CREATING SEMIFLOWS ON SIMPLICIAL COMPLEXES FROM COMBINATORIAL VECTOR FIELDS

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Abstract. Combinatorial vector fields on simplicial complexes introduced by Robin Forman constitute a combinatorial analogue of classical flows. They have found numerous and varied applications in recent years. Yet, their formal relationship to classical dynamical systems has been less clear. In this paper we prove that for every combinatorial vector field on a finite simplicial complex $\mathcal{X}$ one can construct a semiflow on the underlying polytope X which exhibits the same dynamics. The equivalence of the dynamical behavior is established in the sense of Conley-Morse graphs and uses a tiling of the topological space X which makes it possible to directly construct isolating blocks for all involved isolated invariant sets based purely on the combinatorial information.

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1. Introduction

Combinatorial vector fields on CW complexes were introduced in 1998 by R. Forman [25] as a tool in the construction of a discrete analogue of classical Morse theory. Originally needed only in the gradient setting of Morse theory, they were further studied as an analogue of a flow in [26] where Forman presented a combinatorial counterpart of Conley's result [13] on the decomposition of a flow into chain recurrent and gradient dynamics. Forman's study of the combinatorial analogues of the concepts in dynamics covered several further directions [27, 29, 30, 28, 31]. At the outset, these results seem to have been loosely motivated by the corresponding classical results. Since then his results have been used successfully in their own right in a number of applications, such as visualization and computer graphics [12, 14, 15, 40, 58], networks and sensor networks analysis [17, 38, 64, 63], homology computation [34, 51], astrophysics [43, 59], neuroscience [19], algebra [36] and computational geometry [8].

A fundamental question arises whether these combinatorial analogues may be tied in some formal way to their classical counterparts. More specifically, there are two interesting, mutually inverse, questions:

(1) **Flow modeling:** Given a flow on a smooth manifold M, can one model its dynamics by a combinatorial vector field on a triangulation of M or approximate it in some sense by a sequence of triangulations of M and combinatorial vector fields?
(2) **Flow reconstruction:** Given a combinatorial vector field on a CW complex does it model the dynamics of a classical flow or semiflow on the polytope of the complex?

Questions of this type are of inherent interest, as it seems natural to use a combinatorial vector field or one of its generalizations such as generalized Morse matching [32] or combinatorial multivector field [53, 41], as a discretization tool for the rigorous study of differential equations, see for example [50, 53], as well as a tool to investigate dynamical systems known only from finite samples [1, 16, 49, 52].

Surprisingly, so far there are few answers to such questions. What concerns the flow modeling question Gallais [33] proved in the gradient situation of Morse theory that given a smooth manifold M with a Morse function $F : M \to \mathbb{R}$ there is a triangulation of M and a gradient combinatorial vector field on this triangulation whose critical cells are in one-to-one correspondence with critical points of F. This was strenghtened by Benedetti [11] who proved that the result applies to rth barycentric subdivision of every PL triangulation with a suitably chosen integer r (see [39, Section 7.5] for an overview of these results). The general, non-gradient case seems to be significantly more challenging and, to our best knowledge, remains untouched. The problem here is the diversity and complexity of general dynamics for which a correspondence to finite dynamics is not sufficient and must be replaced by an approximation scheme. Moreover, as indicated in [53], a more general concept of combinatorial vector field may be needed.

In this paper we address the equally significant flow reconstruction question. The question is particularly important in the dynamical systems known only from finite samples. We prove that for a given combinatorial vector field $\mathcal{V}$ on a simplicial complex $\mathcal{X}$ there exists a continuous-time dynamical system φ on the underlying polytope X such that for every Morse decomposition of the combinatorial vector field $\mathcal{V}$ there is a Morse decomposition of φ with the same Conley-Morse graph. This result requires some ideas of asymptotic dynamics excluded from the original Forman's study of combinatorial dynamics and added to the theory in [37, 53, 7], in particular the concepts of isolated invariant sets, Morse decompositions and their topological invariants: Conley index and Conley-Morse graph.

As we already pointed out, to achieve useful results on a formal correspondence between combinatorial and classical dynamics, combinatorial vector fields are too specific and a more general concept is needed. Combinatorial multivector fields introduced in [53] and further generalized and studied in [41] seem to be a remedy. For instance, unlike combinatorial vector fields, they may be used to model such dynamical phenomena as heteroclinic connections or chaotic invariant sets. Thus, they constitute a natural candidate to construct an approximation scheme for classical dynamics. What concerns the results of this paper formulated and proved for combinatorial vector fields, they directly apply to combinatorial multivector fields, because every combinatorial multivector field may be subdivided into a combinatorial vector field in a way which preserves Morse decomposition. Thus, the flow constructed for the combinatorial vector field shares the Conley-Morse graph with the combinatorial multivector field. An interesting observation here is that the there is no unique way to subdivide which is related to the known phenomenon of non-uniqueness of connection matrices in classical dynamics. This will be discussed in [54].

The work in [37, 7] attempts to address the flow reconstruction question, but the answers provided there are not in terms of flows, naturally expected in the context of combinatorial vector fields, but in terms of a multivalued dynamical system with discrete time instead of the expected classical flow. This makes these earlier results fundamentally less satisfactory. Moreover, the results of the present paper may lead towards a clue how to answer the modeling question. This is because the present results are based on a specific cellular tiling of the polytope associated with the combinatorial vector field which, on one hand, via transversality conditions on the boundary of the tiles, provides a direct tool to construct isolating blocks for all involved isolated invariant sets based purely on the combinatorial information, but, on the other hand, suggests how to construct a combinatorial multivector field modeling a given differential equation just from the transversality conditions. Research in this direction is in progress.

The remainder of this paper is organized as follows. Section 2 provides the description of the main results of the paper together with examples. In Section 3 we collect necessary background material on topology, on simplicial complexes and their representations, on semiflows and the classical Conley index, as well as on combinatorial vector fields. Section 4 then demonstrates that the underlying polytope X of a given simplicial complex $\mathcal{X}$ can be subdivided in a canonical way into tiles, based on the concept of barycentric coordinates. These tiles form the cell decomposition used in Theorems 2.1–2.3 and are the basic building blocks of our semiflow construction. In addition, we introduce the notion of an admissible semiflow on X for a combinatorial vector field $\mathcal{V}$, which has to satisfy certain transversality conditions on the tile boundaries, as well as the condition of strong admissibility which additionally puts restrictions on the flow behavior in arrow tiles. After that, in Section 5 we recall the concepts of isolated invariant sets and Morse decompositions for a combinatorial vector field $\mathcal{V}$, and we show that any semiflow on X which is admissible has the same isolated invariant sets and Conley indexes, while every strongly admissible semiflow has the same Conley-Morse graph as the combinatorial vector field $\mathcal{V}$, thereby establishing Theorems 2.1 and 2.2. The construction of a specific strongly admissible semiflow is the subject of Section 6, and this finally implies Theorem 2.3. The semiflow construction follows the design decisions made earlier in this section.

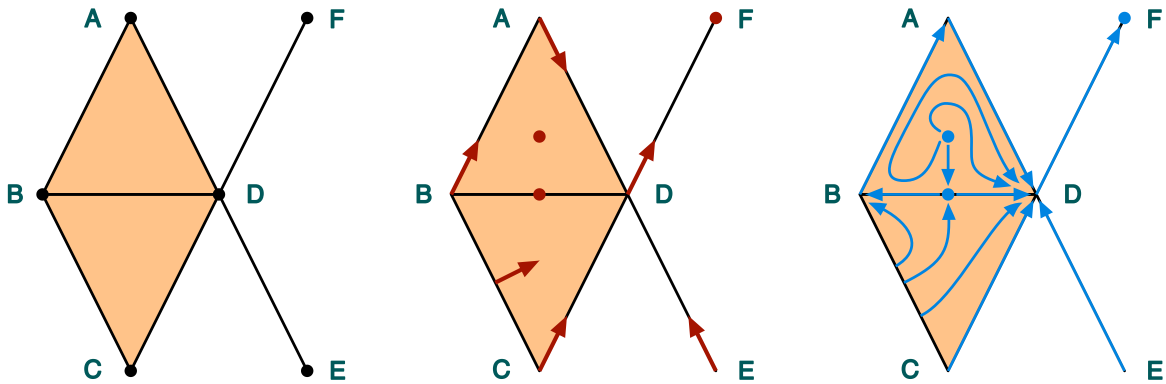

FIGURE 1. Sample simplicial complex $\mathcal{X}$, together with a combinatorial vector field $\mathcal{V}$ and the induced semiflow. The left panel shows a two-dimensional simplicial complex consisting of six vertices, seven edges, and two triangles. The panel in the middle depicts a combinatorial vector field on $\mathcal{X}$. Critical cells are indicated as red dots, arrows of $\mathcal{V}$ are marked in red. Finally, the right-most panel sketches an induced semiflow on the geometric representation X of $\mathcal{X}$.

2. Main Results

We recall that a *combinatorial vector field* $\mathcal{V}$ on a *simplicial complex* $\mathcal{X}$ may be interpreted as a certain partition of $\mathcal{X}$ into subsets of cardinality at most two (see Sections 3.2 and 3.10 for precise definitions). Singletons in $\mathcal{V}$ are interpreted as *critical cells*. Doubletons (sets of cardinality two) are required to consist of an n-dimensional simplex $\sigma \in \mathcal{X}$ and one of its $(n-1)$-dimensional faces $\tau \subset \sigma$. A doubleton $\{\tau, \sigma\} \in \mathcal{V}$ is interpreted as a *vector* or *arrow* with tail τ and head σ, denoted $\tau \to \sigma$.

As an example consider the two-dimensional simplicial complex

$$\mathcal{X} := \{A,\ B,\ C,\ D,\ E,\ F,\ AB,\ AD,\ BC,\ BD,\ CD,\ DE,\ DF,\ ABD,\ BCD\},$$

where for the sake of readability we write $A = \{A\}$, $AB = \{A, B\}$, $ABD = \{A, B, D\}$, etc. As a sample combinatorial vector field in $\mathcal{X}$ we take

$$\mathcal{V} := \{\{F\}, \{BD\}, \{ABD\}, \{A, AD\}, \{B, AB\}, \{BC, BCD\}, \{C, CD\}, \{D, DF\}, \{E, DE\}\}.$$

Thus, the critical cells of $\mathcal{V}$ are the vertex F, the edge BD and the triangle ABD. The arrows are $A \to AD$, $B \to AB$, $BC \to BCD$, $C \to CD$, $D \to DF$, and $E \to DE$. The simplicial complex $\mathcal{X}$ is visualized in the left image of Figure 1 and the combinatorial vector field $\mathcal{V}$ is visualized in the middle of this figure. We note that each of the fifteen simplices of $\mathcal{X}$ appears exactly once, either in an arrow or as a critical cell.

It has already been pointed out in [37] that for most combinatorial vector fields $\mathcal{V}$ on a simplicial complex $\mathcal{X}$ one can intuitively draw a continuous-time dynamical system on the underlying polytope X of $\mathcal{X}$ which mimics the behavior of $\mathcal{V}$. For our example, selected solutions of such a dynamical system are shown in the right-most panel of Figure 1. Notice that the three critical cells of $\mathcal{V}$ give rise to three equilibrium solutions. The Morse index of these stationary states is given by the dimension of the underlying simplex, which is due to the intuition that on a critical simplex the flow should move towards its boundary. Thus, in Figure 1 there is an unstable equilibrium of index two, which has a local two-dimensional unstable manifold, as well as an index one equilibrium with a local one-dimensional unstable manifold. Between these two stationary states, there exists a heteroclinic solution. Finally, there is a stable equilibrium at the vertex labelled F, and almost all solutions of the system converge to this stable equilibrium in forward time — except for the two unstable equilibria, the points on their heteroclinic connection, and the points on a unique solution which starts on the edge BC and converges to the index one equilibrium.

While the construction of a continuous-time dynamical system is fairly straightforward for small simplicial complexes, higher-dimensional examples quickly become difficult. It is therefore necessary to develop a general construction technique which leads to an easily analyzable dynamical system. While this is the main subject of the current paper, the example of Figure 1 demonstrates that there are a number of *design decisions* that have to be made first.

(D1) In a perfect world, we would like to be able to define a continuous-time dynamical system on the underlying polytope X of the simplicial complex $\mathcal{X}$ through a smooth differential equation. This should clearly be possible in neighborhoods of the three equilibrium solutions shown in Figure 1. Therefore, in general, our first design goal is to *define the continuous-time dynamical system via smooth differential equations whenever feasible*.

(D2) A closer look at our example shows that there generally are solutions of the dynamical system which do not exist for all negative time. Along the edge BC or at the vertex E, the combinatorial vector field points into the relative interior of the simplex, and therefore the flow should enter with positive velocity. In other words, our goal has to be to *define a continuous semiflow in which all solutions exist for all $t \geq 0$, but not necessarily in backward time*.

(D3) Since our space X is the underlying polytope of a simplicial complex, the local dimension of X can change. One such point is vertex D in Figure 1, and due to the choice of $\mathcal{V}$, solutions of the sought-after semiflow reaching this vertex from the edge DE or the two triangles should flow through D with positive speed and enter the edge DF. This implies that our goal has to be to *allow for solutions that can merge in finite forward time, and can do this with jumps in velocity*.

Based on the above three design decisions, the goal of this paper is the construction of a continuous semiflow on the topological space X which is piece-wise smooth. While our construction is related in spirit to Filippov systems, see for example [18, 24], we cannot directly apply Filippov's theory in our setting. This will be described in more detail later on.

In order to construct a semiflow with the above properties on the underlying polytope X of an arbitrary simplicial complex $\mathcal{X}$ and an arbitrary combinatorial vector field $\mathcal{V}$ on $\mathcal{X}$, we proceed in two steps. As a first step, the space X is subdivided into cells, each of which uniquely corresponds to an arrow or a critical cell of $\mathcal{V}$. For the example introduced in Figure 1, the associated cell decomposition is shown in the left panel of Figure 2. Notice that most of these cells intersect a number of different simplices of $\mathcal{X}$. We then call a semiflow *admissible for the combinatorial vector field* $\mathcal{V}$ (see Definition 4.8), if it is transverse to the cell boundaries in a certain way. For our example, these transversality directions are indicated in the right panel of Figure 2. While the detailed definitions of the cell decomposition and

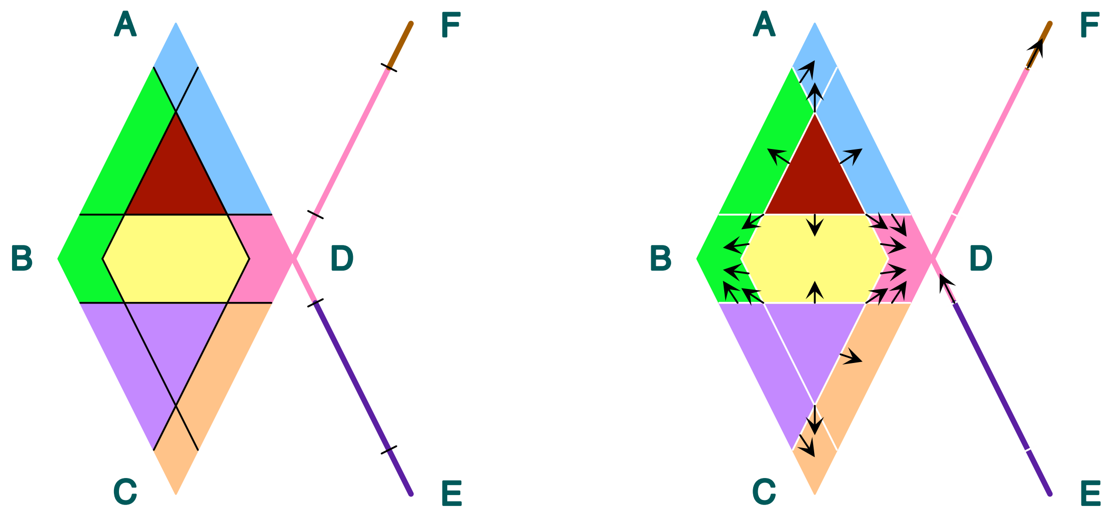

Figure 2. Cell decomposition and flow behavior across cell boundaries for the example of Figure 1. The left image shows the nine different cells of X associated with the six arrows and three critical cells of $\mathcal{V}$, highlighted in different colors. The panel on the right indicates the flow behavior across cell boundaries which is induced by $\mathcal{V}$, and which leads to the notion of admissible semiflow.

the notion of admissibility will be given later in this paper, they lead to the following first result, whose precise form is presented in Theorem 5.9.

Theorem 2.1 (Admissible Semiflows Inherit Isolated Invariant Sets and Conley Indices)**.** *Let $\mathcal{X}$ denote a simplicial complex, and let X be the underlying polytope of $\mathcal{X}$. Furthermore, let $\mathcal{V}$ denote a combinatorial vector field on $\mathcal{X}$ in the sense of Definition 3.11. Then there exist a cell decomposition of X and $\mathcal{V}$-induced flow directions across the boundaries of the cell decomposition such that the following holds. If $\varphi : \mathbb{R}_0^+ \times X \to X$ denotes a continuous dynamical system whose flow is transverse to the boundaries of the cell decomposition and moves in the prescribed directions, then for every isolated invariant set of the combinatorial vector field $\mathcal{V}$ there exists a corresponding isolated invariant set for the semiflow φ which has the same Conley index.*

While prescribing flow directions across the cell decomposition boundaries is sufficient for establishing the equivalence of isolated invariant sets and their Conley indices, this condition is not enough to carry over Morse decompositions, i.e., the global structure of the dynamics. For this we need to introduce the notion of *strong admissibility* (see Definition 4.8), which in addition limits the semiflow behavior on cells associated with arrows of $\mathcal{V}$. This leads to our second result, whose precise form is presented later in Theorem 5.10.

Theorem 2.2 (Strongly Admissible Semiflows Exhibit the Same Dynamics)**.** *Let $\mathcal{X}$ denote a simplicial complex, and let X be the underlying polytope of $\mathcal{X}$. Furthermore, let $\mathcal{V}$ denote a combinatorial vector field on $\mathcal{X}$ in the sense of Definition 3.11. Then there exists a cell decomposition of X such that the following holds. If $\varphi : \mathbb{R}_0^+ \times X \to X$ denotes a strongly admissible continuous dynamical system (in the sense made precise in Definition 4.8), then for every*

Morse decomposition of the combinatorial vector field $\mathcal{V}$ *there exists a Morse decomposition for* φ *which has the same Conley-Morse graph.*

While we are aware that many of the terms used in the formulation of these two theorems have not yet been introduced, they will be in the course of this paper. At the moment, these results only serve to show that under certain transversality and strong admissibility conditions which are induced by $\mathcal{V}$, and with respect to a cell decomposition of X which is induced by $\mathcal{X}$, continuous semiflows exhibit the same dynamics as the combinatorial vector field $\mathcal{V}$. Verifying that one can actually construct such dynamical systems φ is the subject of our following final result. Its precise form can be found in Theorem 6.12.

Theorem 2.3 (Existence of Strongly Admissible Semiflows)**.** *Let* $\mathcal{X}$ *denote a simplicial complex, and let* X *be the underlying polytope of* $\mathcal{X}$*. Furthermore, let* $\mathcal{V}$ *denote a combinatorial vector field on* $\mathcal{X}$ *in the sense of Definition 3.11. Then one can explicitly construct a continuous semiflow* φ *which satisfies all the assumptions of Theorems 2.1 and 2.2, and which conforms to our design decisions (D1) through (D3).*

Combined, the above three theorems show that every combinatorial vector field $\mathcal{V}$ on an abstract simplicial complex $\mathcal{X}$ gives rise to a continuous semiflow on the underlying polytope X of $\mathcal{X}$ which exhibits the same dynamics in the sense of Conley theory. For the example in Figure 1 the semiflow constructed in the above result is shown in Figure 3.

In the special case when the combinatorial vector field is gradient in the sense of Forman [26] one can conclude even more. In this case every strongly admissible semiflow, in particular the semiflow in Theorem 2.3, is strongly gradient like in the sense of Conley [13, Section II.6.3]. Moreover, since the combinatorial vector field and the strongly admissible semiflow share the same Conley-Morse graph, the critical cells of the combinatorial vector field of dimension n are in one-to-one correspondence with the rest points of the semiflow with Morse index n. We conjecture that this correspondence extends to connection matrices and Morse complexes, and even inside Morse sets one can see some correspondence of recurrent, in particular periodic, behavior. This is currently under investigation.

3. Preliminaries

3.1. **Sets, Maps, and Topology.**

We denote the sets of real numbers, strictly negative real numbers, strictly positive real numbers, non-positive real numbers, non-negative real numbers, and integers, respectively, by $\mathbb{R}$, $\mathbb{R}^-$, $\mathbb{R}^+$, $\mathbb{R}_0^-$, $\mathbb{R}_0^+$, and $\mathbb{Z}$. Given a finite set X, we write $\#X$ for the number of elements of X. We say that a set A is a *doubleton* if $\#A = 2$. By a $\mathbb{Z}$-interval we mean a subset I of $\mathbb{Z}$ such that $x, z \in I$ and $x \le y \le z$ imply $y \in I$.

We write $\gamma : X \nrightarrow Y$ for a partial map from X to Y, that is, a map defined on a subset $\operatorname{dom}\gamma \subset X$, called the *domain* of γ, and such that the set of values of γ, denoted $\operatorname{im}\gamma$, is contained in Y. For functions $\gamma : I \to Y$ with $I = \operatorname{dom}\gamma \subset \mathbb{Z}$ we use the sequence-type notation γ_n to denote the value $\gamma(n)$.

Given a topological space X and $A \subset X$ we write $\operatorname{int}_X A$, $\operatorname{cl}_X A$, and $\operatorname{bd}_X A$, respectively, for the interior, the closure, and the boundary of A in X. We drop the subscript in this notation if the space X is clear from the context. By a topological pair we mean a pair (X, A) of topological spaces such that $A \subset X$. Given a topological pair (A, B) we write A/B for the quotient space with quotient topology and we denote by $[B]$ the point in A/B resulting from collapsing B.

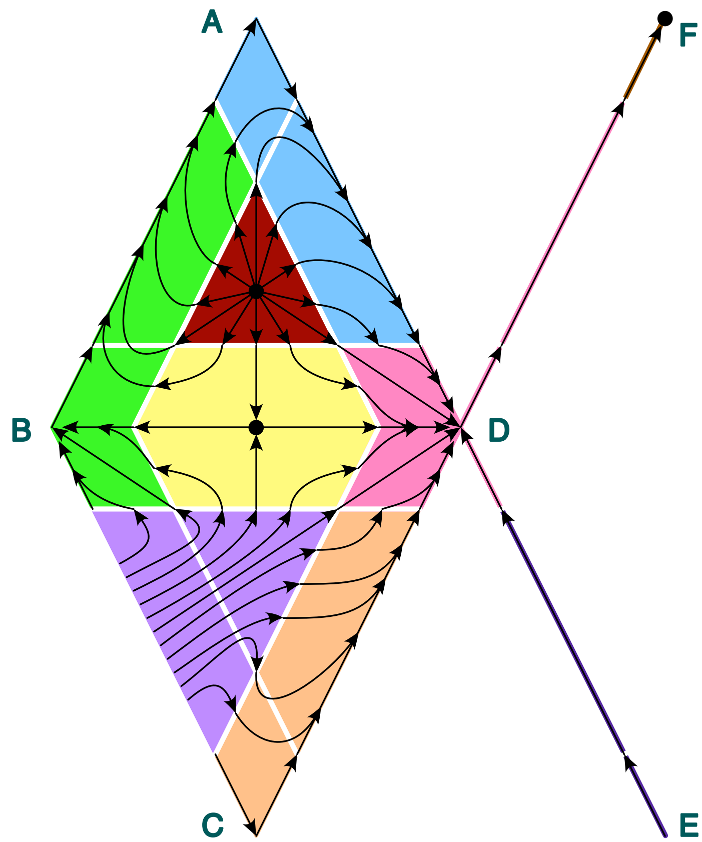

FIGURE 3. Sketch of a strongly admissible semiflow for the example introduced in Figure 1, as constructed in the proof of Theorem 2.3.

3.2. **Abstract Simplicial Complexes.** The terminology, notation and conventions we use with respect to simplicial complexes is based on [35, 55]. We summarize here the main ideas.

By an *abstract simplicial complex* $\mathcal{X}$ we mean a finite collection of nonempty, finite sets such that if $\sigma \in \mathcal{X}$, then every nonempty subset of σ also belongs to $\mathcal{X}$. We refer to the elements of $\mathcal{X}$ as the *simplices* of the simplicial complex. For every simplex $\sigma \in \mathcal{X}$ we define its *dimension* as $\dim \sigma := \#\sigma - 1$. We refer to the 0-, 1-, and 2-dimensional simplices as the *vertices*, *edges*, and *triangles*, respectively. The union of the simplices in $\mathcal{X}$ is called the *vertex set* of $\mathcal{X}$ and denoted $\mathcal{X}_0$. Formally speaking, a vertex $v \in \mathcal{X}_0$ is different from a vertex $\{v\}$ considered as a zero-dimensional simplex. But, it will always be clear from the context what we mean by a vertex. For a simplex $\sigma \in \mathcal{X}$, any nonempty subset $\tau \subset \sigma$ is called a *face of* σ, and in this case σ is referred to as a *coface of* τ. The face τ is called a *proper face* if $\tau \neq \sigma$. Furthermore, a face τ of a simplex σ is called a *facet of* σ, if $\dim \tau = \dim \sigma - 1$. Finally, the *dimension* of the simplicial complex is the maximum of the dimensions of all simplices in $\mathcal{X}$, and it is denoted by $\dim \mathcal{X}$. Two abstract simplicial complexes $\mathcal{X}$ and $\mathcal{X}'$ are called *isomorphic* if there is a bijection $f : \mathcal{X}_0 \to \mathcal{X}'_0$ such that $\sigma \in \mathcal{X}$ if and only if $f(\sigma) \in \mathcal{X}'$.

3.3. **Geometric Simplicial Complexes.** A *geometric n-simplex* σ in $\mathbb{R}^N$ is the convex hull of $n+1$ affinely independent points $v_0, v_1, \ldots, v_n \in \mathbb{R}^N$, that is, points such that the n

vectors $v_i - v_0$ for $i = 1, \ldots, n$ are linearly independent. A *standard n-simplex* is a geometric n-simplex $\Delta^n \subset \mathbb{R}^{n+1}$ spanned by all *versors* of $\mathbb{R}^{n+1}$ where the ith versor in $\mathbb{R}^{n+1}$ is a vector in $\mathbb{R}^{n+1}$ whose ith coordinate is one and all other coordinates are zero. We use the abbreviated notation $\sigma = \langle v_0, v_1, \ldots, v_n \rangle$ to indicate that σ is the geometric simplex spanned by the points $v_0, v_1, \ldots, v_n$. The *vertex set* of σ is the set $\{ v_0, v_1, \ldots, v_n \}$. The elements of this set are the *vertices* of σ. The number n is the *dimension* of σ. A *face* of σ is a geometric simplex whose vertices constitute a subset of $\{ v_0, v_1, \ldots, v_n \}$. The concepts of *proper face*, *coface*, and *facet* in the geometric setting are defined analogously to the abstract case.

We now turn our attention to the representation of points in geometric simplices. Every point $x \in \sigma = \langle v_0, v_1, \ldots, v_n \rangle$ has a unique representation of the form

$$x = \sum_{i=0}^{n} t_{v_i}(x) v_i, \quad \text{where} \quad \sum_{i=0}^{n} t_{v_i}(x) = 1 \quad \text{and} \quad t_{v_i}(x) \geq 0. \tag{1}$$

The number $t_{v_i}(x)$ is called the *barycentric coordinate* of x with respect to the vertex v_i. While this definition introduces barycentric coordinates via functions t_{v_i}, we sometimes also make use of the abbreviated notation

$$x_{v_i} := t_{v_i}(x) \quad \text{for all} \quad x \in \sigma = \langle v_0, v_1, \ldots, v_n \rangle \quad \text{and} \quad i = 0, \ldots, n, \tag{2}$$

in which we express the barycentric coordinates as actual coordinates of x. Given a geometric n-simplex $\sigma = \langle v_0, v_1, \ldots, v_n \rangle$, the associated *cell* $\overset{\circ}{\sigma}$ consists of all points in σ whose barycentric coordinates are all strictly positive. Note that for every geometric simplex σ we have

$$\sigma = \bigcup \{ \overset{\circ}{\tau} \mid \tau \text{ a face of } \sigma \}. \tag{3}$$

A *geometric simplicial complex* in $\mathbb{R}^N$ consists of a finite collection $\mathcal{K}$ of geometric simplices in $\mathbb{R}^N$ such that every face of a simplex in $\mathcal{K}$ is in $\mathcal{K}$, and the intersection of two simplices in $\mathcal{K}$ is their common face. The *underlying polytope* or briefly *polytope* of a geometric simplicial complex $\mathcal{K}$ is the union of all simplices in $\mathcal{K}$ considered as a topological space with the topology inherited from $\mathbb{R}^N$. The *vertex scheme* of a geometric simplicial complex $\mathcal{K}$ is an abstract simplicial complex whose abstract simplices are the vertex sets of the geometric simplices in $\mathcal{K}$.

3.4. **Subcomplexes and Combinatorial Closures.** A *subcomplex* of a simplicial complex $\mathcal{X}$, abstract or geometric, is a collection of simplexes in $\mathcal{X}$ which itself is a simplicial complex. The *combinatorial closure* of a subset $\mathcal{S}$ of a simplicial complex $\mathcal{X}$ is the set of faces of all dimensions of simplices in $\mathcal{S}$. It is denoted by $\operatorname{Cl} \mathcal{S}$. We say that a collection of simplices $\mathcal{S}$ in $\mathcal{X}$ is *combinatorially closed*, if $\operatorname{Cl} \mathcal{S} = \mathcal{S}$. Hence, the combinatorial closure $\operatorname{Cl} \mathcal{S}$ is the smallest subcomplex of $\mathcal{X}$ which contains $\mathcal{S}$, and the set $\mathcal{S}$ is combinatorially closed if and only if it is a subcomplex of $\mathcal{X}$. For a simplex $\sigma \in \mathcal{X}$ we denote the set of all proper faces of σ by $\operatorname{Bd} \sigma$ and call it the *combinatorial boundary* of the simplex. For any vertex $v \in \mathcal{X}_0$, the *star of* v consists of all simplices of $\mathcal{X}$ which contain v as a vertex.

A reader familiar with finite topological spaces will immediately notice that the phrases "combinatorial closure" and "combinatorially closed" may be replaced respectively by "closure" and "closed" with respect to the Alexandrov topology [2] induced on a finite simplicial complex $\mathcal{X}$, abstract or geometric, considered as a poset of simplexes ordered by inclusion [6, Definition 1.4.10]. Similarly, the star of a vertex is the smallest open set in the Alexandrov topology containing the vertex. However, we notice that the combinatorial boundary of a simplex need not be its topological boundary in the Alexandrov topology.

3.5. **Geometric Realizations.** A geometric simplicial complex $\mathcal{K}$ is a *geometric realization* of an abstract simplicial complex $\mathcal{X}$ if the vertex scheme of $\mathcal{K}$ is isomorphic to $\mathcal{X}$. One fundamental property of abstract simplicial complexes is the following theorem, see for example [35, Proposition 1.9.3 and Theorem 1.9.5], [45, Proposition 1.5.4], or [55, Theorem 3.1].

Theorem 3.1. *Every abstract simplicial complex admits a geometric realization and any two of its geometric realizations are piecewise linear isomorphic. In particular, their underlying polytopes are homeomorphic.* □

Theorem 3.1 states that in topological terms the geometric realization of an abstract simplicial complex is unique. Two geometric realizations of the same abstract simplicial complex may differ geometrically but the underlying polytopes are homeomorphic. In particular, by the *underlying polytope* of an abstract simplicial complex $\mathcal{X}$ we mean the underlying polytope of any geometric realization of $\mathcal{X}$. Note that the underlying polytope of an abstract simplicial complex is unique up to a homeomorphism.

Among the many geometric realizations of an abstract simplicial complex $\mathcal{X}$ there is one, typically used in the proofs of existence, constructed as follows. We set $d := \#\mathcal{X}_0$, identify the vertices $v_1, v_2, \dots v_d$ of $\mathcal{X}_0$ in a fixed order with versors $e_1, e_2, \dots e_d$ of $\mathbb{R}^d$ and take as the geometric realization of $\mathcal{X}$ the subcomplex of the standard $(d-1)$-simplex in $\mathbb{R}^d$ consisting of faces $\langle e_{i_0}, e_{i_1}, \dots, e_{i_n} \rangle$ such that $\{v_{i_0}, v_{i_1}, \dots, v_{i_n}\}$ is a simplex in $\mathcal{X}$. We refer to this geometric realization of $\mathcal{X}$ as the *standard* geometric realization. For convenience, in this paper we typically work with the standard geometric realization. We emphasize, however, that the results of the paper are purely topological and apply to every geometric realization.

Depending on the context, a simplex σ of $\mathcal{X}$ is interpreted as either an abstract simplex $\{\, v_0, \dots, v_n \,\}$, or the corresponding geometric simplex $\langle v_0, \dots, v_n \rangle \subset \mathbb{R}^d$. For example, if we write $\sigma \subset \mathcal{X}_0$, we always mean an abstract simplex. Whenever for an arbitrary point $x \in \mathbb{R}^d$ we write $x \in \sigma$, then σ is interpreted as the geometric simplex. If v is a vertex, writing $v \in \sigma$ makes equal sense in both cases.

Given the underlying polytope X of $\mathcal{X}$ the barycentric coordinates introduced earlier can be extended to a well-defined continuous functions $t_v : X \to [0,1]$ which assigns to each point $x \in X$ its barycentric coordinate with respect to the vertex v, whenever x belongs to a simplex in the star of v, and zero otherwise.

With every subset $\mathcal{S} \subset \mathcal{X}$ we associate a subset $|\mathcal{S}|$ of the underlying polytope X given by

$$|\mathcal{S}| \;:=\; \bigcup_{\sigma \in \mathcal{S}} \overset{\circ}{\sigma} \;\subset\; X \;=\; |\mathcal{X}|.$$

Note that in the case when $\mathcal{S}$ is combinatorially closed, that is, $\mathcal{S}$ is a subcomplex of $\mathcal{X}$, we get from (3) that

$$|\mathcal{S}| = \bigcup_{\sigma \in \mathcal{S}} \sigma. \tag{4}$$

In particular, if $\mathcal{S}$ is a subcomplex of $\mathcal{X}$ then $|\mathcal{S}|$ is the underlying polytope of $\mathcal{S}$.

As an example consider $\mathcal{S} = \{\{v\}\} \subset \mathcal{X}$ consisting of a singleton of a vertex in $\mathcal{X}_0$. Then $|\mathcal{S}| = \{v\}$ is the associated singleton in the underlying polytope X. But, if we have $\mathcal{S} =$

$\{\{v, w\}\} \subset \mathcal{X}$, then $|\mathcal{S}|$ is the line segment between the points v and w, but not including either of the endpoints. Moreover, one can easily see that the following holds.

Lemma 3.2. *Let $\mathcal{X}$ denote a simplicial complex, and let $X = |\mathcal{X}|$ be the underlying polytope. Then a subset $\mathcal{S} \subset \mathcal{X}$ is combinatorially closed if and only if the set $|\mathcal{S}| \subset |\mathcal{X}|$ is closed.* $\square$

This simple lemma will be useful for constructing isolating blocks for admissible semiflows, as it allows us to translate properties for isolated invariant sets in the combinatorial setting directly to the classical dynamical systems framework.

3.6. **Representable Sets and Convex Partitions.** The construction we present in this paper relies on the class of triangulable topological spaces to guarantee some properties of homology modules as explained in Section 3.8. We recall that a topological space is *triangulable* if it is homeomorphic to the polytope of a geometric simplicial complex, and a topological pair (X, A) is *triangulable* if X is homeomorphic to the polytope of a geometric simplicial complex $\mathcal{K}$ and A is homeomorphic to the polytope of a subcomplex of $\mathcal{K}$. To ensure that certain sets and pairs in our construction are triangulable we need some definitions and results presented in this section.

Let $C \subset \mathbb{R}^N$ be a convex set. We recall that the *dimension* of C is the dimension of the affine hull of C (see [56, Section 1]) and C is *relatively open* (see [56, Section 6]) if it is open in its affine hull. A simple argument based on [56, Theorem 6.1] proves the following proposition.

Proposition 3.3. *Assume $C \subset \mathbb{R}^N$ is a relatively open, convex set of dimension n. Then the topological pair $(\operatorname{cl} C, \operatorname{cl} C \setminus C)$ is homeomorphic to the pair (B^n, S^{n-1}) where B^n denotes the closed unit ball in $\mathbb{R}^n$ and S^{n-1} denotes the boundary of B^n.* $\square$

By a *partition* $\mathcal{C}$ of a set X we mean a finite family of pairwise disjoint, non-empty subsets of X such that $\bigcup \mathcal{C} = X$. Given a partition $\mathcal{C}$ of X we say that a subset $A \subset X$ is $\mathcal{C}$-*representable* if $A = \bigcup \mathcal{C}'$ for a $\mathcal{C}' \subset \mathcal{C}$. We say that a partition $\mathcal{C}$ of a compact set X is *topologically closed* if $\operatorname{cl} C$ is $\mathcal{C}$-representable for every $C \in \mathcal{C}$. The following proposition is straightforward.

Proposition 3.4. *Let $\mathcal{C}$ be a partition of a set X. Then the union, intersection, and set difference of two $\mathcal{C}$-representable sets is $\mathcal{C}$-representable. Moreover, if X is compact and $\mathcal{C}$ is topologically closed, then the closure, interior, and boundary of a $\mathcal{C}$-representable set is $\mathcal{C}$-representable.* $\square$

By a *convex partition* of a compact set $X \subset \mathbb{R}^N$ we mean a topologically closed partition $\mathcal{C}$ of X such that every $C \in \mathcal{C}$ is a relatively open, convex set. As an easy consequence of Proposition 3.3 we obtain the following theorem whose triangulability part follows from [42, Theorem 2.1, Theorem 1.7].

Theorem 3.5. *Assume $X \subset \mathbb{R}^N$ admits a convex partition $\mathcal{C}$. Then X is a regular CW complex whose closed cells are the closures of the elements of $\mathcal{C}$. Moreover, every closed, $\mathcal{C}$-representable subset of X is also a regular CW complex and, in consequence, it is triangulable. The same applies to pairs (A, B) of closed $\mathcal{C}$-representable subsets of X.* $\square$

3.7. **The Family of $\mathcal{D}^d_\varepsilon$-Representable Sets.** We will now introduce a class of representable sets needed in this paper. We first need an auxiliary proposition whose elementary proof is left to the reader.

Proposition 3.6. *Convex partitions have the following properties:*

(i) *If $\mathcal{C}$ and $\mathcal{C}'$ are convex partitions respectively of compact sets $X \subset \mathbb{R}^N$ and $X' \subset \mathbb{R}^{N'}$ then*
$$\mathcal{C} \mathbin{\bar{\times}} \mathcal{C}' := \{\, C \times C' \mid C \in \mathcal{C},\, C' \in \mathcal{C}' \,\}$$
is a convex partition of $X \times X' \subset \mathbb{R}^{N+N'}$.

(ii) *If $\mathcal{C}$ is a convex partition of a compact set $X \subset \mathbb{R}^N$ and $A \subset \mathbb{R}^N$ is a convex set such that $X \cap A$ is compact, then*
$$\mathcal{C} \mathbin{\bar{\cap}} A := \{\, C \cap A \mid C \in \mathcal{C},\, C \cap A \neq \emptyset \,\}$$
is a convex partition of $X \cap A$.

(iii) *If $\mathcal{C}$ is a convex partition of a compact set $X \subset \mathbb{R}^N$ and $Y \subset X$ is a $\mathcal{C}$-representable, closed set, then*
$$\mathcal{C}_Y := \{\, C \in \mathcal{C} \mid C \subset Y \,\}$$
is a convex partition of Y. □

Given an $\varepsilon \in (0,1)$ one easily verifies that
$$\mathcal{C}^1_\varepsilon := \{\{0\}, (0,\varepsilon), \{\varepsilon\}, (\varepsilon,1), \{1\}\}$$
is a convex partition of $[0,1] \subset \mathbb{R}$. Using Proposition 3.6(i) and proceeding recursively, we construct a convex partition $\mathcal{C}^d_\varepsilon$ of $[0,1]^d$ from a convex partition $\mathcal{C}^{d-1}_\varepsilon$ of $[0,1]^{d-1}$ by setting $\mathcal{C}^d_\varepsilon := \mathcal{C}^{d-1}_\varepsilon \mathbin{\bar{\times}} \mathcal{C}^1_\varepsilon$. Thus, by Proposition 3.6(ii) we also have a convex partition $\mathcal{D}^d_\varepsilon := \mathcal{C}^d_\varepsilon \mathbin{\bar{\cap}} \Delta^d$ of the standard d-simplex $\Delta^d \subset \mathbb{R}^{d+1}$.

We have the following corollary of Proposition 3.4 and Theorem 3.5.

Corollary 3.7. *The family of $\mathcal{D}^d_\varepsilon$-representable sets has the following properties.*

(i) *The family is closed under the set-theoretic operations of union, intersection, and difference, as well as under the topological operations of closure, interior, and boundary.*

(ii) *Every pair of sets in this family is triangulable.* □

3.8. **Homology.** Given an abstract simplicial complex $\mathcal{X}$ and its subcomplex $\mathcal{A}$ we denote the simplicial homology of the simplicial pair $(\mathcal{X}, \mathcal{A})$ by $H_*(\mathcal{A}, \mathcal{B})$ (see [55, Section 5]). For a compact metric topological space X and its closed subsets $B \subset A \subset X$, unless explicitly specified otherwise, by $H_*(A, B)$ we mean the singular homology of the pair (A, B) (see [55, Section 29]). We reduce this notation to $H_*(A, b)$ if $B = \{b\}$ is a singleton. We note that for triangulable pairs singular homology is isomorphic to Steenrod homology [44, 60, 61], because they both satisfy the Eilenberg-Steenrod axioms (see [22, Theorem 10.1.c]). Since Steenrod homology satisfies the so-called strong excision property [48, Axiom 8, Theorem 5], we get the following theorem for singular homology of triangulable pairs.

Theorem 3.8 (Strong Excision for Triangulable Pairs)**.** *If $f : (X, A) \to (Y, B)$ is a relative homeomorphism of compact, metric, triangulable pairs, that is, a continuous map $f : X \to Y$ which carries A into B and $X \setminus A$ homeomorphically onto $Y \setminus B$, then $f_* : H(X, A) \to H(Y, B)$ is an isomorphism.* □

In particular, we have $H_*(A, B) \cong H_*(A/B, [B])$, a property useful in the definition of the Conley index.

We also recall the following theorem which is a straightforward consequence of the Vietoris-Begle Theorem [9, 10] for Steenrod homology [3, 62](see also [21, 23]), the Five Lemma, the exactness, and homotopy axioms for homology and the isomorphism between Steenrod and singular homology for triangulable pairs.

Theorem 3.9 (Relative Vietoris-Begle Theorem for Singular Homology)**.** *Assume that the map $f : (X, A) \to (Y, B)$ is a continuous surjection of compact, metric, triangulable pairs, satisfying $f^{-1}(B) = A$ and such that $f^{-1}(y)$ is contractible for every $y \in Y$. Then the induced map $f_* : H(X, A) \to H(Y, B)$ is an isomorphism.* □

To ensure triangulability, in this paper we apply singular homology only to pairs of closed, $\mathcal{D}^d_\varepsilon$-representable sets. Since by Theorem 3.5 such pairs are triangulable, the singular and Steenrod homology of such pairs, as well as the simplicial homology of any triangulation of such a pair, are isomorphic. This lets us easily switch between singular, Steenrod, and simplicial homology according to the context and our needs.

We note that a pair of abstract simplicial complexes $(\mathcal{X}, \mathcal{A})$ may also be considered as a pair of finite topological spaces with Alexandrov topology [2] induced by the face poset. Therefore, singular homology of the pair $(\mathcal{X}, \mathcal{A})$ is well-defined. Also here there is no ambiguity, because, by McCord's Theorem [47], the singular homology of the pair $(\mathcal{X}, \mathcal{A})$ is isomorphic to its simplicial homology.

3.9. **Semiflows.** Our main tool for establishing the connections between combinatorial and classical dynamics is based on Conley theory. Hence, we now recall some basic facts of this theory, see for example [57] for more details.

Consider a semiflow on a compact metric space X, i.e., a continuous map $\varphi : \mathbb{R}^+_0 \times X \to X$ satisfying $\varphi(0, x) = x$ for all $x \in X$ and $\varphi(s, \varphi(t, x)) = \varphi(s+t, x)$ for all $x \in X$ and $s, t \in \mathbb{R}^+_0$. Let $I \subset \mathbb{R}$ be an interval. We say that a map $\gamma : I \to X$ is a *solution* of φ if for any $t \in I$ and any $s \in \mathbb{R}^+_0$ such that $s + t \in I$ we have $\varphi(s, \gamma(t)) = \gamma(s + t)$. A solution γ is a *full solution* if $I = \mathbb{R}$ is satisfied. The α- *and* ω*-limit sets* of a full solution γ are given respectively by

$$\alpha(\gamma) := \bigcap_{\tau \leq 0} \operatorname{cl} \gamma((-\infty, \tau]) \quad \text{and} \quad \omega(\gamma) := \bigcap_{\tau \geq 0} \operatorname{cl} \gamma([\tau, \infty)).$$

We say that a solution $\gamma : I \to X$ is a solution *through* $x \in X$ if $0 \in I$ and $\gamma(0) = x$. Given an arbitrary subset N of X we define

$$\begin{aligned}
\operatorname{Inv}^+(N, \varphi) &:= \left\{ x \in N \mid \varphi(\mathbb{R}^+_0, x) \subset N \right\}, \\
\operatorname{Inv}^-(N, \varphi) &:= \left\{ x \in N \mid \text{there exists a solution } \gamma : \mathbb{R}^-_0 \to N \text{ of } \varphi \text{ through } x \text{ in } N \right\}, \\
\operatorname{Inv}(N, \varphi) &:= \left\{ x \in N \mid \text{there exists a solution } \gamma : \mathbb{R} \to N \text{ of } \varphi \text{ through } x \text{ in } N \right\}.
\end{aligned}$$

If the considered semiflow φ is clear from context, we simplify this notation to $\operatorname{Inv}^- N$, $\operatorname{Inv}^+ N$, and $\operatorname{Inv} N$, respectively. One can easily see that $\operatorname{Inv} N = \operatorname{Inv}^+ N \cap \operatorname{Inv}^- N$. We say that a subset $S \subset X$ is *invariant*, if we have $\operatorname{Inv} S = S$.

Of fundamental importance for Conley theory is the notion of isolation. A closed invariant set S is called an *isolated invariant set* if there exists a closed neighborhood N of S such that

$$\operatorname{Inv} N \;=\; S \;\subset\; \operatorname{int} N.$$

In this case, the set N is called an *isolating neighborhood*. Isolating neighborhoods play an important role in Conley theory, since they allow one to make assertions about S by studying the dynamics of φ close to the boundary of N. One approach is centered around specific isolating neighborhoods called isolating blocks. To define this notion, consider a closed subset $B \subset X$ and let $x \in \operatorname{bd} B$ be an arbitrary boundary point. Then x is called a

- *strict egress point*, if for every solution $\gamma : [\delta_1, \delta_2] \to X$ through x with $\delta_1 \le 0 < \delta_2$ there exists a neighborhood J of 0 in $[\delta_1, \delta_2]$ with $\gamma(t) \notin B$ for all $t \in J \cap \mathbb{R}^+$, as well as $\gamma(t) \in \operatorname{int} B$ for all $t \in J \cap \mathbb{R}^-$,
- *strict ingress point*, if for every solution $\gamma : [\delta_1, \delta_2] \to X$ through x with $\delta_1 \le 0 < \delta_2$ there is a neighborhood J of 0 in $[\delta_1, \delta_2]$ with $\gamma(t) \in \operatorname{int} B$ for all $t \in J \cap \mathbb{R}^+$ and $\gamma(t) \notin B$ for all $t \in J \cap \mathbb{R}^-$,
- *bounce-off point*, if for every solution $\gamma : [\delta_1, \delta_2] \to X$ through x with $\delta_1 \le 0 < \delta_2$ there exists a neighborhood J of 0 in $[\delta_1, \delta_2]$ with $\gamma(t) \notin B$ for all $t \in J \setminus \{0\}$,

where again $\mathbb{R}^\pm$ denotes the set of all strictly positive/negative real numbers. The set of all strict egress, strict ingress, and bounce-off points of B are denoted by B^e, B^i, and B^b, respectively. We define the *exit set* of B by

$$B^- := B^e \cup B^b. \tag{5}$$

Then the closed set $B \subset X$ is called an *isolating block* if we have

$$\operatorname{bd} B = B^e \cup B^i \cup B^b \tag{6}$$

and

$$\text{the exit set } B^- \text{ is closed in } X. \tag{7}$$

One can readily see that every compact isolating block B is an isolating neighborhood for the invariant set $S = \operatorname{Inv} B$, since no full solution in B can touch the boundary $\operatorname{bd} B$ due to the lack of internal tangencies of solutions at the boundary of B.

Knowledge of an isolating block often suffices to make statements about the isolated invariant set S, even if S is unknown. For this, Conley [13] defined the *homotopy Conley index of S* as the homotopy type of the pointed space

$$h(S) \; := \; \left[B/B^-, \; [B^-] \right].$$

The Conley index is well-defined, because, given S, an isolating block such that $S = \operatorname{Inv} B$ always exists and the homotopy type is independent of the choice of B up to homotopy equivalence. It only depends on the underlying isolated invariant set S. In this paper we exclusively use the derived *homological Conley index of S* defined as the Steenrod homology of the pointed space

$$CH_*(S) \; := \; H_* \left(B/B^-, \; [B^-] \right),$$

where the Steenrod homology on the right-hand side may be replaced with singular homology when the pair (B, B^-) is triangulable. Since we work with compact metric spaces, the strong excision property of Steenrod homology enables us to rewrite the Conley index as the relative homology

$$CH_*(S) \; = \; H_*(B, B^-).$$

Note that for the computation of the Conley index only one isolating block is necessary. The homological Conley index is a graded Abelian group, i.e., the notation $CH_*(S)$ denotes a sequence $(CH_k(S))_{k=0}^\infty$ of Abelian groups $CH_k(S)$. The celebrated Ważewski principle can then be stated as follows: If at least one of the homology groups $CH_k(S)$ is non-trivial, then we necessarily have $S \neq \emptyset$.

While the Conley index allows one to study specific isolated invariant sets, we are frequently interested in a finer decomposition of a given isolated invariant set into smaller ones. For this, we need to introduce the notion of Morse decomposition for semiflows. Let $S \subset X$ denote an isolated invariant set for the semiflow φ. A *Morse decomposition* of S (see [46]) is a collection $M = \{M_p \mid p \in \mathbb{P}\}$ of mutually disjoint isolated invariant sets $M_p \subset S$ and a strict partial order $>$ on $\mathbb{P}$ which satisfy the following property. For every $x \in S$, we either have $x \in M_p$ for some $p \in \mathbb{P}$, or there exist $p, q \in \mathbb{P}$ with $p > q$ and a full solution γ through x such that the α- and ω-limit sets satisfy

$$\alpha(\gamma) \subset M_p \quad \text{and} \quad \omega(\gamma) \subset M_q.$$

3.10. **Combinatorial Vector Fields.** We first recall the original definition of combinatorial vector field by Forman [25].

Definition 3.10 (Combinatorial Vector Field as a Map)**.** *A combinatorial vector field on a simplicial complex $\mathcal{X}$ is a map $\mathcal{V} : \mathcal{X} \to \mathcal{X} \cup \{0\}$ such that*

(i) *if $\mathcal{V}(\sigma) \neq 0$, then σ is a facet of $\mathcal{V}(\sigma)$,*
(ii) *if $\tau \in \operatorname{im} \mathcal{V} \setminus \{0\}$, then $\mathcal{V}(\tau) = 0$,*
(iii) *for $\tau \in \mathcal{X}$ the cardinality of $\mathcal{V}^{-1}(\tau)$ is at most one.*

In this paper we use the following, equivalent definition.

Definition 3.11 (Combinatorial Vector Field as a Partition)**.** *A* combinatorial vector field $\mathcal{V}$ *on a simplicial complex $\mathcal{X}$ is a partition of $\mathcal{X}$ into singletons and doubletons such that each doubleton consists of a simplex and one of its facets.*

The equivalence of the two definitions is established by the following proposition, which is straightforward to verify.

Proposition 3.12. *If $\mathcal{V}$ is a combinatorial vector field in the sense of Definition 3.10, then*

$$\{\, \{\sigma, \mathcal{V}(\sigma)\} \mid \mathcal{V}(\sigma) \neq 0 \,\} \;\cup\; \{\, \{\sigma\} \,:\, \mathcal{V}(\sigma) = 0,\ \mathcal{V}^{-1}(\sigma) = \emptyset \,\}$$

is a combinatorial vector field in the sense of Definition 3.11. If $\mathcal{V}$ is a combinatorial vector field in the sense of Definition 3.11, then

$$\tau \mapsto \begin{cases} \sigma & \text{if } \{\tau, \sigma\} \in \mathcal{V} \text{ and } \tau \text{ is a facet of } \sigma, \\ 0 & \text{otherwise,} \end{cases}$$

is a combinatorial vector field in the sense of Definition 3.10. Moreover, the two constructions are mutually inverse. □

Assume now that $\mathcal{V}$ is a fixed combinatorial vector field on $\mathcal{X}$. We say that a simplex $\sigma \in \mathcal{X}$ is a *critical cell* if $\{\sigma\} \in \mathcal{V}$. A doubleton $\{\tau, \sigma\} \in \mathcal{V}$ is an *arrow* of $\mathcal{V}$. The facet relation in an arrow of $\mathcal{V}$ lets us write an arrow in the form $\tau \to \sigma$ meaning that $\{\tau, \sigma\} \in \mathcal{V}$ is an arrow and τ is a facet of σ. If $\tau \to \sigma$ is an arrow of $\mathcal{V}$, then we say that τ is the *tail* of σ and σ is the *head* of τ. We say that $\tau \in \mathcal{X}$ is a *tail* if it is the tail of some $\sigma \in \mathcal{X}$. We say that $\sigma \in \mathcal{X}$ is a *head* if it is the head of some $\tau \in \mathcal{X}$. We denote the set of critical cells, tails, and heads of $\mathcal{V}$ by $\operatorname{Crit} \mathcal{V}$, $\operatorname{Tail} \mathcal{V}$, and $\operatorname{Head} \mathcal{V}$, respectively. Note that

$$\mathcal{X} = \operatorname{Crit} \mathcal{V} \cup \operatorname{Tail} \mathcal{V} \cup \operatorname{Head} \mathcal{V},$$

and all of these three sets are mutually disjoint. Finally, for a simplex $\sigma \in \mathcal{X}$ we write

$$\sigma^- \quad := \quad \begin{cases} \tau & \text{if } \tau \text{ is a facet of } \sigma \text{ and } \{\tau, \sigma\} \in \mathcal{V}, \\ \sigma & \text{otherwise,} \end{cases} \tag{8}$$

$$\sigma^+ \quad := \quad \begin{cases} \tau & \text{if } \sigma \text{ is a facet of } \tau \text{ and } \{\tau, \sigma\} \in \mathcal{V}, \\ \sigma & \text{otherwise.} \end{cases} \tag{9}$$

Thus, σ is a critical cell if $\sigma^- = \sigma^+$, a tail if $\sigma = \sigma^- \neq \sigma^+$, and a head if $\sigma = \sigma^+ \neq \sigma^-$. We extend this notation to an $\omega \in \mathcal{V}$ by setting $\omega^- := \omega^+ := \sigma$ if ω is a critical cell $\{\sigma\}$ and $\omega^- := \tau$, $\omega^+ := \sigma$ if ω is an arrow $\tau \to \sigma$.

4. Cell Decompositions and Admissible Semiflows

In this section we lay the groundwork for the semiflow extension problem which was outlined in the introduction. We begin with recalling the cell decomposition which forms the foundation of our approach and which was introduced and used in [7, 37]. In addition, we define the notions of admissible and strongly admissible semiflow.

In this and the following sections we assume that $\mathcal{X}$ is a fixed simplicial complex and $\mathcal{V}$ is a fixed combinatorial vector field on $\mathcal{X}$. Furthermore, we suppose that $X := |\mathcal{X}|$ is the underlying polytope of the standard geometric realization of $\mathcal{X}$ and ε is a fixed constant satisfying

$$0 < \varepsilon < \frac{1}{1 + \dim \mathcal{X}}. \tag{10}$$

4.1. A First Cell Decomposition of the Underlying Polytope.

The goal of this paper is the construction of a continuous-time semiflow which mimics the behavior of the underlying combinatorial vector field. For example, in the situation shown in Figure 1 we would like the critical triangle to correspond to an unstable equilibrium of index two. As we intend to use Conley theory to formalize the connection between the two frameworks, it will be necessary to work with *isolated invariant sets*, i.e., with invariant sets which in some sense can be separated from the surrounding dynamics via neighborhoods. Another glance at the rightmost image in Figure 1 shows that this can easily be done for the index two equilibrium at the center of the triangle ABD. However, the index one equilibrium on the edge BD is another matter. While in the picture one can clearly isolate this stationary state via a small neighborhood, this neighborhood necessarily has to cover parts of the adjacent two-dimensional simplices ABD and BCD. In other words, relying purely on the decomposition of the polytope X given by the simplices in $\mathcal{X}$ will not be enough to design an easily implementable construction of isolating neighborhoods in the general case.

This situation is similar to the one encountered in our previous papers [7, 37], and it was resolved by the introduction of a new cell decomposition of the polytope X. While this decomposition is inherently connected to the simplices in $\mathcal{X}$, it leads to cells which have nontrivial intersection with all the cofaces of a given simplex. Since we will use the same cell decomposition as the foundation for our semiflow completion problem, we recall a few definitions and results from the above-cited papers.

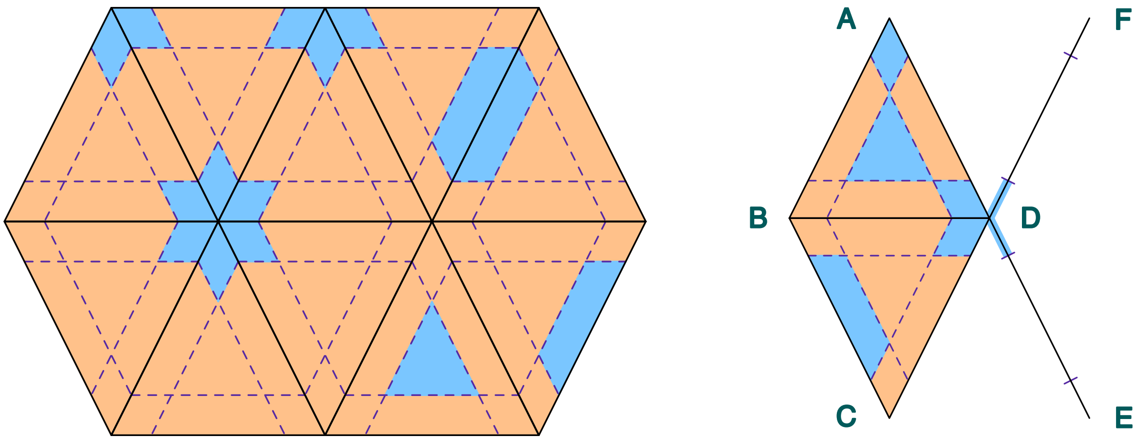

FIGURE 4. Sample ε-cell decomposition boundaries for a simplicial complex $\mathcal{X}$, as introduced in Definition 4.1. The left panel shows a complex $\mathcal{X}$ which consists of ten vertices, nineteen edges, and ten triangles, and whose polytope X is homeomorphic to a closed disk. The dashed purple lines indicate the boundaries of ε-cells, and the image shows six specific ε-cells as blue polygons. The three cells in the left half of the diagram correspond to vertices in $\mathcal{X}$, while the two right-most cells are for edges. Finally, the blue triangle on the lower right is associated with a 2-simplex. All of these cells are open subsets of $X = |\mathcal{X}|$. Similarly, the right panel shows the decomposition into ε-cells for the complex $\mathcal{X}$ of Figure 1. Note in particular the ε-cell corresponding to the vertex D, which reaches into all cofaces, regardless of their dimension.

Definition 4.1 (ε-Cell Associated with a Simplex)**.** *For every simplex $\sigma \in \mathcal{X}$ its associated* ε-cell *is defined as the set*

$$\langle\sigma\rangle_\varepsilon \;:=\; \{x \in X \;|\; t_v(x) > \varepsilon \;\text{ for all }\; v \in \sigma \quad\text{and}\quad t_v(x) < \varepsilon \;\text{ for all }\; v \notin \sigma\} \;\subset\; X.$$

This definition is illustrated in Figure 4. The left panel shows a simplicial complex $\mathcal{X}$ which consists of ten vertices, nineteen edges, and ten triangles, and whose polytope X is homeomorphic to a closed disk. The dashed purple lines indicate the boundaries of ε-cells, and they consist of points $x \in X$ which have at least one barycentric coordinate equal to ε. Six specific ε-cells are shown as blue polygons. The three cells in the left half of the figure correspond to vertices in $\mathcal{X}$, while the two right-most cells are for edges. Finally, the blue triangle on the lower right is associated with a 2-simplex. The right panel of the figure depicts sample ε-cells for the complex of Figure 1.

All but one ε-cells in Figure 1 are homeomorphic to an open Euclidean ball. The cell showing that in general ε-cells need not be homeomorphic to open Euclidean balls is the blue cell containing the vertex D in the right panel. But, ε-cells are finite unions of relatively open, convex sets belonging to the family $\mathcal{D}^d_\varepsilon$ defined in Section 3.7. In other words, we have the following proposition.

Proposition 4.2 (Representability of ε-Cells)**.** *Every ε-cell is $\mathcal{D}^d_\varepsilon$-representable.*

Proof. Consider $\sigma, \tau \in \mathcal{X}$ and an $x \in X$. Then $x \in \langle\sigma\rangle_\varepsilon \cap \overset{\circ}{\tau}$ if and only if for every $v \in \mathcal{X}_0$ we have $t_v(x) \in I_v^{\varepsilon,\tau,\sigma}$ where

$$I_v^{\varepsilon,\tau,\sigma} := \begin{cases} \{0\} & \text{for } v \notin \tau, \\ (0,\varepsilon) & \text{for } v \in \tau \setminus \sigma, \\ (\varepsilon, 1] & \text{for } v \in \tau \cap \sigma. \end{cases}$$

Hence, $\langle\sigma\rangle_\varepsilon \cap \overset{\circ}{\tau} \neq \emptyset$ implies $\langle\sigma\rangle_\varepsilon \cap \overset{\circ}{\tau} \in \mathcal{D}_\varepsilon^d$. Since we have

$$\langle\sigma\rangle_\varepsilon = \bigcup_{\tau \in \mathcal{X}} \langle\sigma\rangle_\varepsilon \cap \overset{\circ}{\tau}$$

for every $\sigma \in \mathcal{X}$, the conclusion follows. □

The following result states a number of elementary properties of ε-cells which were established in [37, Lemma 4.5]. In particular, it contains an explicit characterization of the closures of ε-cells which will be crucial later on.

Lemma 4.3 (Properties of ε-Cells)**.** *The ε-cells introduced in Definition 4.1 for different simplices in $\mathcal{X}$ are disjoint. Moreover, for every simplex $\sigma \in \mathcal{X}$ the ε-cell $\langle\sigma\rangle_\varepsilon$ is a nonempty open subset of the topological space X, and its topological closure can be characterized as*

$$\operatorname{cl}\langle\sigma\rangle_\varepsilon \;=\; \{x \in X \;|\; t_v(x) \geq \varepsilon \;\text{ for all }\; v \in \sigma \quad\text{and}\quad t_v(x) \leq \varepsilon \;\text{ for all }\; v \notin \sigma\}\,.$$

□

It is clear from this result, see also Figure 4, that the ε-cells provide a decomposition of a certain subset of X, but not of the whole polytope. However, by considering the closures of ε-cells one can easily show that

$$X \;=\; |\mathcal{X}| \;=\; \bigcup_{\sigma \in \mathcal{X}} \operatorname{cl}\langle\sigma\rangle_\varepsilon, \tag{11}$$

which is a cell decomposition of the polytope X into closed cells which intersect at most on their boundaries. This cell decomposition forms the backbone for our semiflow construction, and it requires us to have a comprehensive understanding and characterization of how the closures of ε-cells intersect, and which underlying simplices σ lead to intersections. We therefore recall both the following definition and the simple result from [37, Lemma 4.3].

Definition 4.4 (ε-Characteristic Simplices)**.** *Let $x \in X$ be an arbitrary point in the polytope X. Then the* minimal and maximal ε-characteristic simplices of x *are defined by*

$$\sigma^\varepsilon_{\min}(x) \quad:=\quad \{v \in \mathcal{X}_0 \;|\; t_v(x) > \varepsilon\} \qquad \text{and} \tag{12}$$

$$\sigma^\varepsilon_{\max}(x) \quad:=\quad \{v \in \mathcal{X}_0 \;|\; t_v(x) \geq \varepsilon\}\,, \tag{13}$$

respectively, and the set of ε-characteristic simplices *is defined as*

$$\mathcal{X}^\varepsilon(x) \;:=\; \{\sigma \in \mathcal{X} \;|\; t_v(x) \geq \varepsilon \;\text{ for all }\; v \in \sigma \quad\text{and}\quad t_v(x) \leq \varepsilon \;\text{ for all }\; v \notin \sigma\}\,. \tag{14}$$

We also set

$$\sigma^0(x) := \{\, v \in \mathcal{X}_0 \;|\; t_v(x) > 0 \,\}. \tag{15}$$

We note that $\sigma^0(x)$ is the smallest simplex $\sigma \in X$ which satisfies $x \in \sigma$, and it is also the unique simplex $\sigma \in \mathcal{X}$ such that $x \in \overset{\circ}{\sigma}$. The following result follows from [37, Lemma 4.5].

Lemma 4.5 (Upper Semi-Continuity of the Set of ε-Characteristic Simplices)**.** *Consider the set of ε-characteristic simplices introduced in Definition 4.4. Then for all $x \in X$ we have $\mathcal{X}^\varepsilon(x) \neq \emptyset$. Moreover, there exists a neighborhood U of the point x such that the inclusion $\mathcal{X}^\varepsilon(y) \subset \mathcal{X}^\varepsilon(x)$ is satisfied for all $y \in U$. This may be rephrased by saying that the mapping $x \mapsto \mathcal{X}^\varepsilon(x)$ is strongly upper semi-continuous, see* [5, Definition 3.3]. □

This result allows us to close the circle and reveal the connection between ε-characteristic simplices and the notion of ε-cells introduced in Definition 4.1. To see this, notice that the strong upper semicontinuity of $\mathcal{X}^\varepsilon(x)$ with respect to x implies that if at a given point $x \in X$ the set $\mathcal{X}^\varepsilon(x)$ consists of exactly one simplex σ, then we have to have $\mathcal{X}^\varepsilon(y) = \{\sigma\}$ for all points y in an open neighborhood of x. In other words, the subset of X which consists of points with exactly one ε-characteristic simplex is open. In fact, the following result based on [37] shows that its connected components are precisely the ε-cells $\langle\sigma\rangle_\varepsilon$.

Lemma 4.6 (Alternative Characterization of ε-Cells and their Closure)**.** *For every simplex $\sigma \in \mathcal{X}$ the ε-cell from Definition 4.1 can be characterized as*

$$\langle\sigma\rangle_\varepsilon \;=\; \{x \in X \;:\; \mathcal{X}^\varepsilon(x) = \{\sigma\}\} \,. \tag{16}$$

In addition, the following three statements are pairwise equivalent:

(a) *The simplex σ belongs to $\mathcal{X}^\varepsilon(x)$,*
(b) *the inclusions $\sigma^\varepsilon_{\min}(x) \subset \sigma \subset \sigma^\varepsilon_{\max}(x)$ hold,*
(c) *the point x belongs to $\operatorname{cl}\langle\sigma\rangle_\varepsilon$.*

In other words, one can characterize the closure of an ε-cell via ε-characteristic simplices. □

While the first part of the lemma follows easily from the definitions, the proof of the second part can be found in [37, Corollary 4.6].

4.2. **Flow Tiles and Strongly Admissible Semiflows.** The decomposition of the polytope X into the closures of ε-cells as in (11) is only a first step in the derivation of a cell decomposition which can be used in our setting. So far, this decomposition depends only on the underlying simplicial complex $\mathcal{X}$. Yet, with respect to the fixed combinatorial vector field $\mathcal{V}$ on $\mathcal{X}$, we will use a slightly coarser decomposition, which is defined as follows.

Definition 4.7 (Flow Tiling for a Combinatorial Vector Field)**.** *The* flow tiling associated with $\mathcal{V}$ *is defined as the collection $\mathcal{C}$ of compact subsets of X called* flow tiles, *which in turn are given by*

$$C_\omega := \operatorname{cl}\langle\omega^-\rangle_\varepsilon \cup \operatorname{cl}\langle\omega^+\rangle_\varepsilon \qquad \textit{for all} \qquad \omega \in \mathcal{V}, \tag{17}$$

where the ε-cells $\langle\omega^-\rangle_\varepsilon$, $\langle\omega^+\rangle_\varepsilon$ are defined in Definition 4.1.

Since either $\omega^- = \omega^+$ or $\omega^- \to \omega^+$ is an arrow, there are two types of flow tiles:

- For every critical cell $\sigma \in \mathcal{X}$ of $\mathcal{V}$ the associated flow tile is $C_{\{\sigma\}} = \operatorname{cl}\langle\sigma\rangle_\varepsilon$.
- For every arrow $\tau \to \sigma$ of $\mathcal{V}$ the associated flow tile is $C_{\{\tau,\sigma\}} = \operatorname{cl}\langle\tau\rangle_\varepsilon \cup \operatorname{cl}\langle\sigma\rangle_\varepsilon$.

We distinguish between these two types by calling them *critical flow tiles* and *arrow flow tiles*, respectively. Thus, the flow tiling is obtained from the cell decomposition in (11) by simply combining the closures of ε-cells of arrows. This is illustrated in the left image of Figure 2, where each flow tile is marked with a different color. Compare also with the right panel of Figure 4.

We are finally in a position to complete the first step outlined in the introduction. In the next definition, we introduce the concept of an admissible semiflow on the polytope X for a combinatorial vector field $\mathcal{V}$.

Definition 4.8 (Admissible and Strongly Admissible Semiflows)**.** *Consider the flow tiling $\mathcal{C}$ associated with $\mathcal{V}$ from Definition 4.7. A continuous semiflow $\varphi : \mathbb{R}_0^+ \times X \to X$ on the polytope X is called an* admissible semiflow for $\mathcal{V}$*, if for every $x \in X$ which is contained in at least two flow tiles from $\mathcal{C}$, and for every solution $\gamma : [t_-, t_+] \to X$ of the semiflow φ through x with $t_- \le 0 < t_+$ there exists an open neighborhood U of $t = 0$ in $[t_-, t_+]$ (in the subspace topology) such that*

$$\begin{aligned} \gamma(t) &\in \langle \sigma^\varepsilon_{\max}(x) \rangle_\varepsilon \quad \textit{for all} \quad t \in U \cap \mathbb{R}^-, \ \textit{and} \\ \gamma(t) &\in \langle \sigma^\varepsilon_{\min}(x) \rangle_\varepsilon \quad \textit{for all} \quad t \in U \cap \mathbb{R}^+, \end{aligned} \tag{18}$$

where $\mathbb{R}^\pm$ denotes the set of all strictly positive/negative real numbers. The semiflow is called a strongly admissible semiflow for $\mathcal{V}$ *if in addition every forward solution which originates in an arrow flow tile exits the tile in finite forward time, and every solution through a point in an arrow flow tile which exists for all negative times exits the flow tile in finite backward time.*

While the above definition of admissibility might be strange at first sight, we can easily illuminate it using Lemma 4.6. Since a point x lies on the boundary of at least two flow tiles only if it lies on the boundaries of at least two ε-cells, this lemma shows that there are at least two simplices in the set $\mathcal{X}^\varepsilon(x)$. Moreover, the simplices in this set are in one-to-one correspondence with the ε-cells $\langle \sigma \rangle_\varepsilon$ which have x on their boundary. Thus, the condition in (18) requires that the solution through x has to come from the ε-cell associated with the largest simplex σ_1 in $\mathcal{X}^\varepsilon(x)$ and has to move into the ε-cell for the smallest simplex σ_2 in $\mathcal{X}^\varepsilon(x)$. According to Lemma 4.6*(b)* the simplex σ_2 is a face of σ_1, and this means that an admissible semiflow, when crossing the boundary between tiles, always flows towards the boundary of a simplex. Notice also that in general solutions of the semiflow φ are allowed to merge in finite time, and therefore the condition (18) has to be satisfied for every solution γ which passes through the point x. In this sense, solutions through flow tile boundaries exhibit well-defined exit and entrance behavior.

For the combinatorial vector field in Figure 1 the corresponding flow tiles and flow directions along the boundaries between flow tiles are shown in Figure 2. Notice, in particular, that the notion of admissibility from Definition 4.8 does not prescribe any flow directions in the interior of flow tiles — not even on the boundary between the two ε-cells which comprise an arrow flow tile. Only under the assumption of strong admissibility do we impose restrictions on the semiflow in the interior of arrow flow tiles.

The main results of the next section shows that any admissible semiflow in the sense of Definition 4.8 exhibits the same isolated invariant sets as the combinatorial vector field $\mathcal{V}$, while every strongly admissible semiflow exhibits the same global dynamics in terms of Morse decompositions and Conley-Morse graphs.

If we take another look at the example from Figure 1 and the associated flow tiling in Figure 2, then one can easily see that every critical flow tile $C \in \mathcal{C}$ is an isolating block, and the associated Conley index is the one for an equilibrium whose Morse index is the dimension n of the underlying critical simplex. Thus, the Ważewski principle implies that *any admissible semiflow* φ has a nontrivial isolated invariant set in C which on the level of the Conley index acts like an index n equilibrium. Note that in order to verify the properties of an isolating block, we can easily use the admissibility conditions from Definition 4.8. In the next section we show that the flow tiles associated with $\mathcal{V}$ can be used in a straightforward way to construct isolating blocks for more complicated isolated invariant sets.

5. Isolated Invariant Sets and Conley-Morse Graphs

In this section we show that for every combinatorial vector field $\mathcal{V}$ on a simplicial complex $\mathcal{X}$ and any associated strongly admissible semiflow φ on $X = |\mathcal{X}|$, their dynamics is equivalent in the sense of Conley theory. For this, we first recall the combinatorial notions of isolated invariant sets and Morse decompositions in Section 5.1, based on our results in [7, 37]. This is followed in Section 5.2 by the explicit construction of isolating blocks for admissible semiflows φ on X from the combinatorial information, as well as as the verification in Section 5.3 that the associated homological Conley indices are isomorphic. Finally, in Section 5.4 we demonstrate that under the assumption of strong admissibility every Morse decomposition of $\mathcal{V}$ gives rise to a Morse decomposition for φ with isomorphic Conley-Morse graphs.

5.1. **Morse Decompositions for Combinatorial Vector Fields.** We begin by recalling the qualitative dynamical theory for combinatorial vector fields which has been developed in [7, 37]. In its original form, a combinatorial vector field $\mathcal{V}$ on a simplicial complex $\mathcal{X}$ does not create a dynamical system. However, based on the intuition that we laid out in the previous sections, one can easily associate with $\mathcal{V}$ a suitable multivalued discrete-time dynamical system on $\mathcal{X}$, which exhibits the following behavior.

- Critical cells allow for both fixed points and for flow towards the combinatorial boundary of the simplex.
- Arrow tails, i.e., simplices $\sigma \in \operatorname{Tail} \mathcal{V}$ lead to flow towards the simplex $\sigma^+ = \mathcal{V}(\sigma)$.
- Arrow heads, i.e., simplices $\sigma \in \operatorname{Head} \mathcal{V}$ always lead to flow towards the boundary of σ, but not towards the face $\sigma^- = \mathcal{V}^{-1}(\sigma)$.

This behavior can be formalized with the introduction of a multivalued map $\Pi_\mathcal{V} : \mathcal{X} \multimap \mathcal{X}$ defined by

$$\Pi_\mathcal{V}(\sigma) := \begin{cases} \operatorname{Cl} \sigma & \text{if} \quad \sigma \in \operatorname{Crit} \mathcal{V}, \\ \{\mathcal{V}(\sigma)\} & \text{if} \quad \sigma \in \operatorname{Tail} \mathcal{V}, \\ \operatorname{Bd} \sigma \setminus \{\mathcal{V}^{-1}(\sigma)\} & \text{if} \quad \sigma \in \operatorname{Head} \mathcal{V}. \end{cases} \tag{19}$$

Iteration of the multivalued map $\Pi_\mathcal{V}$ defines a discrete-time dynamical system on the simplicial complex $\mathcal{X}$ in the usual way. More precisely, a *solution* ϱ of the combinatorial vector field $\mathcal{V}$ is a partial map $\varrho : \mathbb{Z} \nrightarrow \mathcal{X}$, where $\operatorname{dom} \varrho$ is a $\mathbb{Z}$-interval, such that

$$\varrho_{k+1} \in \Pi_\mathcal{V}(\varrho_k) \qquad \text{for all} \qquad k, k+1 \in \operatorname{dom} \varrho.$$

As in the classical case, a *solution through* $\sigma \in \mathcal{X}$ is a solution such that $\varrho_0 = \sigma$ and a *full solution* is a solution satisfying $\operatorname{dom} \varrho = \mathbb{Z}$.

In the qualitative theory of dynamical systems, solutions themselves are not the primary target. Rather one concentrates on specific collections of solutions, which comprise invariant sets. Borrowing directly from the classical setting, we call a set $\mathcal{S} \subset \mathcal{X}$ an *invariant set* for the associated multivalued flow map $\Pi_\mathcal{V}$, if for each simplex $\sigma \in \mathcal{S}$ there exists a full solution $\varrho : \mathbb{Z} \to \mathcal{X}$ through σ which lies completely in the set $\mathcal{S}$. We would like to point out that in general, there are many solutions of $\Pi_\mathcal{V}$ which pass through a given simplex σ, and while some of them might be full solutions, not all of them have to be. For the notion of invariance, however, all that matters is the existence of (at least) one full solution through σ which stays in $\mathcal{S}$. For examples of invariant sets, we refer the reader to the discussion in [37].

One of the crucial insights of Conley [13] is the observation that general invariant sets are difficult to study. While in the classical dynamical systems case this is due to their sensitivity to perturbations, it was pointed out in [37] that even in the combinatorial setting invariance alone is too weak a concept. This leads to the following definition.

Definition 5.1 (Isolated Invariant Set)**.** *Let $\mathcal{S} \subset \mathcal{X}$ denote an invariant set for the multivalued map $\Pi_\mathcal{V}$ defined in (19). Define the* exit set *or* mouth *of $\mathcal{S}$ by*

$$\operatorname{Mo}\mathcal{S} := \operatorname{Cl}\mathcal{S} \setminus \mathcal{S}.$$

Then the invariant set $\mathcal{S}$ is called an isolated invariant set*, if the following two conditions are satisfied:*

(a) *The mouth of $\mathcal{S}$ is combinatorially closed in the simplicial complex $\mathcal{X}$, i.e., for every simplex $\sigma \in \operatorname{Mo}\mathcal{S}$ the mouth contains all faces of σ.*

(b) *There exists no solution $\varrho : [-1, 1] \cap \mathbb{Z} \to \mathcal{X}$ of $\Pi_\mathcal{V}$ such that $\varrho_{-1}, \varrho_1 \in \mathcal{S}$ and $\varrho_0 \in \operatorname{Mo}\mathcal{S}$.*

If $\mathcal{S}$ is an isolated invariant set, then the combinatorial closure $\operatorname{Cl}\mathcal{S}$ *is called an* isolating block *for $\mathcal{S}$.*

The above definition is inspired by the classical notion of isolating block as introduced in [13, 57], see also our discussion in Section 3.9 concerning isolating blocks. Considering $\operatorname{Cl}\mathcal{S}$ as the combinatorial counterpart of the isolating bloock for $\mathcal{S}$, we see that condition *(a)* directly corresponds to condition (7) and condition *(b)* is a combinatorial version of the exclusion of internal flow tangencies implied by condition (6) of the definition of an isolating block. One can easily see that there are combinatorial vector fields with invariant sets which are not isolated. Such examples can be found in [37], and they demonstrate that the two conditions in Definition 5.1 are in fact independent.

From a practical perspective, the above definition of isolated invariant set is not optimal. While the combinatorial closedness of the mouth of $\mathcal{S}$ can easily be verified in the simplicial complex $\mathcal{X}$, the verification of *(b)* necessitates the use of the multivalued map $\Pi_\mathcal{V}$. However, it was shown in [37] that this condition can be reformulated using the given combinatorial vector field $\mathcal{V}$, and this leads to the following result (see [37, Proposition 3.7]).

Lemma 5.2 (Characterization of Isolated Invariant Sets)**.** *Let $\mathcal{S} \subset \mathcal{X}$ denote an invariant set for the multivalued map $\Pi_\mathcal{V}$ defined in (19). Then $\mathcal{S}$ is an isolated invariant set if and only if the mouth $\operatorname{Mo}\mathcal{S}$ is combinatorially closed, and every arrow of $\mathcal{V}$ either lies completely in $\mathcal{S}$ or completely outside of $\mathcal{S}$.* □

The lemma is illustrated in Figure 5. While the left image shows a simplicial complex $\mathcal{X}$ which triangulates a hexagon, together with a combinatorial vector field $\mathcal{V}$, the right panel depicts a sample isolated invariant set for $\mathcal{V}$ in light blue. One can verify that its mouth is given by the simplices shown in dark blue, and that the assumptions of Lemma 5.2 are satisfied.

We would like to point out that in contrast to the classical case, an isolating block in the combinatorial setting does not determine the associated isolated invariant set. To see this,

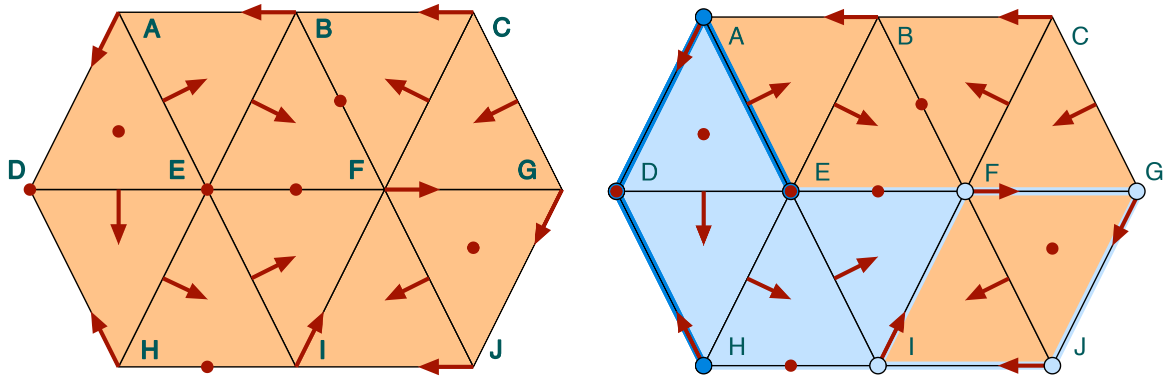

FIGURE 5. Sample combinatorial vector field with an isolated invariant set. The left figure shows a simplicial complex $\mathcal{X}$ which triangulates a hexagon, together with a combinatorial vector field. Critical cells are indicated by red dots, vectors of the vector field are shown as red arrows. The right image depicts a sample isolated invariant set for this combinatorial vector field. The simplices which belong to the isolated invariant set $\mathcal{S}$ are indicated in light blue, and are given by four vertices, nine edges, and four triangles. Its mouth $\operatorname{Mo}\mathcal{S}$ is shown in dark blue, and it consists of four vertices and three edges.

take another look at Figure 5. Both sets

$$\mathcal{S}_1 = \{EF\} \qquad \text{and} \qquad \mathcal{S}_2 = \{EF, E\}$$

are isolated invariant sets for $\mathcal{V}$ according to Lemma 5.2, and in both cases we obtain the same isolating block $\operatorname{Cl}\mathcal{S}_1 = \operatorname{Cl}\mathcal{S}_2 = \{EF, E, F\}$. Nevertheless, it is still possible to distinguish between the two isolated invariant sets, and this leads to the notion of Conley index.

Definition 5.3 (Conley Index and Poincaré Polynomial)**.** *Let $\mathcal{S} \subset \mathcal{X}$ denote an isolated invariant set for the multivalued map $\Pi_{\mathcal{V}}$ defined in (19). Then the* Conley index of $\mathcal{S}$ *is defined as the relative homology*

$$CH_*(\mathcal{S}) := H_*\left(\operatorname{Cl}\mathcal{S}, \operatorname{Mo}\mathcal{S}\right).$$

Moreover, the associated Poincaré polynomial of $\mathcal{S}$ *is given by*

$$p_{\mathcal{S}}(t) := \sum_{k=0}^{\infty} \beta_k(\mathcal{S}) t^k, \quad \textit{where} \quad \beta_k(\mathcal{S}) = \operatorname{rank} CH_k(\mathcal{S}).$$

Notice that in the above definition, both sets $\operatorname{Cl}\mathcal{S}$ and $\operatorname{Mo}\mathcal{S}$ are combinatorially closed, which in the latter case is due to the fact that $\mathcal{S}$ is an isolated invariant set. This implies that both sets are simplicial subcomplexes of $\mathcal{X}$, and the relative homology in the definition is therefore just standard simplicial homology.

Returning to our above example, it is now possible to distinguish between the isolated invariant sets $\mathcal{S}_1$ and $\mathcal{S}_2$. One can easily see that on the one hand we have $p_{\mathcal{S}_1}(t) = t$, and on the other hand one obtains $p_{\mathcal{S}_2}(t) = 0$. While the first Poincaré polynomial corresponds in the classical theory to an equilibrium of index one, the second one corresponds to the index of an empty set. This is in accordance with our intuition, since the second case mimics the

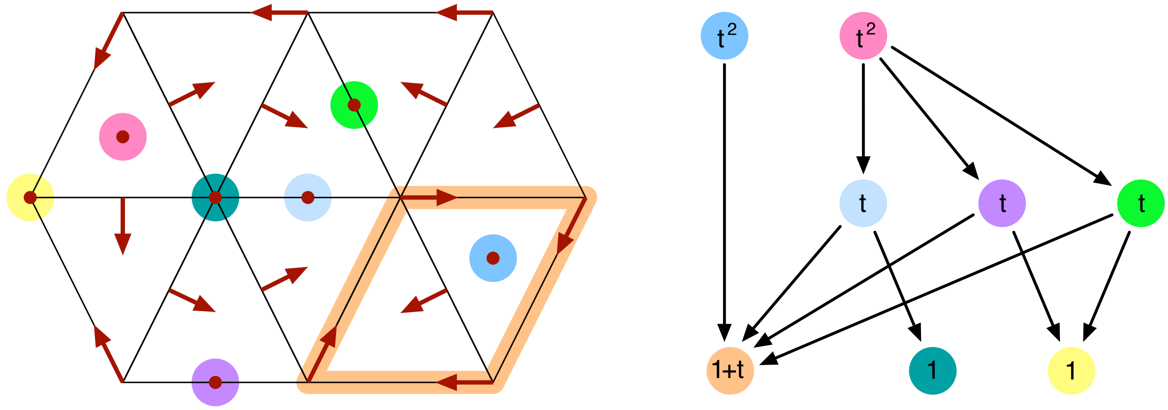

FIGURE 6. Morse decomposition for the example shown in the left panel of Figure 5. For this example, one can find eight minimal Morse sets, which are indicated in the left image in different colors. The right image shows the associated Morse graph.

case of an attractor-repeller pair with connecting solution, which can disappear through a saddle-node bifurcation upon perturbation.

To close this subsection, we now recall how the global dynamics of a combinatorial vector field can be decomposed. For this, we first need to introduce some notation. Consider a solution $\varrho : [a, \infty) \cap \mathbb{Z} \to \mathcal{X}$, which is defined on a $\mathbb{Z}$-interval which is unbounded to the right. We define the *ω-limit set of ϱ* as the set

$$\omega(\varrho) := \bigcap_{n \geq a} \{\varrho_k \;\mid\; k \geq n\}\,.$$

Similarly, for a solution $\varrho : (-\infty, a] \cap \mathbb{Z} \to \mathcal{X}$ of $\Pi_{\mathcal{V}}$ defined on a $\mathbb{Z}$-interval which is unbounded to the left, we define the *α-limit set of ϱ* via

$$\alpha(\varrho) := \bigcap_{n \leq a} \{\varrho_k \;\mid\; k \leq n\}\,.$$

Since the underlying simplicial complex is assumed to be finite, one can easily see that both the α- and the ω-limit set of a solution are nonempty, whenever they are defined.

After these preparations, we can finally present the notions of Morse decomposition and Conley-Morse graph which were introduced in [7].

Definition 5.4 (Morse Decomposition and Conley-Morse Graph)**.** *Let $\mathbb{P}$ be a poset. The family $\mathcal{M} = \{\mathcal{M}_p \,|\, p \in \mathbb{P}\}$ of disjoint, non-empty isolated invariant subsets of $\mathcal{X}$ is called a* Morse decomposition of $\mathcal{X}$ with respect to a combinatorial vector field $\mathcal{V}$, *if the following three statements hold:*

(a) *If ϱ is a solution such that its α-limit set is defined, then $\alpha(\varrho) \subset \mathcal{M}_p$ for some $p \in \mathbb{P}$. Analogously, this statement also has to hold for ω-limit sets.*
(b) *For every full solution ϱ of the map $\Pi_{\mathcal{V}}$ we have both $\alpha(\varrho) \subset \mathcal{M}_p$ and $\omega(\varrho) \subset \mathcal{M}_q$ for some indices $p \geq q$, that is ϱ is a* connection *from $\mathcal{M}_p$ to $\mathcal{M}_q$.*
(c) *If in (b) we have $p = q$, then the given full solution ϱ has to satisfy* $\operatorname{im} \varrho \subset \mathcal{M}_p$.

Finally, the associated Conley-Morse graph *is the partial order induced on $\mathcal{M}$ by the existence of connections, and it is represented as a directed graph labelled with the Conley indices of the isolated invariant sets in $\mathcal{M}$ in terms of their Poincaré polynomials.*

Given a combinatorial vector field $\mathcal{V}$ on a simplicial complex $\mathcal{X}$, the strongly connected components of the multivalued flow map $\Pi_{\mathcal{V}} : \mathcal{X} \multimap \mathcal{X}$ considered as a digraph form the unique finest Morse decomposition of $\mathcal{V}$ (see [16, Theorem 4.1]). As mentioned earlier, the Poincaré polynomials of the Morse sets can be determined via simplicial homology. For the example shown in the left panel of Figure 5 this procedure leads to the Morse decomposition depicted in Figure 6.

5.2. **Construction of Isolating Blocks for Admissible Semiflows.** After the preparations of the last section, we can now easily construct isolating blocks for an admissible semiflow φ based on the combinatorial information encoded by $\mathcal{V}$. Let $\mathcal{S} \subset \mathcal{X}$ be an isolated invariant set for the multivalued map $\Pi_{\mathcal{V}}$ defined in (19). Consider the set

$$B := \bigcup_{\sigma \in \mathcal{S}} \operatorname{cl} \langle \sigma \rangle_{\varepsilon}. \tag{20}$$

The following lemma is a special case of [37, Lemma 5.5].

Lemma 5.5. *If $x \in \operatorname{bd} B \subset X$ then*

$$\mathcal{X}^{\varepsilon}(x) \cap \mathcal{S} \neq \emptyset \qquad \text{and} \qquad \mathcal{X}^{\varepsilon}(x) \cap (\mathcal{X} \setminus \mathcal{S}) \neq \emptyset. \tag{21}$$

□

We then have the following central result.

Proposition 5.6 (Isolating Block Construction)**.** *Let $\mathcal{S} \subset \mathcal{X}$ be an isolated invariant set for the multivalued map $\Pi_{\mathcal{V}}$ defined in (19) and let $\varphi : \mathbb{R}_0^+ \times X \to X$ denote an arbitrary admissible semiflow for $\mathcal{V}$ in the sense of Definition 4.8. Then the set B given by* (20) *is an isolating block for φ.*

Proof. Since the set B is a finite union of compact sets it is clearly compact. Furthermore, according to Lemma 5.2 and Definition 4.7 the set B is in fact a union of flow tiles, since arrows of $\mathcal{V}$ either lie completely inside or completely outside of $\mathcal{S}$. Suppose now that $x \in \operatorname{bd} B$ lies on the boundary of at least two different flow tiles, which immediately implies that x has to be in the closure of at least two different ε-cells. In combination with Lemma 4.6 this in turn yields $\sigma^{\varepsilon}_{\min}(x) \neq \sigma^{\varepsilon}_{\max}(x)$. From the same lemma, it follows that the set $\mathcal{X}^{\varepsilon}(x)$ consists of all simplices $\sigma \in \mathcal{X}$ which satisfy the inclusions $\sigma^{\varepsilon}_{\min}(x) \subset \sigma \subset \sigma^{\varepsilon}_{\max}(x)$, and $\sigma \in \mathcal{X}^{\varepsilon}(x)$ if and only if $x \in \operatorname{cl} \langle \sigma \rangle_{\varepsilon}$.

Finally, let $\gamma : [t_-, t_+] \to X$ denote an arbitrary solution of the semiflow φ through the point x with $t_- \leq 0 < t_+$. Then, according to Definition 4.8, there exists an open neighborhood $U$of 0 in $[t_-, t_+]$ such that we have the inclusions

$$\gamma(t) \in \langle \sigma^{\varepsilon}_{\max}(x) \rangle_{\varepsilon} \quad \text{for} \quad t \in U \cap \mathbb{R}^- \qquad \text{and} \qquad \gamma(t) \in \langle \sigma^{\varepsilon}_{\min}(x) \rangle_{\varepsilon} \quad \text{for} \quad t \in U \cap \mathbb{R}^+, \tag{22}$$

i.e., the solution flows from the ε-cell associated with $\sigma^{\varepsilon}_{\max}(x)$ to the one for $\sigma^{\varepsilon}_{\min}(x)$.

Since either of the two characteristic simplices has to be an element of $\mathcal{S}$ or not, we now distinguish four cases.

Case			Implied Properties	
	$\sigma^\varepsilon_{\max}(x)$	$\sigma^\varepsilon_{\min}(x)$		
(i)	$\in \mathcal{S}$	$\in \mathcal{S}$	Case not possible	
(ii)	$\notin \mathcal{S}$	$\in \mathcal{S}$	$x \in B^i$	$\mathcal{X}^\varepsilon(x) \cap \operatorname{Mo}\mathcal{S} = \emptyset$
(iii)	$\in \mathcal{S}$	$\notin \mathcal{S}$	$x \in B^e$	$\sigma^\varepsilon_{\min}(x) \in \operatorname{Mo}\mathcal{S}$
(iv)	$\notin \mathcal{S}$	$\notin \mathcal{S}$	$x \in B^b$	$\sigma^\varepsilon_{\min}(x) \in \operatorname{Mo}\mathcal{S}$

Table 1. Classification of boundary points of the set B defined in (20) in terms of the location of the simplices $\sigma^\varepsilon_{\max}(x)$ and $\sigma^\varepsilon_{\min}(x)$ with respect to $\mathcal{S}$. For a point $x \in \operatorname{bd} B$ which lies on two different flow tiles, the collection $\mathcal{X}^\varepsilon(x) \subset \mathcal{X}$ defined in (14) has to contain at least two simplices, and it is given by all simplices σ which satisfy $\sigma^\varepsilon_{\min}(x) \subset \sigma \subset \sigma^\varepsilon_{\max}(x)$.

(i) $\sigma^\varepsilon_{\max}(x) \in \mathcal{S}$ and $\sigma^\varepsilon_{\min}(x) \in \mathcal{S}$: Due to (21), there has to be a simplex $\tau \in \mathcal{X}^\varepsilon(x)$ which satisfies $\tau \notin \mathcal{S}$. Together with Lemma 4.6 this implies that the simplex τ is a face of $\sigma^\varepsilon_{\max}(x)$, and therefore we have $\tau \in \operatorname{Mo}\mathcal{S}$. Since $\sigma^\varepsilon_{\min}(x) \subset \tau$ and the mouth is combinatorially closed, this in turn furnishes $\sigma^\varepsilon_{\min}(x) \in \operatorname{Mo}\mathcal{S}$, which is a contradiction. Thus, this case is impossible.

(ii) $\sigma^\varepsilon_{\max}(x) \notin \mathcal{S}$ and $\sigma^\varepsilon_{\min}(x) \in \mathcal{S}$: The definition of the set B and (22) show that in this case, the point x has to be a strict ingress point, see Section 4.2. Moreover, an argument similar to the one in the previous case implies that $\mathcal{X}^\varepsilon(x) \cap \operatorname{Mo}\mathcal{S} = \emptyset$.

(iii) $\sigma^\varepsilon_{\max}(x) \in \mathcal{S}$ and $\sigma^\varepsilon_{\min}(x) \notin \mathcal{S}$: One can easily see that these assumptions, together with (22) imply that x is a strict egress point. In addition, since $\sigma^\varepsilon_{\min}(x)$ is a face of $\sigma^\varepsilon_{\max}(x)$, it has to be in the mouth of $\mathcal{S}$.

(iv) $\sigma^\varepsilon_{\max}(x) \notin \mathcal{S}$ and $\sigma^\varepsilon_{\min}(x) \notin \mathcal{S}$: In this final case, the inclusions in (22) imply that x is a bounce-off point. Furthermore, the first inequality in (21) and Lemma 4.6 show that $\sigma^\varepsilon_{\min}(x)$ has to be in the mouth of $\mathcal{S}$.

The conclusions from these four cases are collected in Table 1.

After these preparations the proof of the proposition can readily be completed. We have to prove that B satisfies properties (6) and (7). The four cases above show that the semiflow φ does not form any internal tangencies with $\operatorname{bd} B$. Therefore, by the admissibility of φ we see that (6) holds. Moreover, directly from the definition of strict egress, strict ingress and bounce-off points we get $B^i \cap (B^e \cup B^b) = \emptyset$. It follows that $x \in B^i$ is possible only in case **(ii)**.

Now let $x \in \operatorname{bd} B \setminus B^-$ be arbitrary. According to $B^- = B^e \cup B^b$ and the last comment, this implies $x \in B^i$. By Lemma 4.5 there exists a neighborhood U of x in X such that the inclusion $\mathcal{X}^\varepsilon(y) \subset \mathcal{X}^\varepsilon(x)$ holds for all $y \in U$. Moreover, since $x \in B^i$ is possible only in case **(ii)**, we obtain $\mathcal{X}^\varepsilon(x) \cap \operatorname{Mo}\mathcal{S} = \emptyset$. Therefore, we have

$$\mathcal{X}^\varepsilon(y) \cap \operatorname{Mo}\mathcal{S} = \emptyset \quad \text{for all} \quad y \in U \cap \operatorname{bd} B. \tag{23}$$

Another glance at Table 1 then shows that every point $y \in U \cap \operatorname{bd} B$ has to be in B^i, since otherwise $\sigma^\varepsilon_{\min}(y) \in \operatorname{Mo}\mathcal{S}$, which contradicts (23). This finally implies that $\operatorname{bd} B \setminus B^- = B^i$ is open in $\operatorname{bd} B$, i.e., the set B^- is closed. Thus, also (7) is satisfied. Therefore, B is indeed an isolating block. $\square$

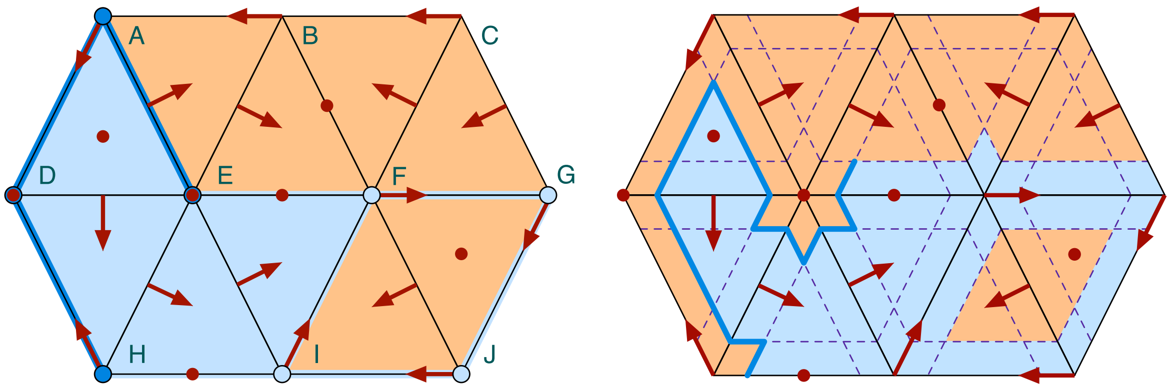

FIGURE 7. Sample construction of an isolating block. The left panel shows the isolated invariant set $\mathcal{S}$ from Figure 5 in light blue, while its mouth $\mathrm{Mo}\,\mathcal{S}$ is indicated in dark blue. The panel on the right depicts the associated isolating block B constructed in Proposition 5.6 in light blue, with its mouth B^- shown in medium dark blue.

The proposition is illustrated in Figure 7. The left panel reproduces the isolated invariant set $\mathcal{S}$ from Figure 5 in light blue, with its mouth $\mathrm{Mo}\,\mathcal{S}$ indicated in dark blue. The panel on the right depicts in light blue the associated isolating block B constructed in Proposition 5.6. The corresponding mouth B^- is shown in medium dark blue.

5.3. **Admissibility and the Equivalence of Conley Indices.** The results of the last section, particularly Proposition 5.6, show that for every isolated invariant set $\mathcal{S}$ in the combinatorial setting we can construct an isolating block B which isolates an isolated invariant set $S_\varphi = \mathrm{Inv}(B, \varphi)$ for every admissible semiflow φ. Therefore, it is natural to wonder whether we have

$$H_*(\mathrm{Cl}\,\mathcal{S}, \mathrm{Mo}\,\mathcal{S}) \cong H_*(B, B^-), \tag{24}$$

i.e., whether the Conley index of the combinatorial isolated invariant set $\mathcal{S}$ computed via simplicial homology is isomorphic to the singular homology of the index pair (B, B^-), which in turn is the Conley index of the isolated invariant set S_φ. In the remainder of this section, we will show that (24) holds.

In order to verify that the Conley indices in the combinatorial and the classical settings are isomorphic, we adapt arguments from our recent work [7]. For any subset $\mathcal{A} \subset \mathcal{X}$ of simplices and an $\varepsilon > 0$ we define the compact set

$$N_\varepsilon(\mathcal{A}) := \bigcup_{\sigma \in \mathcal{A}} \mathrm{cl}\,\langle\sigma\rangle_\varepsilon \subset X. \tag{25}$$

Now let $\mathcal{S} \subset \mathcal{X}$ denote an isolated invariant set for the combinatorial vector field $\mathcal{V}$. We use the topological pairs in X given by

$$P_1 := N_\varepsilon(\mathcal{S}) \quad \text{and} \quad P_2 := N_\varepsilon(\mathrm{Mo}\,\mathcal{S}) \cap \mathrm{bd}\, N_\varepsilon(\mathcal{S}), \tag{26}$$

as well as

$$Q_1 := N_\varepsilon(\mathrm{Cl}\,\mathcal{S}) \quad \text{and} \quad Q_2 := N_\varepsilon(\mathrm{Mo}\,\mathcal{S}). \tag{27}$$

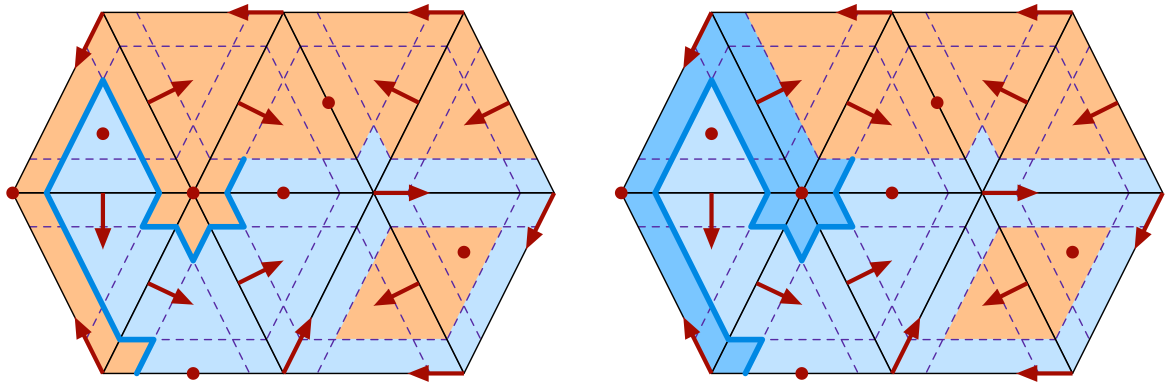

Figure 8. Auxiliary pairs for the Conley index equivalence proof. The left panel shows the pair (P_1, P_2) defined in (26) for the isolated invariant set $\mathcal{S}$ from Figure 5. While P_2 is indicated in dark blue, the set P_1 comprises all points colored in any shade of blue. In the right panel we illustrate the auxiliary pair (Q_1, Q_2) defined in (27). The set Q_2 consists of points in medium and dark blue, while Q_1 contains all points colored in any shade of blue. Notice that we clearly have $(P_1, P_2) \subset (Q_1, Q_2)$ as pairs.

They are illustrated in Figure 8. Using these two auxiliary pairs the Conley index equivalence will be established in several steps. We begin with a simple lemma.

Lemma 5.7. *Let $\mathcal{S} \subset \mathcal{X}$ denote an isolated invariant set for the multivalued map $\Pi_{\mathcal{V}}$ defined in (19), and consider the topological pair (B, B^-) given by (20) and (5) and the two topological pairs (P_1, P_2) and (Q_1, Q_2) as defined in (26) and (27), respectively. Then (B, B^-), (P_1, P_2), (Q_1, Q_2) are pairs of $\mathcal{D}^d_\varepsilon$-representable sets and we have*

$$H_*(B, B^-) = H_*(P_1, P_2) \cong H_*(Q_1, Q_2). \tag{28}$$

Proof. By Proposition 4.2, the sets $\langle\sigma\rangle_\varepsilon$ are $\mathcal{D}^d_\varepsilon$-representable. By Corollary 3.7(i), the closures $\operatorname{cl}\langle\sigma\rangle_\varepsilon$ are $\mathcal{D}^d_\varepsilon$-representable. Therefore, again by Corollary 3.7(i), the sets P_1, Q_1, and Q_2 are $\mathcal{D}^d_\varepsilon$-representable as unions of $\mathcal{D}^d_\varepsilon$-representable sets, and P_2 is $\mathcal{D}^d_\varepsilon$-representable as the intersection of a $\mathcal{D}^d_\varepsilon$-representable set with the boundary of another $\mathcal{D}^d_\varepsilon$-representable set. Moreover, we clearly have $P_1 = B$, and Proposition 5.6, in view of Table 1, shows that also $B^- = B^e \cup B^b = P_2$. Hence, both B and B^- are $\mathcal{D}^d_\varepsilon$-representable as well, and the first equality in (28) holds trivially.

Finally, the definitions of (P_1, P_2) and (Q_1, Q_2) in (26) and (27), respectively, immediately give $P_1 \subset Q_1$ and $P_2 \subset Q_2$, as well as both $P_1 \setminus P_2 = Q_1 \setminus Q_2$. The result now follows from an application of Theorem 3.8, i.e., the strong excision property of singular homology for representable pairs. $\square$

The following second step shows that the enlarged pair (Q_1, Q_2) has the same homology as the combinatorial isolated invariant set $\mathcal{S}$. It makes use of the Vietoris-Begle theorem.

Lemma 5.8. *Let $\mathcal{S} \subset \mathcal{X}$ denote an isolated invariant set for the multivalued map $\Pi_{\mathcal{V}}$ defined in (19), and consider the topological pair (Q_1, Q_2) defined in (27). Then we have*

$$H_*(Q_1, Q_2) \cong H_*(\operatorname{Cl}\mathcal{S}, \operatorname{Mo}\mathcal{S}).$$

Proof. In order to prove the lemma it suffices to verify the assumptions of Theorem 3.9. For this, we will construct a map $\psi_\varepsilon : (Q_1, Q_2) \to (|\operatorname{Cl}\mathcal{S}|, |\operatorname{Mo}\mathcal{S}|)$ which is a continuous surjection with $\psi_\varepsilon^{-1}(|\operatorname{Mo}\mathcal{S}|) = Q_2$, and which has contractible fibers. We note that, as we verified in Lemma 5.7, the sets Q_1 and Q_2 are $\mathcal{D}_\varepsilon^d$-representable. Also the sets $|\operatorname{Cl}\mathcal{S}|$ and $|\operatorname{Mo}\mathcal{S}|$ are $\mathcal{D}_\varepsilon^d$-representable as unions of ε-cells which are representable by Proposition 4.2. Therefore, by Corollary 3.7(ii) both pairs (Q_1, Q_2) and $(\operatorname{Cl}\mathcal{S}, \operatorname{Mo}\mathcal{S})$ are triangulable.

Thus, we only need to construct the map ψ_ε and verify its properties. We recall that $\varepsilon > 0$ is a fixed constant satisfying (10). The construction of the map ψ_ε closely follows a similar consideration in [7], and therefore we only present the essential steps. To begin with, define a function $\varphi_\varepsilon : [0,1] \to [0,1]$ via

$$\varphi_\varepsilon(t) \;=\; \begin{cases} 0 & \text{for} \quad 0 \le t \le \varepsilon, \\ \dfrac{t-\varepsilon}{1-\varepsilon} & \text{for} \quad \varepsilon \le t \le 1. \end{cases}$$

This continuous function maps the interval $[0,\varepsilon]$ to zero and the interval $[\varepsilon, 1]$ homeomorphically onto $[0,1]$. In addition, consider the map $\psi_\varepsilon : X \to X$ defined as

$$\psi_\varepsilon(x) \;=\; \sum_{u \in \mathcal{X}_0} \frac{\varphi_\varepsilon(t_u(x))}{\sum\limits_{v \in \mathcal{X}_0} \varphi_\varepsilon(t_v(x))}\, u,$$

where $\mathcal{X}_0$ denotes the set of all vertices of the simplicial complex $\mathcal{X}$. Notice that due to the assumed constraint (10) on ε the sum in the denominator of the above formula is strictly positive for all $x \in X$, and this readily implies that ψ_ε is well-defined and continuous. In addition, one can easily establish the following properties of ψ_ε:

(i) For every simplex $\sigma \in \mathcal{X}$ and any $x \in \operatorname{cl}\langle\sigma\rangle_\varepsilon$ we have $\psi_\varepsilon(x) \in \sigma$.

This statement follows from the fact that according to Lemma 4.3 one has $t_u(x) \le \varepsilon$ for all vertices $u \notin \sigma$, and therefore $\varphi_\varepsilon(t_u(x)) = 0$.

(ii) For every simplex $\sigma \in \mathcal{X}$ and any $y \in \mathring{\sigma}$ we have $\psi_\varepsilon^{-1}(y) \subset \operatorname{cl}\langle\sigma\rangle_\varepsilon$.

To see this, note that if $\psi_\varepsilon(x) = y$, then the definition of ψ_ε implies $\varphi_\varepsilon(t_u(x)) = 0$ for all vertices $u \notin \sigma$, and therefore $t_u(x) \le \varepsilon$ for all $u \notin \sigma$. On the other hand, for all vertices $u \in \sigma$ we have $\varphi_\varepsilon(t_u(x)) > 0$, i.e., one obtains $t_u(x) > \varepsilon$. The statement now follows from Lemma 4.3.

(iii) For every simplex $\sigma \in \mathcal{X}$, any $x \in \operatorname{cl}\langle\sigma\rangle_\varepsilon$, and every vertex $u \in \sigma$ we have

$$t_u(\psi_\varepsilon(x)) = \frac{t_u(x) - \varepsilon}{1 - \varepsilon(1 + \dim\sigma) - \sum\limits_{w \notin \sigma} t_w(x)},$$

while $t_u(\psi_\varepsilon(x)) = 0$ for all $u \notin \sigma$.

For this identity, one uses the fact that $t_v(x) \ge \varepsilon$ for all vertices $v \in \sigma$, and therefore the definitions of φ_ε and ψ_ε imply

$$t_v(\psi_\varepsilon(x)) = \frac{\varphi_\varepsilon(t_v(x))}{\sum\limits_{w \in \mathcal{X}_0} \varphi_\varepsilon(t_w(x))} = \frac{t_v(x) - \varepsilon}{(1-\varepsilon) \sum\limits_{w \in \mathcal{X}_0} \varphi_\varepsilon(t_w(x))} \quad \text{for all} \quad v \in \sigma, \tag{29}$$

which then immediately leads to

$$t_v(x) = \varepsilon + t_v(\psi_\varepsilon(x))(1-\varepsilon) \sum_{w\in\mathcal{X}_0} \varphi_\varepsilon(t_w(x)). \tag{30}$$

Since by (i) we have $\sum_{v\in\sigma} t_v(\psi_\varepsilon(x)) = 1$, summing both sides of (30) for all $v \in \sigma$ we obtain

$$\sum_{v\in\sigma} t_v(x) = \varepsilon(1+\dim\sigma) + (1-\varepsilon) \sum_{w\in\mathcal{X}_0} \varphi_\varepsilon(t_w(x)).$$

Substituting $\sum_{v\in\mathcal{X}_0} t_v(x) = 1 - \sum_{w\notin\sigma} t_w(x)$ and rearranging we obtain

$$(1-\varepsilon) \sum_{w\in\mathcal{X}_0} \varphi_\varepsilon(t_w(x)) = 1 - \varepsilon(1+\dim\sigma) - \sum_{w\notin\sigma} t_w(x).$$

Combined with (29) this implies the claimed formula.

(iv) For all $\sigma \in \mathcal{X}$ we have $\sigma = \psi_\varepsilon(\sigma \cap \operatorname{cl}\langle\sigma\rangle_\varepsilon)$.

In view of (i) one only has to show that the left-hand side is contained in the image on the right-hand side. Since the right-hand side is clearly compact and $\sigma = \operatorname{cl} \mathring{\sigma}$, it suffices to prove that $\mathring{\sigma}$ is contained in the right-hand side. Let $y \in \mathring{\sigma}$ and define $x := \sum_{v\in\sigma}(\varepsilon + t_v(y)(1-\varepsilon(1+\dim\sigma)))v \in \sigma$. Then (iii) implies that $\psi_\varepsilon(x) = y$, and the statement follows with (ii).

(v) For any simplex $\sigma \in \mathcal{X}$ and arbitrary $y \in \mathring{\sigma}$ we have $\psi_\varepsilon(x) = y$ if and only if

$$t_v(x) = \varepsilon + t_v(y)\left(1 - \varepsilon(1+\dim\sigma) - \sum_{w\notin\sigma} t_w(x)\right) \quad \text{for all} \quad v \in \sigma,$$

as well as $t_v(x) \le \varepsilon$ for all $v \notin \sigma$.

This statement follows immediately from (ii) and the explicit formula in (iii).

(vi) $\psi_\varepsilon(Q_1) = |\operatorname{Cl}\mathcal{S}|$ and $\psi_\varepsilon(Q_2) = |\operatorname{Mo}\mathcal{S}|$.

Using (i) we get $\psi_\varepsilon(Q_1) \subset |\operatorname{Cl}\mathcal{S}|$ and $\psi_\varepsilon(Q_2) \subset |\operatorname{Mo}\mathcal{S}|$. The opposite inclusions follow from (iv)

(vii) $\psi_\varepsilon^{-1}(|\operatorname{Mo}\mathcal{S}|) = Q_2$.

From (vi) we we get $Q_2 \subset \psi_\varepsilon^{-1}(\psi_\varepsilon(Q_2)) = \psi_\varepsilon^{-1}(|\operatorname{Mo}\mathcal{S}|)$. To see the opposite inclusion take an $x \in \psi_\varepsilon^{-1}(|\operatorname{Mo}\mathcal{S}|)$. Let $y := \psi_\varepsilon(x)$ and let $\sigma \in \operatorname{Mo}\mathcal{S}$ be such that $y \in \mathring{\sigma}$. Then, by (ii), $x \in \psi_\varepsilon^{-1}(y) \subset \operatorname{cl}\langle\sigma\rangle_\varepsilon \subset Q_2$.

Property (vi) implies that the restriction of $\psi_\varepsilon : X \to X$ to Q_1 is a well-defined, continuous map of pairs $\psi_\varepsilon|_{Q_1} : (Q_1, Q_2) \to (|\operatorname{Cl}\mathcal{S}|, |\operatorname{Mo}\mathcal{S}|)$. Thus, in view of (vii), in order to apply the Vietoris-Begle theorem to prove the lemma, we only have to show that ψ_ε has contractible fibers.

Thus, let $y \in X$ and let $x \in \psi_\varepsilon^{-1}(y)$. Set $\sigma := \sigma^0(y)$, where σ^0 is given by (15). Then $y \in \mathring{\sigma}$. For $\theta \in [0,1]$ and $v \in \mathcal{X}_0$ set

$$s_{x,\theta,v} := \begin{cases} \varepsilon + t_v(y)\left(1 - \varepsilon(1+\dim\sigma) - \sum_{w\notin\sigma} \theta t_w(x)\right) & \text{for } v \in \sigma, \\ \theta t_v(x) & \text{for } v \notin \sigma. \end{cases} \tag{31}$$

It follows from (10) that $s_{x,\theta,v} \geq 0$. Moreover, one easily verifies that $\sum_{v\in\mathcal{X}_0} s_{x,\theta,v} = 1$. Hence, $s_{x,\theta,v} \in [0,1]$ and we have a well-defined point $z_{x,\theta} := \sum_{v\in\mathcal{X}_0} s_{x,\theta,v} v \in X$, because clearly $t_v(z_{x,\theta}) > 0$ implies $t_v(x) > 0$, that is, $\sigma^0(z_{x,\theta}) \subset \sigma^0(x) \subset X$. Moreover, we easily get from (v) that $z_{x,\theta} \in \psi_\varepsilon^{-1}(y)$. Thus, we have a well-defined homotopy

$$h_y : [0,1] \times \psi_\varepsilon^{-1}(y) \to \psi_\varepsilon^{-1}(y).$$

Again using (v) one can easily see that $h_y(1,x) = x$ for all $x \in \psi_\varepsilon^{-1}(y)$. Finally, for every x in the fiber $\psi_\varepsilon^{-1}(y)$ formula (31) shows that the point $h_y(0,x)$ is independent of x, i.e., the map $h_y(0,\cdot)$ is constant. This proves the contractibility of $\psi_\varepsilon^{-1}(y)$ and the result follows. $\square$

After these preparations, we collect the results of this and the last section in the following theorem, which is valid under the weak notion of admissible flow.

Theorem 5.9 (Isolated Invariant Sets and Conley Index Equivalence)**.** *Let $\mathcal{S} \subset \mathcal{X}$ be an isolated invariant set for the multivalued map $\Pi_\mathcal{V}$ defined in (19) and let $\varphi : \mathbb{R}_0^+ \times X \to X$ denote an arbitrary admissible semiflow for $\mathcal{V}$ in the sense of Definition 4.8. Then the set*

$$B := N_\varepsilon(\mathcal{S}) = \bigcup_{\sigma\in\mathcal{S}} \operatorname{cl}\langle\sigma\rangle_\varepsilon$$

is an isolating block for φ and we have

$$H_*(\operatorname{Cl}\mathcal{S}, \operatorname{Mo}\mathcal{S}) \cong H_*(B, B^-).$$

In other words, every combinatorial isolated invariant set gives rise to an isolated invariant set for φ with the same Conley index in the classical setting.

Proof. The result follows immediately from Proposition 5.6 and Lemmas 5.7 and 5.8. $\square$

5.4. **Equivalence of Morse Decompositions and Conley-Morse Graphs.** In this final part of Section 5 we demonstrate that under the assumption of strong admissibility, any semiflow φ on X exhibits the same global dynamics as the underlying combinatorial vector field. More precisely, we have the following result.

Theorem 5.10 (Morse Decomposition Equivalence)**.** *Suppose we are given a Morse decomposition $\mathcal{M} = \{\mathcal{M}_p \,|\, p \in \mathbb{P}\}$ in the sense of Definition 5.4. Let $\varphi : \mathbb{R}_0^+ \times X \to X$ denote an arbitrary strongly admissible semiflow for $\mathcal{V}$ in the sense of Definition 4.8 and let*

$$M_p := \operatorname{Inv}\left(\bigcup_{\sigma\in\mathcal{M}_p} \operatorname{cl}\langle\sigma\rangle_\varepsilon\right) \qquad \textit{for all} \qquad p \in \mathbb{P}.$$

Then the collection $M = \{M_p \,|\, p \in \mathbb{P}\}$ is a Morse decomposition for the semiflow φ. Moreover, its Conley-Morse graph is isomorphic to the Conley-Morse graph of $\mathcal{M}$.

Proof. In view of (25) we have $M_p = \operatorname{Inv}(N_\varepsilon(\mathcal{M}_p))$ for all $p \in \mathbb{P}$. According to Theorem 5.9 the sets $N_\varepsilon(\mathcal{M}_p)$ are isolating blocks for φ. Since the sets $\mathcal{M}_p$ are pairwise disjoint, the sets $N_\varepsilon(\mathcal{M}_p)$ can only intersect along their boundaries. This immediately implies that the sets M_p are disjoint isolated invariant sets for φ. Moreover, Theorem 5.9 also shows that the Poincaré polynomials of M_p and $\mathcal{M}_p$ agree.

It remains to show that the α- and ω-limit sets of any solution for φ (as long as they are defined) are contained in Morse sets with the correct order relationship, and that if both of these limit sets are contained in the same Morse set, then the whole solution is contained in the Morse set. In the following, we only consider the case of full solutions for φ, as the case of forward solutions can be treated completely analogously.

Assume therefore that $\gamma : \mathbb{R} \to X$ denotes an arbitrary full solution of φ. Due to the compactness of X, both its α- and its ω-limit sets exist, are nonempty, and invariant. Let $\mathcal{C}$ denote the set of flow tiles associated with the combinatorial vector field $\mathcal{V}$, as defined in Definition 4.7. Then we can define a multivalued map $\eta : \mathbb{R} \multimap \mathcal{C}$ via

$$\eta(t) := \{C \in \mathcal{C} \mid \gamma(t) \in C\} \qquad \text{for all} \qquad t \in \mathbb{R}.$$

Then the following hold:

- Due to our definition of admissibility, if $\eta(t_0)$ contains more than one flow tile, then there exists a $\delta > 0$ such that $\gamma(t)$ is single-valued for all $t \in (t_0 - \delta, t_0 + \delta) \setminus \{t_0\}$.
- According to the continuity of φ and the characterization of the boundaries of ε-cells given in Lemma 4.3, if $\eta(t_0)$ contains exactly one flow tile, then there exists a $\delta > 0$ such that $\gamma(t)$ is single-valued for all $t \in (t_0 - \delta, t_0 + \delta)$.
- Combined, these two facts show that the times at which $\eta(t)$ contains more than one flow tile do not have any accumulation points. Thus, there exists a $\mathbb{Z}$-interval $I \subset \mathbb{Z}$ and a strictly increasing sequence $(t_k)_{k \in I} \subset \mathbb{R}$ without accumulation points such that η is multivalued on $R = \{t_k \mid k \in I\}$ and single-valued on $\mathbb{R} \setminus R$.

Based on the above observations we can now choose a $\mathbb{Z}$-interval $J \subset \mathbb{Z}$ whose cardinality equals the number of connected components of $\mathbb{R} \setminus R$, which is bounded from below if $\mathbb{R}^- \setminus R$ has a component of infinite length, and bounded from above if $\mathbb{R}^+ \setminus R$ has a component of infinite length. Furthermore, for every $k \in J$ we can choose precisely one point t_k^* in each of the connected components of $\mathbb{R} \setminus R$ such that $t_k^* < t_j^*$ whenever $k < j$. In other words, as k increases through J the points t_k^* hit every connected component of $\mathbb{R} \setminus R$ precisely once, with respect to the standard linear increasing order. Then $\eta(t_k^*)$ contains exactly one flow tile associated with a simplex $\sigma_k \in \mathcal{X}$. Without loss of generality we may assume that $\sigma_k = \sigma_k^-$. Now define a function $\Gamma : J \to \mathcal{X}$ via $\Gamma(k) := \sigma_k = \sigma_k^-$. Thus, as k ranges through J in increasing order, the sequence $\Gamma(k)$ visits the lowest-dimensional simplices associated with the flow tiles which are traversed by γ, in the correct order, and with every arrow flow tile being represented by the arrow tail simplex. In particular, we have $\Gamma(k) \neq \Gamma(k+1)$ for all arguments $k, k+1 \in J$.

If $k, k+1 \in J$, then the solution γ exits $\operatorname{cl} \langle \sigma_k \rangle_\varepsilon$ at a time $\bar{t} \in (t_k^*, t_{k+1}^*)$. Then $\gamma(\bar{t}) \in \operatorname{cl} \langle \sigma_k \rangle_\varepsilon$ and from Lemma 4.6 we get $\sigma_{\min}^\varepsilon(\gamma(\bar{t})) \subset \sigma_k$. By (18) we know that γ enters $\langle \sigma_{\min}^\varepsilon(\gamma(\bar{t})) \rangle_\varepsilon$ when crossing $\bar{t}$. Therefore, $\sigma_{k+1} = \sigma_{\min}^\varepsilon(\gamma(\bar{t}))^-$ and

$$\Gamma(k+1) = \sigma_{k+1} = \sigma_{\min}^\varepsilon(\gamma(\bar{t}))^- \subset \sigma_{\min}^\varepsilon(\gamma(\bar{t})) \subset \sigma_k \subset \sigma_k^+ = \Gamma(k)^+.$$

Thus, we proved that for all $k, k+1 \in J$ the simplex $\Gamma(k+1)$ is a face of $\Gamma(k)^+$. Moreover, if $\eta(t_k^*)$ contains an arrow flow tile, then one necessarily has $\Gamma(k+1) \neq \Gamma(k)^-$. This immediately implies that the arrowhead extension of Γ as defined in [37, Definition 5.2] is a solution of the combinatorial vector field $\mathcal{V}$. In addition, the following two implications hold:

$$\begin{aligned} k^+ := \sup J < \infty \quad &\Rightarrow \quad \Gamma(k^+) \text{ is a critical simplex}, \\ k^- := \inf J > -\infty \quad &\Rightarrow \quad \Gamma(k^-) \text{ is a critical simplex}. \end{aligned}$$

We only verify the first implication, as the second one can be established analogously. If the implication were false, then one would have $\gamma(t) \in C$ for all $t \geq t_{k^+}^*$ for some arrow flow tile $C \in \mathcal{C}$. This, however, contradicts our assumption of strong admissibility of φ (see Definition 4.8).

Based on this discussion, the arrowhead extension of the mapping Γ can be extended to a full solution $\Gamma_{\text{full}} : \mathbb{Z} \to \mathcal{X}$ of $\mathcal{V}$. In the cases $k^+ < \infty$ or $k^- > -\infty$ one only has to pad the respective end with infinite repetitions of the critical simplex $\Gamma(k^+)$ or $\Gamma(k^-)$, respectively. Moreover, the following holds.

- Due to the definition of Morse decomposition for $\mathcal{V}$ there exist Morse sets $\mathcal{M}_p$ and $\mathcal{M}_q$ with $p \geq q$ such that $\alpha(\Gamma_{\text{full}}) \subset \mathcal{M}_p$ and $\omega(\Gamma_{\text{full}}) \subset \mathcal{M}_q$. In addition, if $p = q$ then one has $\Gamma_{\text{full}}(\mathbb{Z}) \subset \mathcal{M}_p$.
- According to our construction, the classical solution γ traverses the flow tiles associated with $\Gamma_{\text{full}}(\mathbb{Z})$ as t ranges through $\mathbb{R}$. This implies $\alpha(\gamma) \subset N_\varepsilon(\mathcal{M}_p)$ and $\omega(\gamma) \subset N_\varepsilon(\mathcal{M}_q)$. Since both limit sets are invariant sets, we finally obtain $\alpha(\gamma) \subset M_p$ and $\omega(\gamma) \subset M_q$. Moreover, if $p = q$ then $\gamma(\mathbb{R}) \subset N_\varepsilon(\Gamma_{\text{full}}(\mathbb{Z})) \subset N_\varepsilon(\mathcal{M}_p)$. As a full solution, we therefore have $\gamma(\mathbb{R}) \subset \operatorname{Inv}(N_\varepsilon(\mathcal{M}_p)) = M_p$.

This completes the proof of the theorem. □

The above theorem is the precise version of Theorem 2.2 from the introduction. Notice that strong admissibility is essential, since without it we would not have been able to conclude that in the case $k^+ < \infty$ (respectively $k^- > -\infty$) the simplex $\Gamma(k^+)$ (respectively $\Gamma(k^-)$) is critical. This in turn was necessary for the construction of the full combinatorial solution Γ_{full}.

6. Existence of Strongly Admissible Semiflows

In this section we show that for any combinatorial vector field $\mathcal{V}$ on a simplicial complex $\mathcal{X}$ one can explicitly construct a strongly admissible semiflow φ on the polytope $X = |\mathcal{X}|$. This construction is based on the three design principles outlined in the introduction and is divided into four parts. In Section 6.1 we present the precise definition of the vector field which generates the semiflow. This definition relies on a family of vector fields f^ω which are indexed by the simplices $\omega \in \mathcal{X}$, each of which describes the semiflow in its associated flow tile. As this vector field definition is somewhat involved, we spend the remainder of this section describing the main features of the semiflow and its geometry on different flow tiles. The next three sections are devoted to showing that the fields f^ω do indeed generate a continuous, strongly admissible semiflow on X. In Section 6.2 we establish that each vector field f^ω generates a continuous semiflow, and discuss some of its elementary properties. This is followed in Section 6.3 with a detailed study of the induced dynamics on a flow file, in particular the behavior near the boundary of the flow tile. In this section we also derive crucial results towards strong admissibility. Finally, Section 6.4 combines the previous results to generate a continuous semiflow on X via gluing the semiflows generated by the vector fields f^ω.

6.1. Vector Field Definition and Basic Geometry.

In view of the three design principles which are described in the introduction, our goal is the construction of a semiflow on the underlying polytope X of an abstract simplicial complex $\mathcal{X}$. On the one hand, we would like this semiflow to be generated by a differential equation, while on the other hand the resulting semiflow generally will have to have velocity jumps. In addition, our definition has to be flexible enough to accommodate different tilings based on different combinatorial vector fields $\mathcal{V}$ on $\mathcal{X}$.

With these considerations in mind, we have adopted the following framework for the construction of a strongly admissible semiflow. First of all, as in the preceding sections, it is convenient to assume that the geometric realization of the abstract simplicial complex $\mathcal{X}$ is the standard geometric realization (see Section 3.5). Recall that the standard geometric realization of $\mathcal{X}$ is a subcomplex of the standard d-simplex in $\mathbb{R}^d$ where $d := \#\mathcal{X}_0$ denotes the number of vertices in $\mathcal{X}$. It is based on an identification of vertices in $\mathcal{X}_0$ with versors of $\mathbb{R}^d$ which lets us write the coordinates of a vector $x \in \mathbb{R}^d$ in the form x_v where $v \in \mathcal{X}_0$. In the standard geometric realization of $\mathcal{X}$ a simplex $\sigma \in \mathcal{X}$ is represented as a geometric simplex consisting of all points $x \in \mathbb{R}^d$ with $\sum_{u\in\sigma} x_u = 1$, as well as $x_u \geq 0$ for all $u \in \sigma$ and $x_u = 0$ for all $u \notin \sigma$. Notice that the notation for the components of x is an extension of the barycentric coordinate notation introduced earlier.

The standard geometric realization has the advantage that we can extend the range of every barycentric coordinate associated with the simplicial complex to take values in the reals, and this in turn will allow us to define a vector field on all of $\mathbb{R}^d$ in such a way that modifications of standard results give the existence of the semiflow, and that this semiflow leaves the underlying polytope of the simplicial complex invariant. We would like to point out, however, that once we have constructed a strongly admissible semiflow on the underlying polytope X of the standard geometric realization, one can easily map it onto underlying polytopes of other geometric realizations of the given abstract simplicial complex.

Having settled on the geometric realization, we now turn our attention to the underlying principles for the definition of the *vector field*:

- Our vector field will be defined piece-wise. In fact, for every simplex $\omega \in \mathcal{X}$ there will be a bounded and measurable vector field $f^\omega : \mathbb{R}^d \to \mathbb{R}^d$ which induces a semiflow through the ordinary differential equation $\dot{x} = f^\omega(x)$, interpreted in the Carathéodory sense.
- The vector fields f^ω only depend on the underlying flow tiles associated with a combinatorial vector field $\mathcal{V}$ on $\mathcal{X}$, i.e., we have $f^\omega = f^{\omega^+} = f^{\omega^-}$ for all $\omega \in \mathcal{X}$, where again we use the notation introduced in (8). In fact, we are only interested in the behavior of f^ω on the flow tile $C_\omega = \operatorname{cl}\langle\omega^-\rangle_\varepsilon \cup \operatorname{cl}\langle\omega^+\rangle_\varepsilon$ defined in (17).
- The final strongly admissible semiflow is defined in the spirit of a Filippov system on X through the vector field family $\{f^\omega\}_{\omega\in\mathcal{X}}$ over the flow tiles in $\mathcal{C}$, and its solutions cross the boundaries of the flow tiles transversally. Unfortunately, however, standard results on Filippov systems do not apply in our situation, and we have to construct the semiflow differently.

We recall that as in the earlier sections, $\varepsilon > 0$ denotes a small positive parameter satisfying (10). So far it was used in the construction of the ε-cells in Section 4.1 and flow tiles in Section 4.2. It will also be used in this section to construct some auxiliary functions needed to present the precise definition of the vector fields f^ω. Starting with Proposition 6.4, apart from assumption (10) we will also require that ε satisfies

$$0 < \varepsilon < \frac{1}{6d} \tag{32}$$

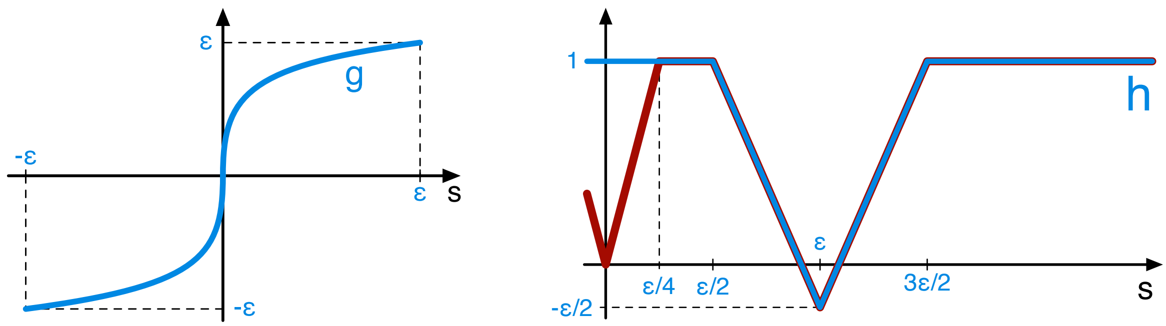

FIGURE 9. Two auxiliary functions for the definition of the vector fields f^ω. The left and right panels depict in blue the functions g and h, respectively, which are defined in (33). The red function in the right panel shows the product $\eta^\omega(s,x)h(s)$ if one has $\sum_{u\notin\omega^+} x_u^2 > 0$, which occurs in the second part of the definition of f^ω in (36).

where $d = \#\mathcal{X}_0$. We begin by defining two auxiliary scalar functions $g, h : \mathbb{R} \to \mathbb{R}$ via

$$g(s) := \sqrt[3]{\varepsilon^2 s} \qquad \text{and} \qquad h(s) := \begin{cases} 1 & \text{if} \quad |s-\varepsilon| \geq \dfrac{\varepsilon}{2}, \\ \left(1+\dfrac{\varepsilon}{2}\right)\dfrac{2}{\varepsilon}\,|s-\varepsilon| - \dfrac{\varepsilon}{2} & \text{if} \quad |s-\varepsilon| \leq \dfrac{\varepsilon}{2}. \end{cases} \tag{33}$$

These two functions are shown in blue in the left and right panels of Figure 9, respectively. Notice that the function g is continuous, but not differentiable at $s = 0$, while the function h is Lipschitz continuous on $\mathbb{R}$. In fact, the function h is of constant value 1 everywhere except in a small neighborhood around $s = \varepsilon$, where it drops to the slightly negative value $-\varepsilon/2$.

In contrast to g and h, which only depend on the small parameter ε, the next two auxiliary functions also depend on the underlying simplex $\omega \in \mathcal{X}$. The function $\theta^\omega : \mathbb{R}^d \to \mathbb{R}$ is given by the definition

$$\theta^\omega(x) := \min\left(\{\varepsilon\} \cup \left\{x_u - \varepsilon \;\mid\; u \in \omega^-\right\}\right). \tag{34}$$

Notice that for any point x in the underlying polytope X of $\mathcal{X}$, we have $\theta^\omega(x) \in [-\varepsilon, \varepsilon]$, and the identity $\theta^\omega(x) = \varepsilon$ is satisfied if and only if $x_u \geq 2\varepsilon$ for all $u \in \omega^-$. The final auxiliary function $\eta^\omega : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is defined as

$$\eta^\omega(s,x) := \begin{cases} 1 & \text{if} \quad \displaystyle\sum_{u\notin\omega^+} x_u^2 = 0 \quad \text{or} \quad |s| > \frac{\varepsilon}{4}, \\ \dfrac{4|s|}{\varepsilon} & \text{if} \quad \displaystyle\sum_{u\notin\omega^+} x_u^2 > 0 \quad \text{and} \quad |s| \leq \frac{\varepsilon}{4}. \end{cases} \tag{35}$$

For the case $\sum_{u\notin\omega^+} x_u^2 > 0$ the product $\eta^\omega(s,x)h(s)$ is shown in red in the right panel of Figure 9.

After these preparations, for every $\omega \in \mathcal{X}$ the vector field $f^\omega : \mathbb{R}^d \to \mathbb{R}^d$ is defined componentwise in the form

$$f_v^\omega(x) := \begin{cases} -g(x_v) & \text{for} \quad v \notin \omega^+, \\ \eta^\omega(x_v, x)\left(h(x_v) + \theta^\omega(x) - \displaystyle\sum_{u \notin \omega^+} x_u\right) & \text{for} \quad v \in \omega^+ \setminus \omega^-, \\ x_v - \dfrac{1}{\#\omega^-}\left(\displaystyle\sum_{u \in \omega^-} x_u + \sum_{u \notin \omega^-} f_u^\omega(x)\right) & \text{for} \quad v \in \omega^-. \end{cases} \tag{36}$$

At first glance, the vector field definition given in formulas (33) through (36) is clearly overwhelming. Thus, before we establish that the so-defined vector fields do indeed generate a strongly admissible semiflow φ on the underlying polytope $X \subset \mathbb{R}^d$ of $\mathcal{X}$, we pause for a brief description of the main features of the induced semiflow on the two types of flow tiles.

(I) The induced semiflow on critical flow tiles. To begin with, suppose that $\omega \in \mathcal{X}$ is a critical cell for the combinatorial vector field $\mathcal{V}$, and let $C_\omega = \operatorname{cl}\langle\omega\rangle_\varepsilon \subset X$ denote the associated flow tile. Then the induced semiflow φ on C_ω is given by the solution of the ordinary differential equation

$$\begin{aligned} \dot{x}_v &= x_v - \frac{1}{1+\dim\omega}\left(\sum_{u\in\omega} x_u - \sum_{u\notin\omega} g(x_u)\right) && \text{for} \quad v \in \omega, \\ \dot{x}_v &= -g(x_v) && \text{for} \quad v \notin \omega. \end{aligned}$$

First consider the intersection $C_\omega \cap \omega$. Then the semiflow further reduces to the linear differential equation

$$\dot{x}_v = x_v - \frac{1}{1+\dim\omega} \qquad \text{for all} \qquad v \in \omega \quad \text{and} \quad x \in C_\omega \cap \omega.$$

This differential equation has a unique equilibrium at the barycenter with barycentric coordinates $x_v := 1/(1+\dim\omega)$ for all $v \in \omega$. This equilibrium is unstable with index $\dim\omega$, and the induced flow is towards the boundary $\operatorname{bd} C_\omega \cap \omega$. See also the illustration in Figure 10.

Consider now solutions which originate at points $x \in C_\omega \setminus \omega$. In this situation, there are vertices $v \notin \omega$ with $x_v \neq 0$, and the behavior of these coordinates is determined by the completely decoupled scalar ordinary differential equations

$$\dot{x}_v = -\sqrt[3]{\varepsilon^2 x_v} \qquad \text{for all} \qquad v \notin \omega \quad \text{and} \quad x \in C_\omega \setminus \omega.$$

One can easily see that these coordinates decay towards zero — and they will in fact reach zero in finite forward time and stay there from then on. Thus, solutions originating in $C_\omega \setminus \omega$ are never constant, and they can enter $C_\omega \cap \omega$ in finite forward time, unless of course they exit the flow tile before that happens. This implies that the semiflow on $C_\omega \setminus \omega$ is attracted towards $C_\omega \cap \omega$ and roughly follows the flow behavior on $C_\omega \cap \omega$, leading to the qualitative semiflow behavior shown in Figure 10. Notice that every critical flow tile contains exactly one equilibrium, whose index is given by $\dim\omega$. We would like to explicitly point out, however, that solutions of the semiflow φ can reach lower-dimensional faces of a simplex in finite forward time.

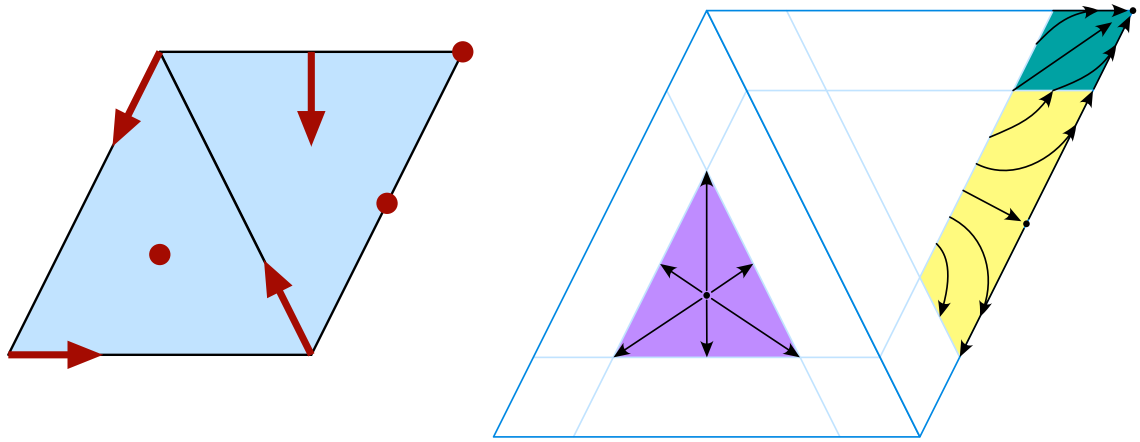

FIGURE 10. Semiflow induced by (36) on critical flow tiles. For the combinatorial vector field $\mathcal{V}$ shown in the left panel, the panel on the right depicts the flow tiles for the three critical cells, together with the strongly admissible semiflow induced by (36) on each of these tiles. Note that in all three cases the flow moves towards the boundary on the intersection of the flow tile and the associated simplex. Solutions which merge in finite forward time can be observed in the tiles associated with the critical cells of Morse index one and zero.

(II) The induced semiflow on arrow flow tiles. Now suppose that the simplex $\omega \in \mathcal{X}$ is part of an arrow for the combinatorial vector field $\mathcal{V}$, i.e., we have $\omega^+ \neq \omega^-$, and the associated flow tile is given by $C_\omega = \mathrm{cl}\,\langle\omega^-\rangle_\varepsilon \cup \mathrm{cl}\,\langle\omega^+\rangle_\varepsilon \subset X$. Then the induced semiflow φ on C_ω is given by the solution of the ordinary differential equation

$$\begin{aligned}
\dot{x}_{v^+} &= \eta^\omega(x_{v^+}, x)\left(h(x_{v^+}) + \theta^\omega(x) - \sum_{u \notin \omega^+} x_u\right) && \text{for} \quad \{v^+\} = \omega^+ \setminus \omega^-, \\
\dot{x}_v &= x_v - \frac{1}{\#\omega^-}\left(\sum_{u \in \omega^-} x_u + \sum_{u \notin \omega^-} f_u^\omega(x)\right) && \text{for} \quad v \in \omega^-, \\
\dot{x}_v &= -g(x_v) && \text{for} \quad v \notin \omega^+.
\end{aligned}$$

The last equation describes the evolution of the vector components x_v for $v \notin \omega^+$, and as in the previous case, these equations are completely decoupled from the other components. These differential equations have a unique equilibrium at zero, and they attract nonzero values in finite forward time. Similarly, the second equation, which describes the evolution of the vector components x_v for $v \in \omega^-$ has a form similar to the one in the critical flow tile case, and it generally leads to flow towards the boundary.

In view of these observations, we focus mainly on the first equation, which describes the flow of the vector component x_{v^+} where v^+ stands for the unique vertex in $\omega^+ \setminus \omega^-$. Also, assume for the moment that the location $x \in C_\omega \subset X$ of the solution satisfies $\sum_{u \notin \omega^+} x_u^2 = 0$, i.e., we

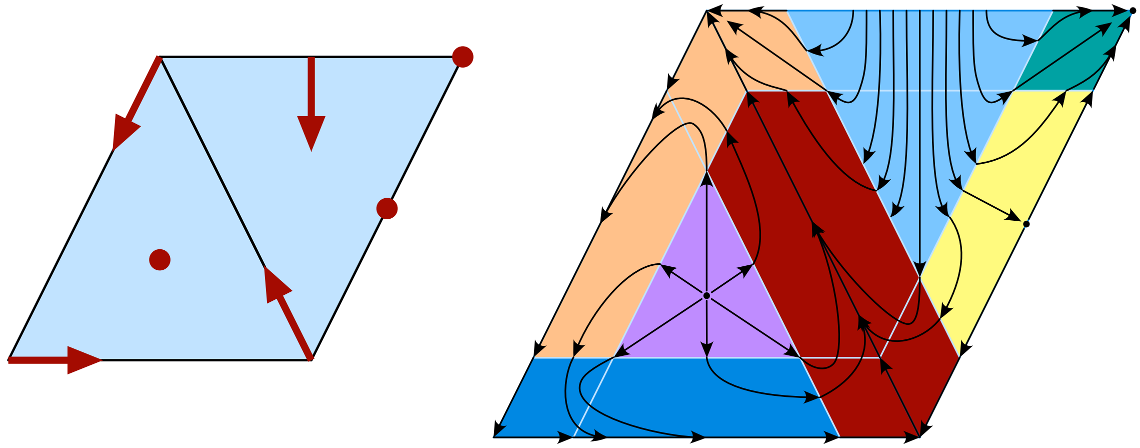

FIGURE 11. Complete strongly admissible semiflow induced by (36). For the combinatorial vector field $\mathcal{V}$ shown in the left panel, the panel on the right illustrates the strongly admissible semiflow induced by (36) on each of the flow tiles. Solutions generally exhibit merging in finite forward time, and the semiflow lowers the dimension of the (smallest) simplex which contains the solution. This can clearly be seen along the four outer edges of the simplicial complex. Moreover, significant discontinuous velocity changes can be observed in arrow flow tiles along the simplex ω^-, see for example the top left vertex, or the two vertices of the lowermost edge.

have $x \in C_\omega \cap \omega^+$. In this case, the identity $\eta^\omega(x_{v^+}, x) = 1$ holds, and the equation for $\dot{x}_{v^+}$ reduces to the first two terms in the parentheses. These terms have different responsibilities:

- The term $h(x_{v^+})$ provides the general profile for the velocity $\dot{x}_{v^+}$, which usually satisfies $\dot{x}_{v^+} > 0$. The only exception is a small neighborhood around $x_{v^+} = \varepsilon$, where the velocity is negative. Note, however, that this can be changed easily by adding a constant larger than $\varepsilon/2$ to the first term, which leads to $\dot{x}_{v^+} > 0$ even if one has $x_{v^+} \approx \varepsilon$.
- Introducing vertical shifts of the function h is the responsibility of the second term. As we have mentioned, this term satisfies $\theta^\omega(x) \in [-\varepsilon, \varepsilon]$, and the identity $\theta^\omega(x) = \varepsilon$ is satisfied if and only if $x_u \geq 2\varepsilon$ for all $u \in \omega^-$. Thus, as long as x is sufficiently far away from the boundary of the flow tile, the velocity $\dot{x}_{v^+}$ is strictly positive. Close to the boundary of C_ω and for $x_{v^+} \approx \varepsilon$ it becomes negative, which is required for the admissibility of the semiflow.

This behavior is illustrated in Figure 11 in the flow tile associated with the vertical arrow whose base is the top edge of the simplicial complex.

We now turn our attention to points $x \in C_\omega \setminus \omega^+$. As long as one also has $x_{v^+} > \varepsilon/4$, the flow description from above still applies. For this one only has to realize that in this case we have $\eta(x_{v^+}, x) = 1$ and the new shift term $\theta^\omega(x) - \sum_{u \notin \omega^+} x_u$ is strictly smaller than $\varepsilon/2$ as long as $x_u \approx \varepsilon$ for some vertex $u \neq v^+$. In other words, close to the boundary of C_ω the flow satisfies the admissibility condition.

The semiflow behavior changes, however, at points $x \in C_\omega \setminus \omega^+$ with $x_{v^+} = 0$. At such points, the prefactor $\eta^\omega(x_{v^+}, x)$ is zero, i.e., it keeps the v^+-component of x fixed at zero until the solution has entered the set $C_\omega \cap \omega^+$. This leads to the "running start" of solutions originating at points with $x_{v^+} = 0$, i.e., to the significant velocity changes which can be observed in all vertices except at the top right vertex in Figure 11. Notice further that the above definition of the flow on arrow flow tiles rules out any equilibria in C_ω. In fact, we will see in the next section that the largest invariant subset of arrow flow tiles is the empty set. Finally, we would like to note that solutions through a point $x \in X$ with $0 < x_u \le \varepsilon$ for some vertex $u \notin \omega^+$ will always lead to the identity $x_u = 0$ in finite forward time, even if the solution leaves C_ω before this happens.

6.2. **Semiflows Induced by the Individual Vector Fields.** We now turn our attention to showing that the vector fields defined in the previous section do in fact generate a continuous, strongly admissible semiflow on the underlying polytope $X \subset \mathbb{R}^d$ of the simplicial complex $\mathcal{X}$. As a first step, in the present section we let $\omega \in \mathcal{X}$ denote an arbitrary but fixed simplex, and we show that for every $\xi \in \mathbb{R}^d$ the initial value problem

$$\dot{x} = f^\omega(x) \qquad \text{with} \qquad x(0) = \xi \tag{37}$$

has a unique solution $\varphi^\omega(\cdot, \xi) : \mathbb{R}_0^+ \to \mathbb{R}^d$, which depends continuously on the initial condition ξ. In this context, a function $\nu : [0, T) \to \mathbb{R}^d$ is called a *solution of the initial value problem (37)* if ν is continuous and satisfies the integral identity

$$\nu(t) = \xi + \int_0^t f^\omega(\nu(\tau))\, d\tau \qquad \text{for all} \qquad 0 \le t < T. \tag{38}$$

We would like to point out that due to the fact that f^ω is not everywhere continuous, we cannot directly apply standard existence and uniqueness results. We begin with a simple lemma which allows us to solve part of the system (37).

Lemma 6.1 (Solution of the Decoupled Components)**.** *Consider the scalar function g defined in (33). Then for every $\zeta \in \mathbb{R}$ the initial value problem*

$$\dot{s} = -g(s) \qquad \textit{with} \qquad s(0) = \zeta \tag{39}$$

has the unique forward solution

$$\psi(t, \zeta) \;:=\; \begin{cases} \operatorname{sgn} \zeta \left(\zeta^{2/3} - \dfrac{2}{3} \varepsilon^{2/3} t \right)^{3/2} & \textit{for} \quad 0 \le t \le \dfrac{3}{2} \left(\dfrac{\zeta}{\varepsilon} \right)^{2/3}, \\[2ex] 0 & \textit{for} \quad t \ge \dfrac{3}{2} \left(\dfrac{\zeta}{\varepsilon} \right)^{2/3}. \end{cases} \tag{40}$$

Furthermore, the mapping $\psi : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous semiflow.

Proof. One can easily verify that the formula in (40) is a differentiable solution of the one-dimensional initial value problem (39) which is defined on $\mathbb{R}_0^+$. Furthermore, since the right-hand side of this problem is continuously differentiable on $\mathbb{R} \setminus \{0\}$, nonuniqueness can only occur once the solution hits zero.

Suppose therefore that $\nu : [t_0, T) \to \mathbb{R}$ is a solution of $\dot{s} = g(s)$ with $\nu(t_0) = 0$. The result follows if we can show that this implies $\nu(t) = 0$ for all $t \in [t_0, T)$. For this, assume that there exists a $t_1 \in (t_0, T)$ such that $\nu(t_1) \ne 0$. We first consider the case $\nu(t_1) > 0$. Then the supremum $t_* := \sup\{0 \le t \le t_1 \mid \nu(t) \le 0\}$ exists and satisfies $t_* \in [t_0, t_1)$ and $\nu(t_*) = 0$, as well as $\nu(t) > 0$ for all $t \in (t_*, t_1]$. This implies

$$\nu(t_1) = \nu(t_*) - \int_{t_*}^{t_1} \underbrace{g(\nu(\tau))}_{\ge 0}\, d\tau \le 0 + \int_{t_*}^{t_1} 0\, d\tau = 0,$$

a contradiction. The case $\nu(t_1) < 0$ can be treated analogously, which proves the lemma. $\square$

The scalar differential equation discussed in the above lemma is precisely the one that describes the evolution of the v-components of the solution of (37) for all $v \notin \omega^+$ with f_v^ω defined by (36). Notice further that the discontinuities of the vector field f^ω are restricted to the subspace of $\mathbb{R}^d$ in which all of these components are zero. Thus, we consider the decomposition $\mathbb{R}^d = Y_\omega \oplus Z_\omega$ with

$$\begin{aligned} Y_\omega \;&:=\; \left\{ x \in \mathbb{R}^d \;\mid\; x_v = 0 \text{ for all } v \notin \omega^+ \right\} \quad \text{and} \\ Z_\omega \;&:=\; \left\{ x \in \mathbb{R}^d \;\mid\; x_v = 0 \text{ for all } v \in \omega^+ \right\}, \end{aligned} \tag{41}$$

and we define

$$\tau^\omega(x) \;:=\; \max\left\{ \frac{3}{2}\left(\frac{x_v}{\varepsilon}\right)^{2/3} \;\mid\; v \notin \omega^+ \right\}. \tag{42}$$

We point out that $\tau^\omega(x) = 0$ if and only if we have $x \in Y_\omega$. Thus, in view of Lemma 6.1 these definitions imply that a solution $x(t)$ to the initial value problem (37) satisfies

$$x(t) \notin Y_\omega \;\text{ for all }\; 0 \le t < \tau^\omega(\xi) \qquad \text{and} \qquad x(t) \in Y_\omega \;\text{ for all }\; t \ge \tau^\omega(\xi).$$

This simple observation lies at the heart of the proof of the following central result of this section.

Proposition 6.2 (Existence of the Simplex-Induced Semiflow)**.** *For each $\xi \in \mathbb{R}^d$ a unique solution $\varphi^\omega(\cdot, \xi) : \mathbb{R}_0^+ \to \mathbb{R}^d$ of the initial value problem (37) exists. It is differentiable everywhere with the exception of at most one point in time, and $\varphi^\omega(t, \xi)$ is contained in the subspace Y_ω defined in (41) if and only if $t \ge \tau^\omega(\xi)$, as introduced in (42). In particular, the map $\varphi^\omega : \mathbb{R}_0^+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a well-defined continuous semiflow for every simplex $\omega \in \mathcal{X}$.*

Proof. At first glance it seems impossible to apply standard existence and uniqueness results for ordinary differential equations to the construction of the semiflow φ^ω. Note, however, that our discussion leading up to the formulation of the proposition pointed out two major points. On the one hand, the anticipated dynamics is divided into two clear-cut regimes — one outside the subspace Y_ω, and one inside of it. On the other hand, the discontinuity of the vector field and its accompanying velocity jumps happen only upon entering this subspace. This allows us to construct the semiflow in three stages.

(i) Semiflow inside the subspace Y_ω. We begin by considering only initial conditions $\xi \in Y_\omega$. On this subspace, the v-components f_v^ω of the vector field vanish for all $v \notin \omega^+$, and therefore we have $f^\omega(x) \in Y_\omega$ for all $x \in Y_\omega$. This immediately implies that the solution of the initial

value problem (37) satisfies the reduced differential equation

$$\begin{aligned} \dot{x}_{v^+} &= h(x_{v^+}) + \theta^\omega(x) && \text{for} \quad \{v^+\} = \omega^+ \setminus \omega^-, \\ \dot{x}_v &= x_v - \frac{1}{\#\omega^-}\left(\sum_{u\in\omega^-} x_u + f^\omega_{v^+}(x)\right) && \text{for} \quad v \in \omega^-. \end{aligned}$$

The right-hand side of this system is clearly globally Lipschitz continuous on Y_ω, since in view of (35) the prefactor $\eta^\omega(x_{v^+}, x)$ reduces to 1. Thus, standard existence and uniqueness results for ordinary differential equations imply that the solutions of this system generate a continuous flow $\Phi^\omega : \mathbb{R} \times Y_\omega \to Y_\omega$.

(ii) Semiflow outside the subspace Y_ω. We now turn our attention to the semiflow outside Y_ω. To study this case we need a new vector field $\bar{f}^\omega : \mathbb{R}^d \to \mathbb{R}^d$ which is a slight modification of the vector field (36). It is given by the formula

$$\bar{f}^\omega_v(x) := \begin{cases} -g(x_v) & \text{for} \quad v \notin \omega^+, \\ \min\left\{1, \dfrac{4}{\varepsilon}\,|x_v|\right\}\left(h(x_v) + \theta^\omega(x) - \displaystyle\sum_{u\notin\omega^+} x_u\right) & \text{for} \quad v \in \omega^+ \setminus \omega^-, \\ x_v - \dfrac{1}{\#\omega^-}\left(\displaystyle\sum_{u\in\omega^-} x_u + \sum_{u\notin\omega^-} \bar{f}^\omega_u(x)\right) & \text{for} \quad v \in \omega^-. \end{cases} \tag{43}$$

Essentially, the only change is the first factor in the product defining $\bar{f}^\omega_v(x)$ for the unique vertex $v \in \omega^+ \setminus \omega^-$ plus the resulting modification in the second sum of the formula defining $\bar{f}^\omega_v(x)$ for $v \in \omega^-$. In particular, $\bar{f}^\omega$ coincides with f^ω on the set $\mathbb{R}^d \setminus Y_\omega$ and one easily verifies that

$$\bar{f}^\omega(x) \;=\; \lim_{Y_\omega \not\ni y \to x} f^\omega(y).$$

While the new vector field is continuous, it is still not Lipschitz continuous due to the presence of the root function g in the v-components for $v \notin \omega^+$. Notice, however, that these components are completely decoupled from the rest of the vector field. Thus, the part of the initial value problem

$$\dot{x} = \bar{f}^\omega(x) \qquad \text{with} \qquad x(0) = \xi \tag{44}$$

which corresponds to v-components with $v \notin \omega^+$ can be solved individually using Lemma 6.1. For the initial condition $\xi^z \in Z_\omega$ this leads to the semiflow $\Psi : \mathbb{R}^+_0 \times Z_\omega \to Z_\omega$ given by

$$\Psi_v\,(t, \xi^z) \;=\; \psi(t, \xi_v) \quad \text{for all} \quad v \notin \omega^+,$$

where ψ is defined in (40). Having solved for the solution components in Z_ω, one can now see that the initial value problem (44) is equivalent to solving the nonautonomous ordinary differential equation problem

$$\dot{y} = \bar{f}^{\omega,y}(y + \Psi(t, \xi^z)) \qquad \text{with} \qquad y(0) = \xi^y, \tag{45}$$

where we decompose the vector field in the form $\bar{f}^\omega(x) = \bar{f}^{\omega,y}(x) + \bar{f}^{\omega,z}(x) \in Y_\omega \oplus Z_\omega$. Note that (45) is an initial value problem which is only defined for $t \geq 0$, because $\Psi(t, \xi^z)$ is defined only for $t \geq 0$. Moreover, it depends both on the initial value $\xi^y \in Y_\omega$ and on the parameter $\xi^z \in Z_\omega$. In addition, the right-hand side of the nonautonomous differential equation is continuous with respect to (t, y, ξ^z), as well as globally Lipschitz continuous with respect to y for every fixed combination of t and ξ^z. For example, the v^+-component of the right-hand side for $\{v^+\} = \omega^+ \setminus \omega^-$ is given explicitly by

$$\bar{f}^\omega_{v^+}(y + \Psi(t, \xi^z)) \;=\; \min\left\{1,\, \frac{4}{\varepsilon}\,|y_{v^+}|\right\} \left(h(y_{v^+}) + \theta^\omega(y + \Psi(t, \xi^z)) - \sum_{u \notin \omega^+} \psi(t, \xi_u) \right), \tag{46}$$

and the nondifferentiable functions g appear only in the v-components of the vector field for the vertices $v \in \omega^-$ and are evaluated at the functions $\psi(t, \xi_v)$.

While recasting (44) in the form (45) might seem a technicality at first, the nonautonomous form of the new equation isolates the non-Lipschitz part of the vector field $\bar{f}^\omega$ in the t-dependent part. This approach only works because we can solve for the Z_ω-component of the solution ahead of time and independently from the rest. Furthermore, the nonautonomous parameter-dependent initial value problem (45) satisfies all the assumptions of [4, Theorem 2.4]. This implies the existence of a unique solution $\Xi(\cdot, \xi^y, \xi^z) : \mathbb{R}_0^+ \to Y_\omega$ of (45), and it also shows that the map $\Xi : \mathbb{R}_0^+ \times Y_\omega \times Z_\omega \to Y_\omega$ is continuous with respect to all variables. Finally, the mapping $\Pi : (t, \xi^y, \xi^z) \mapsto (\Xi(t, \xi^y, \xi^z), \Psi(t, \xi^z))$ satisfies (44) for the initial condition $\xi = \xi^y + \xi^z$, and this solution is continuously differentiable on $\mathbb{R}_0^+$. Moreover, since (44) is autonomous, Π is a semiflow on $\mathbb{R}^d = Y_\omega \oplus Z_\omega \cong Y_\omega \times Z_\omega$.

(iii) Constructing the combined semiflow. While the solution Ξ constructed in the last part solves the initial value problem (44) for all times $t \geq 0$, this initial value problem is different from the one we set out to solve. Based on the discussion leading up to this proposition, we can however use it to find the solution to (37). For this, recall the definition of the time $\tau^\omega(\xi)$ in (42). Due to the specific form of f^ω, any solution of (37) has to solve (44) on the interval $[0, \tau^\omega(\xi)]$, and it has to satisfy the autonomous differential equation studied in *(i)* on the interval $[\tau^\omega(\xi), \infty)$. Thus, the unique forward solution of (37) is given by the composition

$$\varphi^\omega(t, \xi) \;=\; \Phi^\omega\left(\max\{0, t - \tau^\omega(\xi)\}\,,\; \Xi\left(\min\{t, \tau^\omega(\xi)\}\,, \xi^y, \xi^z\right)\right) + \Psi(t, \xi^z). \tag{47}$$

According to the previous two parts of this proof, the mapping $\varphi^\omega : \mathbb{R}_0^+ \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous. In addition, the solution $\varphi^\omega(\cdot, \xi)$ is differentiable everywhere except possibly at $t = \tau^\omega(\xi)$, and it satisfies (38). Since the vector field f^ω is autonomous, this last fact together with the fact that $\varphi^\omega(\cdot, \xi)$ is the unique forward solution of (37) finally shows that φ^ω is a continuous semiflow, which concludes the proof of the proposition. $\square$

The above result shows that the vector field f^ω generates a continuous semiflow on $\mathbb{R}^d$, despite its discontinuities. All of its solutions are uniquely defined in forward time, but they can merge in finite time.

For us, the semiflows $\varphi^\omega : \mathbb{R}_0^+ \times \mathbb{R}^d \to \mathbb{R}^d$ are just a first step towards the construction of a strongly admissible semiflow on the underlying polytope $X \subset \mathbb{R}^d$ of the given simplicial complex $\mathcal{X}$, and with respect to the combinatorial vector field $\mathcal{V}$. As a second step, we need to show that φ^ω leaves appropriate parts of X invariant. This is the subject of the following corollary.

Corollary 6.3 (Relative Forward Invariance of the Semiflow on Flow Tiles)**.** *Let $\omega \in \mathcal{X}$ be an arbitrary simplex and consider the semiflow φ^ω on $\mathbb{R}^d$ guaranteed by Proposition 6.2. Consider the flow tile associated with ω, that is*

$$C_\omega \;=\; \mathrm{cl}\,\langle\omega^-\rangle_\varepsilon \cup \mathrm{cl}\,\langle\omega^+\rangle_\varepsilon \;\subset\; X \;\subset\; \mathbb{R}^d$$

as introduced in (17). Then for every $\xi \in C_\omega$ the solution $\varphi^\omega(\cdot, \xi)$ stays in C_ω until it reaches a point $\xi^* \in C_\omega$ which satisfies $\xi^*_v = \varepsilon$ for at least one vertex $v \in \omega^-$. Such a point necessarily lies on the boundary of the flow tile C_ω.

Proof. The semiflow φ^ω constructed in Proposition 6.2 generates a semiflow on the whole space $\mathbb{R}^d$, and we have already seen that the underlying polytope X is only a subset of $\mathbb{R}^d$ of measure zero. In view of Lemma 4.3 and (17) we can characterize the flow tile C_ω as the set of vectors $x \in \mathbb{R}^d$ satisfying the following four conditions

(a) $\sum_{v\in\mathcal{X}_0} x_v = 1$,
(b) $0 \le x_v \le \varepsilon$ for all $v \notin \omega^+$,
(c) $x_v \ge 0$ for $v \in \omega^+ \setminus \omega^-$,
(d) $x_v \ge \varepsilon$ for all $v \in \omega^-$.

Thus, in order to establish the corollary we only have to show that a solution $\varphi^\omega(\cdot, \xi)$ originating at $\xi \in C_\omega$ cannot exit this flow tile by violating conditions (a), (b), or (c). We verify this claim for each condition separately.

(i) Along the solution, condition (a) cannot be violated. One can easily see that for every point $x \in \mathbb{R}^d$ we have the identity

$$\begin{aligned}\sum_{v\in\mathcal{X}_0} f^\omega_v(x) &= \sum_{v\notin\omega^-} f^\omega_v(x) + \sum_{v\in\omega^-}\left(x_v - \frac{1}{\#\omega^-}\left(\sum_{u\in\omega^-} x_u + \sum_{u\notin\omega^-} f^\omega_u(x)\right)\right) \\ &= \sum_{v\notin\omega^-} f^\omega_v(x) + \sum_{v\in\omega^-} x_v - \left(\sum_{u\in\omega^-} x_u + \sum_{u\notin\omega^-} f^\omega_u(x)\right) \;=\; 0.\end{aligned} \tag{48}$$

Since every solution $\varphi^\omega(\cdot, \xi)$ of (37) satisfies the integral equality (38), we obtain from (48) that $\sum_{v\in\mathcal{X}_0} \varphi^\omega_v(t, \xi)$ is independent of t for $t \ge 0$. In particular, for any $\xi \in C_\omega$ the sum of the components of a solution has to remain equal to one for all times $t \ge 0$.

(ii) Along the solution, condition (b) cannot be violated. In view of $\xi_v \in [0, \varepsilon]$ for all $v \notin \omega^+$, the formulas (47) and (40) immediately imply that $\varphi^\omega_v(t, \xi) \in [0, \varepsilon]$ for all $t \ge 0$ and $v \notin \omega^+$.

(iii) Along the solution, condition (c) cannot be violated. Recall that $\omega^+ \setminus \omega^-$ is either empty or a singleton. If it is empty, there is nothing to be verified. Thus, assume that v^+ is the unique vertex in $\omega^+ \setminus \omega^-$. According to the proof of Proposition 6.2, the system (44), which was solved in the nonautonomous form (45), has the invariant hyperplane

$$H_\omega = \left\{ x \in \mathbb{R}^d \;\mid\; x_{v^+} = 0 \right\}.$$

In other words, the solution Ξ preserves the inequality $x_{v^+} \ge 0$. Once the solution hits the set $Y_\omega \cap C_\omega$, part *(i)* of the proof of Proposition 6.2 shows that $\dot{x}_{v^+} > 0$, i.e., the v^+-component of $\varphi^\omega(\cdot, \xi)$ becomes strictly positive. This proves the result. $\square$

6.3. **Dynamics of the Individual Semiflows on Flow Tiles.** In the last section we have constructed a semiflow φ^ω on $\mathbb{R}^d$ for every simplex $\omega \in \mathcal{X}$. We have also seen that the associated flow tile $C_\omega = \operatorname{cl}\langle\omega^-\rangle_\varepsilon \cup \operatorname{cl}\langle\omega^+\rangle_\varepsilon$ is forward invariant under this semiflow, until a solution reaches a well-defined subset of its boundary in X. But how exactly this boundary is reached, what the vector field f^ω looks like on the boundary, and what other properties forward solutions of φ^ω have in C_ω has been left unexplored. This gap is closed in the present section.

We begin our discussion with the behavior of φ^ω on the flow tile C_ω near its boundary. Recall that according to Definition 4.4 and Lemma 4.6 a point $x \in X$ lies on the boundary of at least two ε-cells, if there exists at least one vertex $v \in \mathcal{X}_0$ such that $x_v = \varepsilon$. In fact, the set of all vertices v for which this identity is satisfied is given by the vertices in $\sigma^\varepsilon_{\max}(x) \setminus \sigma^\varepsilon_{\min}(x)$. In order to guarantee that x lies on the boundary of at least two flow tiles, one needs the somewhat stronger condition that there exists at least one vertex $v \in \mathcal{X}_0$ such that $x_v = \varepsilon$ and $v \notin \omega^+ \setminus \omega^-$. With these observations in mind, we obtain the following result in which, as everywhere in this section, $d := \#\mathcal{X}_0$ and $X \subset \mathbb{R}^d$ denotes the underlying polytope of the standard geometric realization of $\mathcal{X}$.

Proposition 6.4 (Vector Field Bounds near Flow Tile Boundaries)**.** *Consider the vector fields f^ω defined in (33) through (36), the flow tiles C_ω defined in (17), and suppose that* (32) *is satisfied. Then for every simplex $\omega \in \mathcal{X}$ the following is true.*

(a) *For all $v \in \omega^-$ and all $x \in C_\omega$ with $|x_v - \varepsilon| \le \varepsilon$ we have $f^\omega_v(x) \le -1/(4d) < 0$.*
(b) *For all $v \notin \omega^+$ and all $x \in C_\omega$ with $|x_v - \varepsilon| \le \varepsilon/2$ we have $f^\omega_v(x) \le -\varepsilon/2 < 0$.*
(c) *If $\omega^+ \ne \omega^-$ and $\{v^+\} = \omega^+ \setminus \omega^-$, then for all $x \in C_\omega$ with $|x_{v^+} - \varepsilon| \le \varepsilon^2/(8+4\varepsilon)$ and for which there is a vertex $v \ne v^+$ with $|x_v - \varepsilon| \le \varepsilon/8$ we have $f^\omega_{v^+}(x) \le -\varepsilon/8 < 0$.*

Proof. The proof of the proposition is divided into four separate parts.

(i) Verification of (a) for critical cells. Suppose first that the simplex $\omega \in \mathcal{X}$ is a critical cell and let $v \in \omega$ be a fixed vertex. Furthermore, let $x \in C_\omega$ be given with $|x_v - \varepsilon| \le \varepsilon$. Then Lemma 4.3 implies $\varepsilon \le x_v \le 2\varepsilon$, as well as $0 \le x_u \le \varepsilon$ for all vertices $u \notin \omega$. The definition of the function g then further yields $0 \le g(x_u) \le \varepsilon$ for all $u \notin \omega$. Let $m := \#\omega$. We deduce from both $d = \#\mathcal{X}_0$ and $\sum_{u\in\mathcal{X}_0} x_u = 1$, in combination with (32), the estimate

$$\begin{aligned} f^\omega_v(x) &= x_v - \frac{1}{\#\omega}\left(1 - \sum_{u\notin\omega} x_u + \sum_{u\notin\omega} f^\omega_u(x)\right) \le 2\varepsilon - \frac{1}{m} + \frac{1}{m}\sum_{u\notin\omega} x_u + \frac{1}{m}\sum_{u\notin\omega} g(x_u) \\ &\le 2\varepsilon - \frac{1}{m} + \frac{(d-m)\varepsilon}{m} + \frac{(d-m)\varepsilon}{m} = -\frac{1}{m} + \frac{2d\varepsilon}{m} < -\frac{2}{3m} \le -\frac{2}{3d} < -\frac{1}{4d}, \end{aligned}$$

which proves *(a)* for critical cells.

(ii) Verification of (a) for arrow cells. Suppose that the simplex $\omega \in \mathcal{X}$ is part of an arrow, i.e., we have $\omega^+ \ne \omega^-$. Let $v \in \omega^-$ be a fixed vertex, and let $x \in C_\omega$ be given with $|x_v - \varepsilon| \le \varepsilon$. Then Lemma 4.3 implies again $\varepsilon \le x_v \le 2\varepsilon$, as well as $0 \le x_u \le \varepsilon$ and $0 \le g(x_u) \le \varepsilon$ for all vertices $u \notin \omega^+$. Setting $m := \#\omega^-$, we now deduce from both $d = \#\mathcal{X}_0$ and $\sum_{u\in\mathcal{X}_0} x_u = 1$,

in combination with (32) and $\{v^+\} = \omega^+ \setminus \omega^-$, the estimate

$$\begin{aligned}
f_v^\omega(x) &= x_v - \frac{1}{m}\left(\sum_{u\in\omega^-} x_u + \sum_{u\notin\omega^-} f_u^\omega(x)\right) \le 2\varepsilon + \frac{1}{m}\left(-1 + \sum_{u\notin\omega^-} x_u - \sum_{u\notin\omega^-} f_u^\omega(x)\right) \\
&= 2\varepsilon + \frac{1}{m}\left(x_{v^+} - f_{v^+}^\omega(x) - 1 + \sum_{u\notin\omega^+} x_u - \sum_{u\notin\omega^+} f_u^\omega(x)\right) \\
&\le 2\varepsilon + \frac{1}{m}\left(x_{v^+} - f_{v^+}^\omega(x) - 1\right) + \frac{2(d-m-1)\varepsilon}{m} \\
&< \frac{2d\varepsilon}{m} + \frac{1}{m}\left(x_{v^+} - f_{v^+}^\omega(x) - 1\right) < \frac{1}{3m} + \frac{1}{m}\left(x_{v^+} - f_{v^+}^\omega(x) - 1\right). \qquad (49)
\end{aligned}$$

We now turn our attention to the term in parentheses in (49). Due to $x \in C_\omega$, Lemma 4.3 implies $x_w \ge \varepsilon$ for all $w \in \omega^-$, and the definition of $\theta^\omega(x)$ in (34) then yields $\theta^\omega(x) \ge 0$. Furthermore, in view of (35) we have $0 \le \eta^\omega(x_{v^+}, x) \le 1$. Now the second equation in (36) implies

$$\begin{aligned}
-f_{v^+}^\omega(x) &= -\eta^\omega(x_{v^+}, x)h(x_{v^+}) - \eta^\omega(x_{v^+}, x)\theta^\omega(x) + \eta^\omega(x_{v^+}, x)\sum_{u\notin\omega^+} x_u \\
&\le -\eta^\omega(x_{v^+}, x)h(x_{v^+}) + \sum_{u\notin\omega^+} x_u \le -\eta^\omega(x_{v^+}, x)h(x_{v^+}) + (d-m-1)\varepsilon \\
&< \frac{1}{6} - \eta^\omega(x_{v^+}, x)h(x_{v^+}),
\end{aligned}$$

which together with (49) gives

$$f_v^\omega(x) < \frac{1}{2m} + \frac{1}{m}\left(x_{v^+} - \eta^\omega(x_{v^+}, x)h(x_{v^+}) - 1\right). \qquad (50)$$

A glance at the graph of $\eta^\omega(x_{v^+}, x)h(x_{v^+})$ in the right panel of Figure 9, which could either be the blue or the red curve, readily shows that for $0 \le x_{v^+} \le 1$ the distance between $\eta^\omega(x_{v^+}, x)h(x_{v^+})$ and $x_{v^+} - 1$ is minimal for $x_{v^+} = \varepsilon$, and one obtains

$$\eta^\omega(x_{v^+}, x)h(x_{v^+}) - (x_{v^+} - 1) \ge 1 - \frac{3\varepsilon}{2} > \frac{3}{4} \quad \text{for all} \quad 0 \le x_{v^+} \le 1,$$

since we have $\varepsilon < 1/(6d) \le 1/6$. Together with (50) we finally get the estimate

$$f_v^\omega(x) < \frac{1}{2m} + \frac{1}{m}\underbrace{\left(x_{v^+} - \eta^\omega(x_{v^+}, x)h(x_{v^+}) - 1\right)}_{<-3/4} < -\frac{1}{4m} \le -\frac{1}{4d},$$

which completes the proof of *(a)* for arrow cells.

(iii) Verification of (b). For arbitrary $x \in C_\omega$ and $v \notin \omega^+$ we have $0 \le x_v \le \varepsilon$, i.e., the assumption in *(b)* implies $\varepsilon/2 \le x_v \le \varepsilon$. Together with (33) and the first equation in (36) one then obtains $f_v^\omega(x) = -g(x_v) \le -g(\varepsilon/2) < -\varepsilon/2$.

(iv) Verification of (c). Finally, suppose that $\omega^- \ne \omega^+$ and $\{v^+\} = \omega^+ \setminus \omega^-$. Let $x \in C_\omega$ be arbitrary with $|x_{v^+} - \varepsilon| \le \varepsilon^2/(8 + 4\varepsilon)$. Then the definitions of h and η^ω in (33) and (34),

respectively, imply both $h(x_{v^+}) \le -\varepsilon/4$ and $\eta^\omega(x_{v^+}, x) = 1$, where for the latter identity we use the inequality $\varepsilon^2/(8+4\varepsilon) < \varepsilon/2$. According to (36) these statements imply

$$f^\omega_{v^+}(x) \;=\; \eta^\omega(x_{v^+}, x) \left(h(x_{v^+}) + \theta^\omega(x) - \sum_{u \notin \omega^+} x_u \right) \;<\; -\frac{\varepsilon}{4} + \theta^\omega(x) - \sum_{u \notin \omega^+} x_u. \tag{51}$$

Consider first the case when there exists a vertex $v \notin \omega^+$ such that $|x_v - \varepsilon| \le \varepsilon/8$. Then Lemma 4.3 yields $7\varepsilon/8 \le x_v \le \varepsilon$, and (51) implies in combination with $\theta^\omega(x) \le \varepsilon$ the estimate

$$f^\omega_{v^+}(x) \;<\; -\frac{\varepsilon}{4} + \theta^\omega(x) - x_v \;\le\; -\frac{\varepsilon}{8},$$

which establishes *(c)*. In the other case we have $|x_u - \varepsilon| > \varepsilon/8$ for all $u \notin \omega^+$. Then, there has to be a vertex $v \in \omega^-$ for which $|x_v - \varepsilon| \le \varepsilon/8$. In view of Lemma 4.3 this furnishes the inequalities $\varepsilon \le x_v \le 9\varepsilon/8$, and therefore we have $\theta^\omega(x) \le \varepsilon/8$. Now (51) implies

$$f^\omega_{v^+}(x) \;<\; -\frac{\varepsilon}{4} + \frac{\varepsilon}{8} - \sum_{u \notin \omega^+} x_u \;\le\; -\frac{\varepsilon}{8},$$

since $x_u \ge 0$ for all $u \notin \omega^+$. Thus, also in the other case property (c) holds. This completes the proof of the proposition. □

For later reference, we formulate an easy corollary of this result, which describes the vector field behavior on the boundary of flow tiles.

Corollary 6.5 (Vector Field Direction along Flow Tile Boundaries)**.** *Consider the vector fields f^ω defined in (33) through (36), the flow tiles C_ω defined in (17), and suppose that (32) holds. Then for every simplex $\omega \in \mathcal{X}$ and every $x \in C_\omega$ which lies on the boundary of C_ω in X, the vector $f^\omega(x)$ points into the interior of the flow tile $C_{\sigma^\varepsilon_{\min}(x)}$, while $-f^\omega(x)$ points into the interior of $C_{\sigma^\varepsilon_{\max}(x)}$, where $\sigma^\varepsilon_{\min}(x)$ is given by (12) and $\sigma^\varepsilon_{\max}(x)$ by (13).*

Proof. Suppose that $x \in C_\omega$ lies on the boundary of two flow tiles. Then there is at least one vertex $v \in \mathcal{X}_0$ for which $x_v = \varepsilon$, and in fact all vertices for which this identity holds are the vertices in $V_x := \sigma^\varepsilon_{\max}(x) \setminus \sigma^\varepsilon_{\min}(x)$. In addition, if $\omega^+ \ne \omega^-$ and if V_x contains the unique vertex v^+ in $\omega^+ \setminus \omega^-$, then V_x has to contain at least one more vertex $v \ne v^+$. Using Proposition 6.4 one can then show that

$$f^\omega_v(x) < 0 \qquad \text{for all} \qquad v \in V_x = \sigma^\varepsilon_{\max}(x) \setminus \sigma^\varepsilon_{\min}(x),$$

which establishes the corollary. □

Proposition 6.4 describes in detail the behavior of solutions of φ^ω in the flow tile C_ω near its boundary. As we will see in the next section, this result is crucial both for the definition of the final semiflow on X, as well as for its admissibility. In contrast, the next result will be used for establishing strong admissibility.

Proposition 6.6 (Solution Exit from Arrow Flow Tiles)**.** *Consider the vector fields f^ω defined in (33) through (36), as well as the associated semiflow $\varphi^\omega : \mathbb{R}^+_0 \times \mathbb{R}^d \to \mathbb{R}^d$ guaranteed by Proposition 6.2, and suppose that (32) holds. Let $\omega \in \mathcal{X}$ be a simplex which is part of an arrow of the combinatorial vector field $\mathcal{V}$, and let $x \in C_\omega$ be contained in the associated flow tile as defined in (17). Then the forward solution of φ^ω which originates in x exits the tile C_ω*

in finite forward time, and every solution through x which exists for all negative times exits the flow tile C_ω in finite backward time.

Proof. We argue by contradiction. Suppose that there exists a solution of φ^ω which stays in the compact set C_ω for all $t \geq 0$ or for all $t \leq 0$. Then its ω- or its α-limit set has to be nonempty, and standard results for semiflows imply that there exists a full solution $\gamma : \mathbb{R} \to \mathbb{R}^d$ of φ^ω which stays in C_ω for all times, see for example [20, Proposition 2.6, p. 204]. We will prove the following four statements:

(i) For all $t \in \mathbb{R}$ we have $\gamma(t) \in C_\omega \cap \omega^+$.

(ii) For every $v \in \omega^-$ the set $F_v := \{x \in C_\omega \cap \omega^+ \;\mid\; x_v \leq 2\varepsilon\}$ is positively invariant under φ^ω relative to $C_\omega \cap \omega^+$. Furthermore, every solution starting in one of these sets will exit C_ω in finite forward time.

(iii) Every solution originating in $F^* := \{x \in C_\omega \cap \omega^+ \;\mid\; x_w \geq 2\varepsilon \;\text{ for all }\; w \in \omega^-\}$ has to enter a set F_v for some $v \in \omega^-$ in finite forward time.

(iv) Every solution originating in F^* has to exit C_ω in finite forward time.

Proof of (i): Let $x \in \gamma(\mathbb{R})$ be arbitrary and suppose that there exists a vertex $u \notin \omega^+$ such that $x_u \neq 0$. Then according to $x \in X$ we have $x_u > 0$, and the definition of the vector field shows that γ_u solves the differential equation $\dot{y} = -g(y)$ with the positive initial value x_u. Solutions of this initial value problem are uniquely determined in backward time, and one can easily see that they become unbounded as $t \to -\infty$, which contradicts the fact that γ lies in the compact set C_ω. Thus, we have to have $x_u = 0$ for all $u \notin \omega^+$ and *(i)* follows.

Proof of (ii): Let $v \in \omega^-$ be arbitrary but fixed. Then Proposition 6.4*(a)* immediately implies

$$f_v^\omega(x) \;\leq\; -\frac{1}{4d} \;<\; 0 \qquad \text{for all} \qquad x \in F_v \subset C_\omega \cap \omega^+.$$

This leads to the following two observations. On the one hand, since the semiflow φ^ω on C_ω is generated by the vector field f^ω, it shows that the v-component of any solution originating in F_v is decreasing, i.e., the solution stays in F_v for as long as it stays in $C_\omega \cap \omega^+$. On the other hand, since the v-component has a velocity which is bounded away from zero, any such solution has to reach the hyperplane $x_v = \varepsilon$ in finite forward time, at which point it will exit C_ω due to Proposition 6.4 — unless of course the solution exits earlier. This completes the proof of *(ii)*.

Proof of (iii): Suppose that there exists a point $x \in F^*$ with $f^\omega_{v^+}(x) \leq \varepsilon/4$, where v^+ denotes the unique vertex in $\omega^+ \setminus \omega^-$. Then the definition of $f^\omega_{v^+}(x)$ in (36), together with the fact that for all $y \in C_\omega \cap \omega^+$ we have both $\eta^\omega(y_{v^+}, y) = 1$ and $y_w = 0$ for all $w \notin \omega^+$, implies the estimate

$$\frac{\varepsilon}{4} \;\geq\; f^\omega_{v^+}(x) \;=\; h(x_{v^+}) + \theta^\omega(x) \;\geq\; -\frac{\varepsilon}{2} + \theta^\omega(x), \qquad \text{and thus} \qquad \theta^\omega(x) \;\leq\; \frac{3\varepsilon}{4}.$$

According to (34) this yields a vertex $w \in \omega^-$ with $x_w - \varepsilon \leq 3\varepsilon/4 < \varepsilon$, i.e., one has to have the inequality $x_w \leq 7\varepsilon/4 < 2\varepsilon$. This clearly contradicts our choice of $x \in F^*$.

In view of the last paragraph, we therefore have $f^\omega_{v^+}(x) > \varepsilon/4$ for all $x \in F^*$. This in turn implies that any solution of φ^ω which starts in F^* either has to reach one of the sets F_v as desired, or its v^+-component has to reach a point $y \in F^*$ with $y_{v^+} > 1 - 2\varepsilon \cdot \#\omega^-$ in finite forward time. Due to $y \in \omega^+$ this yields $\sum_{u \in \omega^-} y_u = 1 - y_{v^+} < 2\varepsilon \cdot \#\omega^-$, and therefore there has to be a vertex $v \in \omega^-$ with $y_v < 2\varepsilon$. This shows that also in this case the solution enters a set F_v. This completes the proof of *(iii)*.

Proof of (iv): This is an immediate consequence of (ii) and (iii).

Finally, since we clearly have $C_\omega \cap \omega^+ = F^* \cup \bigcup_{v \in \omega^-} F_v$, we see that the statement *(i)* contradicts statements *(ii)* and *(iv)*, which in turn establishes the result. □

To close this section we take a closer look at solutions of φ^ω which eventually exit the associated flow tile C_ω. While the previous result demonstrates that every solution in an arrow flow tile has to exit, every critical flow tile contains points x for which the forward solution $\varphi^\omega(\mathbb{R}_0^+, x)$ is contained in C_ω. Yet, as the following result shows, the set of initial conditions which lead to an exit from the flow tile is always open in C_ω, and the time it takes to exit C_ω varies continuously with the initial condition. This fact will be crucial in the next section.

Lemma 6.7 (Continuity of the Exit Time from Flow Tiles)**.** *Consider the continuous semi-flows $\varphi^\omega : \mathbb{R}_0^+ \times \mathbb{R}^d \to \mathbb{R}^d$ guaranteed by Proposition 6.2, and suppose that (32) holds. For every simplex $\omega \in \mathcal{X}$ we define the exit time*

$$T^\omega(x) \;:=\; \inf\{t > 0 \;\mid\; \varphi^\omega(t,x) \notin C_\omega\} \quad \textit{for all} \quad x \in C_\omega, \tag{52}$$

where C_ω is defined in (17) and the infimum of the empty set is assumed to be $+\infty$. In addition, we let

$$E_\omega \;:=\; \{x \in C_\omega \;\mid\; T^\omega(x) < +\infty\} \tag{53}$$

denote the set of all initial conditions which lead to domain exit from the flow tile C_ω. Then the set E_ω is open in C_ω, and the map $T^\omega : E_\omega \to \mathbb{R}_0^+$ is continuous. Finally, $T^\omega(x) = 0$ if and only if there exists a vertex $v \in \omega^-$ with $x_v = \varepsilon$.

Proof. We first show that E_ω is open in C_ω. For this, let $x \in E_\omega$ be arbitrary. Then there exists a time $\tau > 0$ such that $\varphi^\omega(\tau, x) \notin C_\omega$. Since C_ω is closed and $\varphi^\omega : \mathbb{R}_0^+ \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous, there exists an open neighborhood $U \subset \mathbb{R}^d$ of x such that for all points $y \in U$ we have $\varphi^\omega(\tau, y) \notin C_\omega$. This immediately implies $U \cap C_\omega \subset E_\omega$, and establishes the openness of E_ω in C_ω. Moreover, the characterization of all $x \in E_\omega$ which satisfy $T^\omega(x) = 0$ follows directly from Corollary 6.3 and Proposition 6.4*(a)*.

Now let $\delta > 0$ and $x \in E_\omega$ be arbitrary. Furthermore, choose a time $\tau \in (T^\omega(x), T^\omega(x) + \delta)$ with $\varphi^\omega(\tau, x) \notin C_\omega$. Then there exists an open neighborhood $U \subset \mathbb{R}^d$ of x such that for all points $y \in U$ we have $\varphi^\omega(\tau, y) \notin C_\omega$. This in turn implies $T^\omega(y) \le \tau < T^\omega(x) + \delta$ for all elements $y \in U \cap C_\omega$, and therefore $T^\omega : E_\omega \to \mathbb{R}_0^+$ is upper semicontinuous.

Next we choose an $x \in E_\omega$ with $T^\omega(x) > 0$, and we let $\delta \in (0, T^\omega(x)]$ be arbitrary. In view of Proposition 6.4*(a)* we have the strict inequality $\varphi_u^\omega(t,x) > \varepsilon$ for all vertices $u \in \omega^-$ and all times $t \in [0, T^\omega(x) - \delta]$. The compactness of the latter interval and the continuity of φ^ω then imply the existence of a constant $\varrho > \varepsilon$ such that

$$\varphi_u^\omega(t,x) \ge \varrho > \varepsilon \quad \text{for all} \quad u \in \omega^- \;\text{ and }\; t \in [0, T^\omega(x) - \delta].$$

Since φ^ω is a continuous semiflow, there now exists an $\eta > 0$ such that

$$|\varphi^\omega(t,y) - \varphi^\omega(t,x)| \;<\; \varrho - \varepsilon \quad \text{for all} \quad |y - x| < \eta \;\text{ and }\; t \in [0, T^\omega(x) - \delta],$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. This uniform estimate implies for all $u \in \omega^-$, all $t \in [0, T^\omega(x) - \delta]$, and all $y \in C_\omega$ with $|y - x| < \eta$ the bound

$$\varphi_u^\omega(t,y) \;=\; |\varphi_u^\omega(t,y)| \;\ge\; |\varphi_u^\omega(t,x)| - |\varphi_u^\omega(t,x) - \varphi_u^\omega(t,y)| \;>\; \varrho - (\varrho - \varepsilon) \;=\; \varepsilon.$$

But then Corollary 6.3 yields the inclusion $\varphi^\omega(t,y) \in C_\omega$ for all times $t \in [0, T^\omega(x) - \delta]$ and all points $y \in C_\omega$ with $|y - x| < \eta$, and therefore

$$T^\omega(y) \geq T^\omega(x) - \delta \quad \text{for all} \quad y \in C_\omega \;\text{ with }\; |y - x| < \eta.$$

Since this last estimate is trivially satisfied if $T^\omega(x) = 0$, this shows that the map T^ω is lower semicontinuous. This completes the proof of the lemma. □

6.4. **Gluing and the Final Strongly Admissible Semiflow.** After the preparations of the last two sections, we are finally in a position to construct a strongly admissible semiflow for a given combinatorial vector field $\mathcal{V}$ on the underlying polytope $X \subset \mathbb{R}^d$ of a simplicial complex $\mathcal{X}$. So far, for every simplex $\omega \in \mathcal{X}$ we have constructed a semiflow φ^ω on X which behaves as intended inside the flow tile C_ω. In order to construct a stronlgy admissible semiflow $\varphi : \mathbb{R}_0^+ \times X \to X$, we only have to concatenate solution pieces from the different semiflows φ^ω. We first construct the solution through a fixed point $x \in X$.

Proposition 6.8 (Auxiliary Map)**.** *Consider the semiflows $\varphi^\omega : \mathbb{R}_0^+ \times \mathbb{R}^d \to \mathbb{R}^d$ guaranteed by Proposition 6.2, the associated flow tiles C_ω defined in (17), and suppose that (32) holds. Let $x \in X$ be fixed and for $s \in \mathbb{R}_0^+ \cup \{\infty\}$ set*

$$\Delta(s) := \begin{cases} [0,\infty) & \text{if } s = \infty, \\ [0,s] & \text{otherwise.} \end{cases}$$

There exist sequences $(t_k)_{k\in\mathbb{N}}$ of non-negative real numbers, $(x_k)_{k\in\mathbb{N}}$ of points in X, and $(\chi_k)_{k\in\mathbb{N}}$ of partial maps $\chi_k : \mathbb{R}_0^+ \nrightarrow X$ with $t_0 = 0$ and $x_0 = x$, and such that for arbitrary $k \in \mathbb{N}$ and $\omega_k := \sigma^\varepsilon_{\min}(x_k)$ we have

- (i) $\operatorname{dom} \chi_k = \Delta(t_k + T^{\omega_k}(x_k))$, *and*
- (ii) $\chi_k(t) = \varphi^{\omega_k}(t - t_k, x_k)$ *for all* $t \in \operatorname{dom} \chi_k$ *with* $t \geq t_k$.

Moreover, for arbitrary $k \geq 1$ we have

- (iii) $(\chi_k)|_{\operatorname{dom} \chi_{k-1}} = \chi_{k-1}$, *as well as*
- (iv) $t_k = t_{k-1} + T^{\omega_{k-1}}(x_{k-1})$ *and* $x_k = \varphi^{\omega_{k-1}}(T^{\omega_{k-1}}(x_{k-1}), x_{k-1})$ *if* $T^{\omega_{k-1}}(x_{k-1}) < \infty$.

In particular, $\chi_x := \bigcup_{k\in\mathbb{N}} \chi_k$ is a well-defined partial map with domain $\operatorname{dom} \chi_x = [0, T_x)$, where $T_x = \sup\{\, t_k : k \in \mathbb{N} \,\}$, if the sequence (t_k) is strictly increasing, and $T_x = \infty$ otherwise.

Proof. We define the three sequences (t_k), (x_k), and (χ_k) recursively. It is straightforward to verify that $t_0 := 0$, $x_0 := x$ and

$$\chi_0 : \Delta(T^{\omega_0}(x_0)) \ni t \mapsto \varphi^{\omega_0}(t, x_0) \in X$$

satisfy properties (i-iv). Assume now that t_k, x_k, and χ_k are already defined in such a way that properties (i-iv) are satisfied. Let

$$s_k := \begin{cases} T^{\omega_k}(x_k) & \text{if } T^{\omega_k}(x_k) < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Set $t_{k+1} := t_k + s_k$, $x_{k+1} := \varphi^{\omega_k}(s_k, x_k)$, and define $\chi_{k+1} : \Delta(t_{k+1} + T^{\omega_{k+1}}(x_{k+1})) \to X$ by

$$\chi_{k+1}(t) := \begin{cases} \chi_k(t) & \text{if } t \in \operatorname{dom} \chi_k, \\ \varphi^{\omega_{k+1}}(t - t_{k+1}, x_{k+1}) & \text{otherwise.} \end{cases}$$

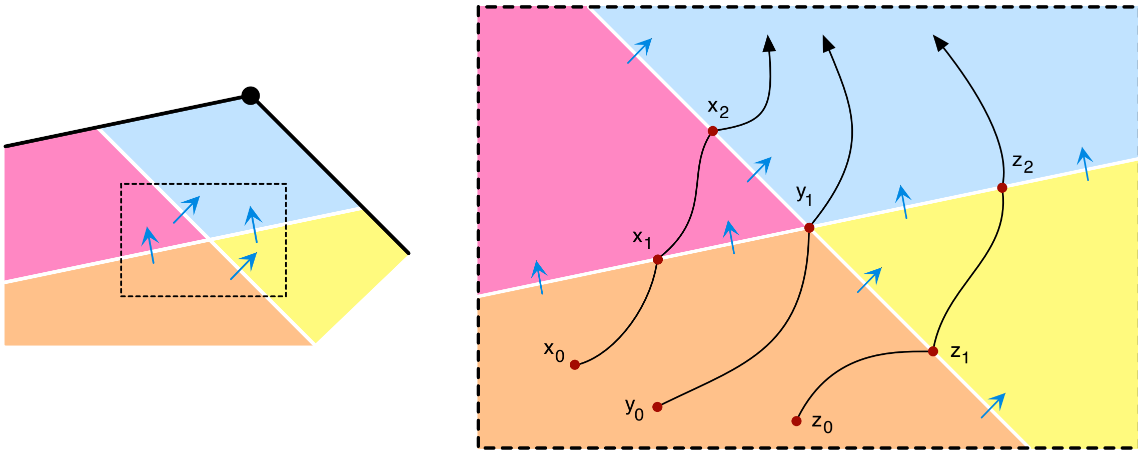

Figure 12. Construction of the final semiflow. Since in points ξ on flow tile boundaries the semiflows φ^ω have to move from $\sigma^\varepsilon_{\max}(\xi)$ to $\sigma^\varepsilon_{\min}(\xi)$, the forward continuation is uniquely determined. The left image shows the region close to a triangle vertex in a two-dimensional simplex, part of its boundary is shown in black. The white lines indicate the ε-cell boundaries. The image on the right is a larger version of the dashed region on the left. In it one can see three solution segments which all start in the lower left cell and end in the upper right cell, but in each case the cells visited along the way differ.

Again, it is straightforward to verify that t_{k+1}, x_{k+1}, and χ_{k+1} satisfy properties (i-iv).

Note that due to $\omega_k = \sigma^\varepsilon_{\min}(x_k)$ the exit time $T^{\omega_k}(x_k)$ defined in (52) is either $+\infty$ or finite and strictly positive. If for some $k \in \mathbb{N}$ we have $T^{\omega_k}(x_k) = +\infty$, then we get $s_k = 0$, and therefore $t_{k+1} = t_k$, $x_{k+1} = x_k$, and $\operatorname{dom}\chi_{k+1} = [0,\infty)$. Consequently, all these sequences are constant for $k' > k$ and $T_x = \infty$. If $T^{\omega_k}(x_k) < \infty$ for all $k \in \mathbb{N}$, then the sequence (t_k) is strictly increasing and $T_x = \sup\{\, t_k \mid k \in \mathbb{N} \,\}$. $\square$

Definition 6.9 (Construction of the Final Semiflow)**.** *Under the assumptions of Proposition 6.8 we set* $U := \{\, (t,x) \in \mathbb{R}^+_0 \times X \mid 0 \le t < T_x \,\}$*, and we define* $\varphi : U \to X$ *for* $(t,x) \in U$ *by* $\varphi(t,x) := \chi_x(t)$.

The construction of φ is visualized in Figure 12. While the intuition for the above definition is straightforward, its precise formulation may seem daunting at first. Therefore, a few comments are in order:

- As it stands, we make no claim yet that Definition 6.9 leads to a strongly admissible semiflow. We still have to show that $T_x = \infty$ for all $x \in X$, and that $\varphi : \mathbb{R}^+_0 \times X \to X$ is continuous. Both of these statements are far from obvious.
- It is, however, straightforward to verify that for all $x \in X$ we have

$$\varphi(t,\varphi(s,x)) = \varphi(t+s,x) \quad \text{for all} \quad s \in [0,T_x) \;\text{ and }\; t \in [0,T_{\varphi(s,x)}). \tag{54}$$

  This follows directly from the definition of φ and the semiflow properties of the involved semiflows φ^ω.

- By choosing the simplices ω_k in Definition 6.9 as $\sigma^\varepsilon_{\min}(x_k)$, we have $t_v(x_k) > \varepsilon$ for all vertices $v \in \omega_k$, and therefore this inequality is satisfied for all $v \in \omega_k^- \subseteq \omega_k$. In view of Lemma 6.7 this is the reason for the strict inequality $T^{\omega_k}(x_k) > 0$.
- Notice further that the choice $\omega_{k+1} = \sigma^\varepsilon_{\min}(x_{k+1})$ is the only possible choice for extending the semiflow definition in view of Corollary 6.5.

The remainder of this section is devoted to showing that Definition 6.9 does indeed define a continuous semiflow on X which is strongly admissible. This will be accomplished by localizing the problem and using the compactness of the underlying space. Our approach is based on the following two lemmas.

Lemma 6.10 (Local Semiflow in the Interior of Flow Tiles)**.** *Consider the mapping φ introduced in Definition 6.9. Furthermore, let $\omega \in \mathcal{X}$ denote an arbitrary simplex and let $x \in \operatorname{int}_X C_\omega$. Then there exists a neighborhood U of x in C_ω and a time $\tau > 0$ such that for all $y \in U$ we have $T_y \geq \tau$, and the map $\varphi : [0,\tau] \times U \to X$ is continuous.*

Proof. Let V denote an open neighborhood of x in C_ω. Choose an open set $U \subset V$ such that its closure $\operatorname{cl} U$ is still contained in V. Since X is compact, the same is true for $\operatorname{cl} U$. Due to the continuity of $\varphi^\omega : \mathbb{R}^+_0 \times \mathbb{R}^d \to \mathbb{R}^d$, which was established in Proposition 6.2, there exists a time $\tau > 0$ such that $\varphi^\omega([0,\tau], \operatorname{cl} U) \subset V$. According to $V \subset C_\omega$, this further implies the identity $\varphi(t,y) = \varphi^\omega(t,y)$ for all $t \in [0,\tau]$ and $y \in U$, and the result follows. $\square$

Lemma 6.10 shows that at an interior point of a flow tile, there is a neighborhood U of the point and a positive time τ such that φ is defined on $[0,\tau] \times U$, as well as continuous on this set. Notice, however, that both the neighborhood and the time do depend on the chosen base point x. The next result establishes an analogous result for points x which lie on the boundary of at least two flow tiles.

Lemma 6.11 (Local Semiflow near the Boundary of Flow Tiles)**.** *Consider the mapping φ introduced in Definition 6.9. Furthermore, let $x \in X$ be contained in at least two flow tiles. Then there exists a neighborhood $U \subset X$ of x and a time $\tau > 0$ such that for all $y \in U$ we have $T_y \geq \tau$, and the map $\varphi : [0,\tau] \times U \to X$ is continuous.*

Proof. Since the point is contained in at least two flow tiles, the vertex set

$$\mathcal{E} = \{ v \in \mathcal{X}_0 \;\mid\; x_v = \varepsilon \}$$

is nonempty, and the simplex set $\mathcal{X}^\varepsilon(x)$ defined in (14) contains at least the two distinct simplices $\sigma^\varepsilon_{\min}(x)$ and $\sigma^\varepsilon_{\max}(x)$ defined in (12) and (13), respectively. For example, Figure 12 illustrates the situation in which $\mathcal{E}$ contains exactly two vertices, the cell associated with $\sigma^\varepsilon_{\max}(x)$ is on the lower left, and the upper right cell is associated with $\sigma^\varepsilon_{\min}(x)$.

According to Proposition 6.4 there exists a neighborhood V of x in X and a positive constant $\alpha > 0$ such that

$$f^\omega_v(y) \leq -\alpha < 0 \quad \text{for all} \quad \omega \in \mathcal{X}^\varepsilon(x), \;\; y \in V \cap \operatorname{cl} \langle \omega \rangle_\varepsilon, \;\; \text{and all} \;\; v \in \mathcal{E}. \tag{55}$$

Since $\tilde{V} := \{ y \in X \;\mid\; v \notin \mathcal{E} \implies y_v \neq \varepsilon \}$ is open in X and clearly $x \in \tilde{V}$, by replacing V with the intersection $\tilde{V} \cap V$ we may also assume that

$$y \in V \;\text{ and }\; y_v = \varepsilon \quad \Rightarrow \quad v \in \mathcal{E}. \tag{56}$$

In addition, despite them not being continuous, all of the vector fields f^ω are uniformly bounded on X, i.e., there exists a constant $\beta > 0$ such that

$$|f^\omega(y)| \le \beta \quad \text{for all} \quad y \in X. \tag{57}$$

Based on these estimates and the definition of φ, which in turn is based on the definition of the semiflows φ^ω as almost everywhere solutions of differential equations with right-hand side f^ω, we see that

(58) the solutions $\varphi(\cdot, y)$ have a maximum speed while they move in V.

We now turn to selecting an open neighborhood $U \subset V$ of x and a time $\tau > 0$ as in the formulation of the lemma. We claim that

$$T^\omega(x) > 0 \quad \text{for all} \quad \omega \in \mathcal{X}^\varepsilon(x) \cap \left\{\sigma^\varepsilon_{\min}(x)^-,\ \sigma^\varepsilon_{\min}(x)^+\right\},$$

as well as

$$T^\omega(x) = 0 \quad \text{for all} \quad \omega \in \mathcal{X}^\varepsilon(x) \setminus \left\{\sigma^\varepsilon_{\min}(x)^-,\ \sigma^\varepsilon_{\min}(x)^+\right\}.$$

To see this assume first that $\omega \in \mathcal{X}^\varepsilon(x) \cap \{\sigma^\varepsilon_{\min}(x)^-,\ \sigma^\varepsilon_{\min}(x)^+\}$. Then in both cases $\omega^- \subset \sigma^\varepsilon_{\min}(x)$, and we get $T^\omega(x) > 0$ from the definition and properties of the exit times T^ω guaranteed by Lemma 6.7. Consider now $\omega \in \mathcal{X}^\varepsilon(x) \setminus \{\sigma^\varepsilon_{\min}(x)^-,\ \sigma^\varepsilon_{\min}(x)^+\}$. We will show that this assumption excludes the inclusion $\omega^- \subset \sigma^\varepsilon_{\min}(x)$. Otherwise, we have $\omega^- \subset \sigma^\varepsilon_{\min}(x) \subset \omega \subset \omega^+$, which in turn implies either $\sigma^\varepsilon_{\min}(x) = \omega^-$ or $\sigma^\varepsilon_{\min}(x) = \omega = \omega^+$. In either case one immediately obtains $\omega \in \{\sigma^\varepsilon_{\min}(x)^-,\ \sigma^\varepsilon_{\min}(x)^+\}$, which is a contradiction. Thus, we have $\omega^- \not\subset \sigma^\varepsilon_{\min}(x)$, and therefore there exists a vertex $v \in \omega^- \setminus \sigma^\varepsilon_{\min}(x)$. Together with $\omega \subset \sigma^\varepsilon_{\max}(x)$ we finally get $T^\omega(x) = 0$, again from Lemma 6.7.

Notice further that at least one of $\sigma^\varepsilon_{\min}(x)^-$ and $\sigma^\varepsilon_{\min}(x)^+$ are contained in $\mathcal{X}^\varepsilon(x)$, namely the simplex $\sigma^\varepsilon_{\min}(x)$, but not necessarily both. Due to the continuity of the exit times, we can therefore find a neighborhood $U \subset V$ of x in X and a time $\tau > 0$ such that

$$T^\omega(y) \ge \tau > 0 \quad \text{for all} \quad \omega \in \mathcal{X}^\varepsilon(x) \cap \left\{\sigma^\varepsilon_{\min}(x)^-,\ \sigma^\varepsilon_{\min}(x)^+\right\} \ \text{ and all } \ y \in U \cap \operatorname{cl}\langle\omega\rangle_\varepsilon, \tag{59}$$

as well as

$$T^\omega(y) < +\infty \quad \text{for all} \quad \omega \in \mathcal{X}^\varepsilon(x) \setminus \left\{\sigma^\varepsilon_{\min}(x)^-,\ \sigma^\varepsilon_{\min}(x)^+\right\} \ \text{ and all } \ y \in U \cap \operatorname{cl}\langle\omega\rangle_\varepsilon. \tag{60}$$

We will prove that by possibly shrinking U further and decreasing the value of τ, we can also achieve that

(61) no solution of φ which originates in U can reach the boundary of V within time τ.

Indeed, we get this easily from (58) if we choose U in such a way that its boundary has a positive smallest distance to the boundary of V, and we then choose $\tau > 0$ smaller than the ratio of this minimum distance and β.

With these choices we will prove that

$$\varphi(\cdot, y) \ \text{ exists at least on } \ [0, \tau] \ \text{ for all } \ y \in U. \tag{62}$$

To see this, we just need to follow the construction of this solution in Definition 6.9 based on Proposition 6.8. Let $y \in U$ be arbitrary. We have to prove that $\tau < T_y$. Assume the contrary. Then $T_y < \infty$ and the sequence (t_k) associated with y as in Proposition 6.8 is strongly increasing. In particular, for every $m \in \mathbb{N}$ we have $t_m < \tau$, which means that, due to (61), the solution always stays in V. Note, however, that, by (56), within V only solution components associated with vertices in $\mathcal{E}$ can cross ε. There are only finitely many of these thresholds and, once the v-component drops below ε, it cannot increase again while in V due to (55). Thus, there exists a $k \in \mathbb{N}$ such that $y_k := \varphi(t_k, y) \in \langle \sigma^\varepsilon_{\min}(x)\rangle_\varepsilon$. Let $\omega_k := \sigma^\varepsilon_{\min}(y_k)$. It follows from (59) that $T^{\omega_k}(y_k) \geq \tau$ and, in consequence, $t_{k+1} = t_k + T^{\omega_k}(y_k) \geq \tau$, a contradiction proving (62).

To summarize, so far we have constructed a neighborhood U of x and a time $\tau > 0$ such that all the numbered statements in this proof are satisfied. In order to complete the proof of the lemma, we only have to show that $\varphi : [0, \tau] \times U \to V$ is continuous. Notice first that in view of (57) and Definition 6.9 one has for all $(t, y), (t_0, y_0) \in [0, \tau] \times U$ the estimate

$$\begin{aligned} |\varphi(t,y) - \varphi(t_0,y_0)| &\leq |\varphi(t,y) - \varphi(t_0,y)| + |\varphi(t_0,y) - \varphi(t_0,y_0)| \\ &\leq \beta\,|t - t_0| + |\varphi(t_0,y) - \varphi(t_0,y_0)|\,. \end{aligned}$$

This immediately implies that it suffices to establish the continuity of $\varphi(t, \cdot) : U \to V$ for every fixed $t \in [0, \tau]$.

Thus, from now on we let $\tau^* \in [0, \tau]$ be arbitrary, but fixed. Furthermore, let $(y^{(k)})_{k\in\mathbb{N}}$ denote a sequence of points in U which converges to $y \in U$. Each solution $\varphi(\cdot, y^{(k)})$ exists on the interval $[0, \tau^*]$, and as outlined in Definition 6.9 based on Proposition 6.8, as t increases through this interval the solution visits a well-defined sequence of flow tiles C_ω. More precisely, in view of (56) and (55), for each k this sequence is a finite chain of nested proper faces which is contained in the simplex face poset interval with maximal simplex $\sigma^\varepsilon_{\max}(x)$ and minimal simplex $\sigma^\varepsilon_{\min}(x)$. Clearly, there are only finitely many possibilities for such chains. Therefore, the sequence $(y^{(k)})_{k\in\mathbb{N}}$ may be split into a finite number of subsequences such that for each of these subsequences the associated sequence of simplex sequences is constant. Thus, it suffices to prove that $\varphi(\tau^*, y_{(k)})$ converges to $\varphi(\tau^*, y)$ under the additional assumption that the associated simplex sequence is the same for each k.

For illustration purposes, we refer the reader again to Figure 12. In this situation, if we consider $y^{(k)} \to y_0$, we would separately discuss two subsequences of points — one which takes the route taken by the rightmost solution, and another one which takes the route of the leftmost. Note that the simplex sequence associated with the limit initial point is not required to be the same as the sequences for the solutions starting at $y^{(k)}$.

Thus, there exists a simplex chain of nested proper faces

$$\sigma^\varepsilon_{\max}(x) \supseteq \omega_0 \supsetneq \omega_1 \supsetneq \ldots \supsetneq \omega_m \supseteq \sigma^\varepsilon_{\min}(x),$$

and for every $k \in \mathbb{N}$ an associated sequence of times

$$0 = t_0^{(k)} < t_1^{(k)} < \ldots t_m^{(k)} \leq t_{m+1}^{(k)} = \tau^*$$

such that for all $\ell = 0, \ldots, m$

$$\varphi(t, y^{(k)}) \in C_{\omega_\ell} \quad \text{for} \quad t_\ell^{(k)} \leq t \leq t_{\ell+1}^{(k)},$$

and the points $y_\ell^{(k)} = \varphi(t_\ell^{(k)}, y^{(k)})$ for $\ell = 0, \ldots, m-1$ satisfy

$$y_{\ell+1}^{(k)} = \varphi^{\omega_\ell}(T^{\omega_\ell}(y_\ell^{(k)}), y_\ell^{(k)}) \quad \text{and} \quad t_{\ell+1}^{(k)} = t_\ell^{(k)} \;+\; T^{\omega_\ell}(y_\ell^{(k)}), \tag{63}$$

as guaranteed by Definition 6.9.

We will prove by induction in ℓ that for every $\ell \in \{1, 2, \ldots, m\}$ the limits $t_\ell := \lim_{k\to\infty} t_\ell^{(k)}$ and $y_\ell := \lim_{k\to\infty} y_\ell^{(k)}$ exist, $y_\ell = \varphi(t_\ell, y)$ and $\ell < m$ implies

$$t_{\ell+1} = t_\ell + T^{\omega_\ell}(y_\ell). \tag{64}$$

Since $t_0^{(k)} = 0$ we see that $\lim_{k\to\infty} t_0^{(k)}$ exists and $t_0 = 0$. Due to our choice of the sequence $(y^{(k)})_{k\in\mathbb{N}}$ we know that $y_0^{(k)} = y^{(k)} \to y$ as $k \to \infty$. Hence, $y_0 = y = \varphi(0, y) = \varphi(t_0, y)$, which means that our claim is satisfied for $\ell = 0$. Now assume that our claim holds for some index $\ell \in \{0, 1, \ldots, m-1\}$. In view of $y_\ell^{(k)} \in C_{\omega_\ell}$ and the closedness of the flow tile, one obtains the inclusion $y_\ell \in C_{\omega_\ell}$. The continuity of T^{ω_ℓ} then implies $T^{\omega_\ell}(y_\ell^{(k)}) \to T^{\omega_\ell}(y_\ell)$. Therefore, property (63) implies the convergence of $(t_{\ell+1}^{(k)})_{k\in\mathbb{N}}$ and, consequently, also property (64). Moreover, the continuity of φ^{ω_ℓ} further yields

$$y_{\ell+1}^{(k)} = \varphi^{\omega_\ell}(T^{\omega_\ell}(y_\ell^{(k)}), y_\ell^{(k)}) \to \varphi^{\omega_\ell}(T^{\omega_\ell}(y_\ell), y_\ell).$$

This proves that $(y_{\ell+1}^{(k)}))_{k\in\mathbb{N}}$ is convergent and we have

$$y_{\ell+1} = \varphi^{\omega_\ell}(T^{\omega_\ell}(y_\ell), y_\ell) = \varphi^{\omega_\ell}(T^{\omega_\ell}(y_\ell), \varphi(t_\ell, y)) = \varphi(t_\ell + T^{\omega_\ell}(y_\ell), y) = \varphi(t_{\ell+1}, y),$$

where the last equality follows from (64) and the definition of φ. Thus, we have verified our claim for $\ell + 1$, which completes our induction argument. We would like to point out that in this argument it is certainly possible that $T^{\omega_\ell}(y_\ell) = 0$. This happens for example in the situation shown in Figure 12.

Now consider the situation when we finally reach $\ell = m$. Then according to our setting we have $T^{\omega_m}(y_m^{(k)}) \geq \tau^* - t_m^{(k)}$, which in the limit $k \to \infty$ implies $T^{\omega_m}(y_m) \geq \tau^* - t_m$. This finally gives

$$\varphi(\tau^*, y^{(k)}) = \varphi^{\omega_m}(\tau^* - t_m^{(k)}, y_m^{(k)}) \to \varphi^{\omega_m}(\tau^* - t_m, y_m),$$

as well as

$$\varphi^{\omega_m}(\tau^* - t_m, \varphi(t_m, y)) = \varphi(\tau^* - t_m, \varphi(t_m, y)) = \varphi(\tau^*, y),$$

where we also used (54). This establishes the continuity of $\varphi(\tau^*, \cdot) : U \to V$, and the proof of the lemma is complete. □

After these preparations, we can now prove the main result of this paper. It shows that the mapping φ constructed at the beginning of this section is indeed a strongly admissible semiflow, which adheres to our design decisions from the introduction.

Theorem 6.12 (Existence of Strongly Admissible Semiflows)**.** *Let $\mathcal{V}$ be a combinatorial vector field on the simplicial complex $\mathcal{X}$, and let $X \subset \mathbb{R}^d$, where $d := \#\mathcal{X}_0$, denote the underlying polytope of the standard geometric realization of $\mathcal{X}$. Furthermore, suppose that we have $\varepsilon \in (0, 1/(6d))$ and consider the mapping φ introduced in Definition 6.9. Then for every $x \in X$ the solution $\varphi(\cdot, x)$ is defined on all of $\mathbb{R}_0^+$, and the resulting map $\varphi : \mathbb{R}_0^+ \times X \to X$ is continuous. In addition, we have*

$$\varphi(t, \varphi(s, x)) = \varphi(t + s, x) \qquad \text{for all} \qquad t, s \in \mathbb{R}_0^+, \tag{65}$$

and φ is strongly admissible in the sense of Definition 4.8.

Proof. Let $x \in X$ be arbitrary. According to Lemmas 6.10 and 6.11 there exists an open neighborhood U_x of x and a positive time $\tau_x > 0$ such that for every $y \in U_x$ the mapping $\varphi(\cdot, y)$ is defined at least on $[0, \tau_x]$, and $\varphi : [0, \tau_x] \times U_x \to X$ is continuous. Since X is compact, we can find a finite set $\{x_1, x_2, \ldots, x_k\} \subset X$ such that

$$X \subset U_{x_1} \cup \ldots \cup U_{x_n}.$$

If we now let

$$\tau := \min \left\{ \tau_{x_1}, \tau_{x_2}, \ldots, \tau_{x_n} \right\},$$

then clearly $\tau > 0$ and one can easily see that for every $x \in X$ the mapping $\varphi(\cdot, x)$ is defined at least on $[0, \tau]$. Iteration of this map now implies that in fact $T_x = +\infty$ for all $x \in X$. The continuity of the resulting map $\varphi : \mathbb{R}_0^+ \times X \to X$ is a direct consequence of the continuity of the restricted maps, and the semiflow property (65) follows from (54). Finally, the strong admissibility of the semiflow φ is a direct consequence of Corollary 6.5 and Proposition 6.6. This completes the proof of the theorem. $\square$

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