Then, applying formal differentiation on both sides, we obtain

$$(tB - A)\,x^{(i+1)}(t) = -(i+1)B^{(i)}(t) + (t - t_0)^{k-i}\left((k - i + 1)\,r_i(t) + (t - t_0)\,r_i'(t)\right) Q\,e.$$

We can fix $r_{i+1} = (k - i + 1)\,r_i(t) + (t - t_0)\,r_i'(t)$; it is obvious that t_0 is not a root of this polynomial. ∎

Evaluating the previous sequence at t_0, we find that $((-1)^i/i!\,x^{(i)}(t_0))$, $i = 0, \ldots, k$ provides us with a sequence of generalized eigenvectors for eigenvalue t_0.

Example 16 In Example 14, the vectors

$$x_1(t) = [0, 0, 0, 0, 0, -1, -t, 0]^T; \quad x_2(t) = [0, -1, -t - 1, 0, 1, -t - 1, -t - 1, 0]^T$$

are eigenvectors of $tB - A$ for every field value, and so they are also generalized eigenvectors for every field value. The vectors

$$-x_1'(t) = [0, 0, 0, 0, 0, 0, 1, 0]^T; \quad -x_2'(t) = [0, 0, 1, 0, 0, 1, 1, 0]^T$$

are also generalized eigenvectors for every field value.

4 Numerical Results

We studied a map on a cloud of 100 points, taken in $S^1 \subset \mathbb{C}$ and then subjected to Gaussian noise with standard deviation varying from $\sigma = 0$ to 0.30. The image of each point z is taken to be the closest point to z^2, so the map is angle-doubling with noise. It is expected that we should find in homology H_1, computed over the field $\mathbb{Z}_{19}$, an eigenvector of long persistence for eigenvalue $t = 2$, but that stronger noise may make it harder to distinguish. Figure 1 shows the persistence barcodes for the eigenvector associated with $t = 2$ along a filtration of complexes indexed with parameter value ε. Since we can identify, at every step along the filtration, whether the eigenvector originates from the singular structure of the pencil or not, we can code the bar with the following colours: red when it does originate from the singular structure, and blue when it does not. It can be seen that as the noise level is increased,

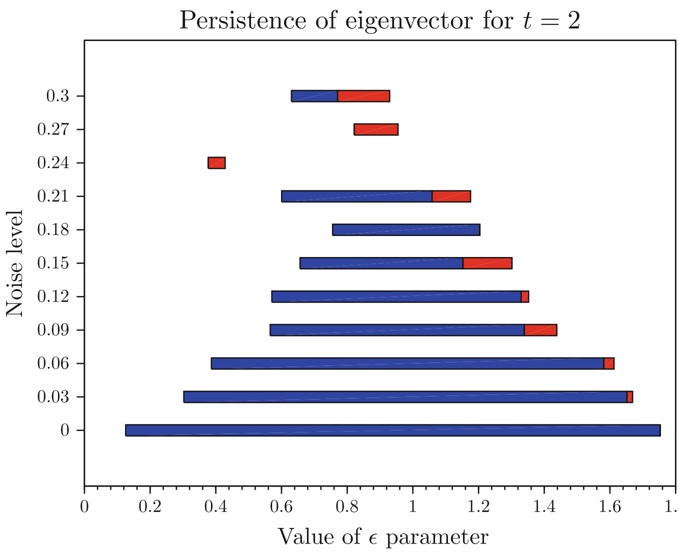

Fig. 1 Persistence of the longest lasting eigenvector associated with $t = 2$ in H_1 persistence over $\mathbb{Z}_{19}$ for several noise levels of a cloud of sample points on S^1, subject to the map $z \mapsto z^2$. Bar is *red* for vectors from singular structure, *blue* otherwise

the persistence of this eigenvector tends to become shorter, being born later and dying earlier, and additionally the eigenvector becomes "degenerate" (associated with the singular structure of the pencil) for a longer term.

Our 3D example uses a map on the torus constructed in the following way. Consider the square $[0, 1]^2$, identifying its left and right edges, as well as its top and bottom edges. Take a randomly selected sample of 200 points on this square, and build the map sending each point (x, y) to the closest point to $A[x, y]^T$, for the 2×2 matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which has eigenvalues 1 and -1. In Fig. 2 we show persistence barcodes in H_1 homology over the field $\mathbb{Q}$ for eigenvalues 0, 1 and -1 for this test case. Here too

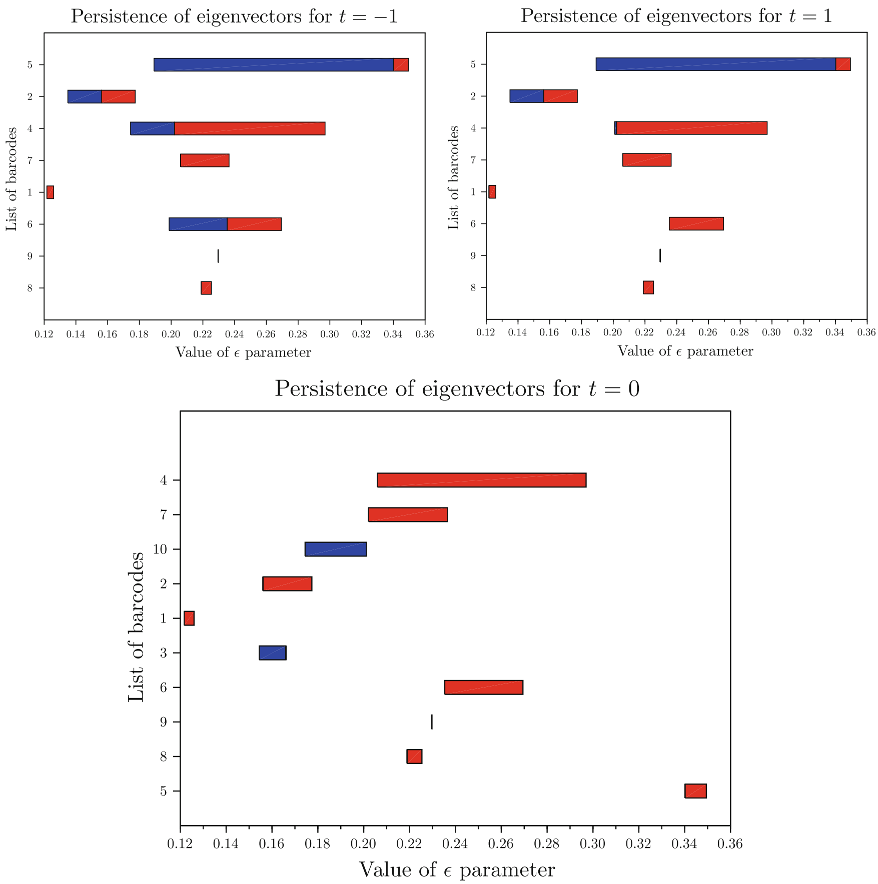

Fig. 2 Persistence barcodes for eigenvalues $t = -1$, $t = 1$ and $t = 0$ in H_1 persistence over $\mathbb{Q}$ for matrix A on the torus. Numbering is arbitrary. Bar is *red* for vectors from singular structure, *blue* otherwise

the bars are colour-coded red if the vector comes from the singular structure of the pencil, and blue if it comes from its finite structure. We notice several long-lasting vectors, but only two of those, one for eigenvalue 1 and one for eigenvalue -1, have a long life as non-singular vectors.

5 Conclusion

Algorithms 9, 11 and 12 positively answer the question asked in [1], on whether it is possible to compute the eigenspace towers of a pair of morphisms between two towers of vector spaces for all eigenvalues simultaneously. This is a necessary condition in applications, where candidate eigenvalues for long-lasting eigenvectors are not and cannot be known. It also makes it possible to study towers of eigenspaces when the spaces are over an infinite field such as $\mathbb{Q}$.

Furthermore, Proposition 15 and the preceding discussion describe a procedure to compute generalized eigenvectors for pairs of maps that does not have any added cost with respect to simply computing eigenvectors themselves. The link between generalized eigenvectors and differentiation is to our knowledge not very well-known, but it can be inferred for example from discussions in [3, Chap. 6].

Finally, being able to split the eigenspace for a pair of maps between a finite and a singular part, with the singular part being represented by polynomial eigenvectors, raises the question whether it is possible to define persistence generally for the Kronecker structure of a tower of maps between spaces. This is not a trivial problem and has links with the non-existence of a simple classification for persistence over modules [6] and with the problem of finding constraints for the persistence diagrams of two towers joined by a morphism.

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