$\mathbb{Z}_2$-HOMOLOGY OF WEAK $(n-2)$-FACELESS n-PSEUDOMANIFOLDS MAY BE COMPUTED IN $O(n)$ TIME

MATEUSZ JUDA AND MARIAN MROZEK

Abstract. We consider the class of weak $(n-2)$-faceless n-pseudomanifolds with bounded boundaries and coboundaries. We show that in this class the Betti numbers with $\mathbb{Z}_2$ coefficients may be computed in time $O(n)$ and the $\mathbb{Z}_2$ homology generators in time $O(nm)$ where n denotes the cardinality of the n-pseudomanifold on input and m is the number of homology generators.

1. Introduction

Many computer assisted proofs in nonlinear dynamics (see for instance [1, 4, 5, 12]) are based on algorithmic computation of topological invariants such as the Conley index [2, 13, 14]. This, in particular, requires efficient algorithms computing homology groups and homology generators. The demand for fast homology algorithms originated about 20 years ago not only from rigorous numerics in nonlinear topological dynamics but also from problems in data and image analysis, electromagnetic engineering, material science and robotics.

The task of computing homology may be easily reduced to finding the Smith normal form of the matrices of the boundary maps [19, Sec. 1.11]. Unfortunately, the supercubical complexity of this process [20] results in the failure of such an approach in the presence of large input. The input in the mentioned applications usually consists of a collection of simplices or, particularly in image analysis and rigorous numerics, of cubical sets (see [10]) and its size is often measured in millions of elements and more. The problem is additionally complicated by the range of the required output, which varies from Betti numbers, through homology generators to matrices of homology maps.

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Under specific conditions homology may be computed more efficiently than by computing the Smith diagonalization. For instance, Donald and Chang [7] formalize a measure of sparseness under which the expected complexity is at most roughly quadratic in the size of input and linear in the dimension. Delfinado and Edelsbrunner show that the Betti numbers of simplicial subcomplexes of a triangulation of $\mathbb{S}^3$ may be computed in time $O(n \log^* n)$ where n is the size of the triangulation and $\log^*$ stands for the inverse of the Ackermann function. However, this requires the input triangulations to be orientable. Moreover, representing a given complex as a subcomplex of $\mathbb{S}^3$, even if such a representation exists, is not straightforward.

However, the recently proposed acyclic subspace homology algorithm [16] and the coreduction homology algorithm [17] indicate via numerical experiments that at least for low dimensional spaces the homology may be computed fast regardless of the embedding dimension. Unfortunately, so far we have no theoretical understanding of the computational complexity of these algorithms.

In the special case of compact, connected, orientable surfaces without boundary the computational complexity of computing Betti numbers may be easily shown to be linear, because in this case it easily reduces to computing the Euler characteristic. Indeed, if X is such a surface, then its first Betti number is two minus the Euler characteristic of X and the other two nonzero Betti numbers are one. Since the Euler characteristic may be computed in time $O(n)$, we conclude that for such surfaces the Betti numbers may be computed in time $O(n)$. Moreover, G. Vegter and C.-K. Yap [21] proved that the generators of the fundamental group and, via the Hurewicz Theorem, the generators of the first homology group of a surface of genus g may be constructed in time $O(n \log n + ng)$. In 2005 Erickson and Whittlesey [8] proved that the minimal homology generators of connected, compact, orientable, 2-manifolds without boundary with genus g may be computed in time $O(n^2 \log n + n^2 g + ng^3)$.

In this paper we study the homology complexity of some n-pseudomanifolds in the purely combinatorial setting of S-complexes introduced in [17]. Recall (see [11, Definition IX.8.1]) that a finite p-pseudomanifold is a regular CW-complex such that the following three conditions are satisfied

(i) Every cell is a face of some p-cell.
(ii) Every $(p-1)$-cell is a face of exactly two p-cells.

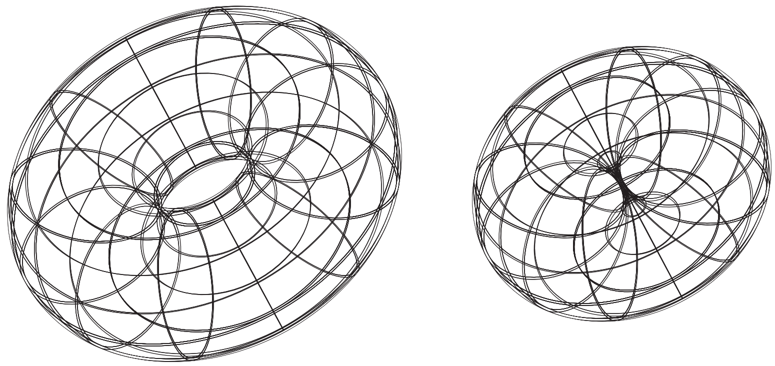

FIGURE 1. A CW-complex with the topology of a torus $S_{r_1} \times S_{r_2}$ (left) and a pseudomanifold which is not a surface (right). The right-hand example may be obtained from a representation of a 2-sphere and glueing together the poles.

(iii) Any two p-cells may be joined by a sequence of p-cells such that any two consecutive cells in the sequence have a common $(p-1)$-face.

We say that a regular CW-complex is a weak p-pseudomanifold if it satisfies the second property in the definition of a p-pseudomanifold and has no q-cells for $q > p$.

By gluing together two or more vertices of a surface we obtain an example of a 2-pseudomanifold which is not a surface (see Figure 1). By gluing two or more surfaces or 2-pseudomanifolds in a vertex we get an example of a weak 2-pseudomanifold which is not a 2-pseudomanifold (see Figure 2).

We define the counterparts of the concepts of a p-pseudomanifold and a weak p-pseudomanifold in the general setting of S-complexes and present an algorithm wich computes the $\mathbb{Z}_2$-Betti numbers of a weak $(n-2)$-faceless n-pseudomanifold in $O(n)$ time, with n denoting the number of cells, and homology generators in $O(nm)$ time, with m denoting the number of homology generators. We then discuss the complexity of a combination of the proposed algorithm with the coreduction homology algorithm presented in [17]. This sheds some light on the numerically observed high efficiency of the coreduction homology algorithm.

The organisation of the paper is as follows. In Section 2 we recall the definition and some properties of an S-complex. In the next section we define and study the connected components of an S-complex. The definition of a weak p-pseudomanifold in terms of S-complexes is given

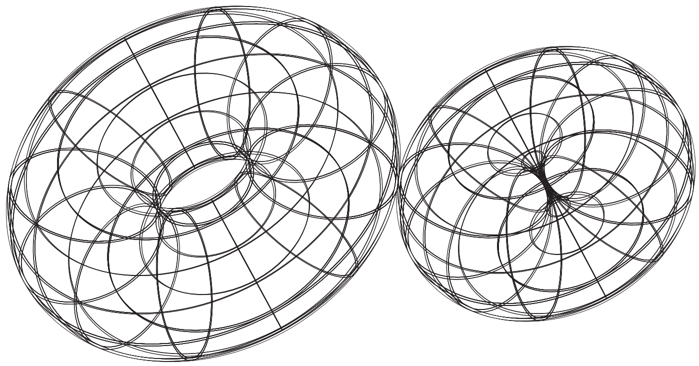

FIGURE 2. The CW-complex obtained by gluing the examples in Figure 1 in a point. It is an example of weak pseudomanifold which is not a pseudomanifold, because it does not satisfy the third condition in the definition of a p-pseudomanifold.

in Section 4. In Section 5 we present the main results concerning weak $(n-2)$-faceless n-pseudomanifolds. We then recall the coreduction homology algorithm and study some its properties in Section 6. In the following section we present the concept and properties of a geometric S-complex. Applications to the case of weak 2-pseudomanifolds are presented in Section 8. The last section contains some final comments.

2. S-COMPLEXES.

We begin with recalling from [17] the concept of an S-complex, a reformulation of chain complex suitable for algorithmic setting. Let R be a ring with unity. Given a finite set A let $R(A)$ denote the free module over R generated by A. To simplify the notation, in the sequel we identify the product of the unity of R and an element $a \in A$ with a.

Let X be a finite set with a gradation X_q for $q \in \mathbb{N}$. Then $R(X_q)$ is a gradation of $R(X)$ in the category of moduli over the ring R. For every element $x \in X$ there exists a unique number q such that $x \in X_q$. This number will be referred to as the *dimension* of x and denoted $\dim x$. For a subset A of X by A_q we mean $\{\, a \in A \mid \dim a = q \,\}$ and by the *dimension* of A denoted $\dim A$ we mean the maximum dimension of its elements. Let $\kappa : X \times X \to R$ be a map satisfying

$$\kappa(s,t) \neq 0 \;\Rightarrow\; \dim s = \dim t + 1.$$

The map κ is referred to as the *coincidence index*. If $\kappa(s,t) \neq 0$, then we say that t is a *face* of s and s is a *coface* of t.

We define the *boundary operator*

$$\partial^\kappa : R(X) \to R(X)$$

given on a generator $s \in X$ by

$$\partial^\kappa(s) := \sum_{t \in X} \kappa(s,t)t.$$

We also define the dual *coboundary operator*

$$\delta^\kappa : R(X) \to R(X)$$

defined on a generator $t \in X$ by

$$\delta^\kappa(t) := \sum_{s \in X} \kappa(s,t)s.$$

We say that (X,κ) is an *S-complex* if $(R(X), \partial^\kappa)$ is a free chain complex with base X.

We use the notation $\langle \cdot, \cdot \rangle : R(X) \times R(X) \to R$ for the scalar product defined on generators by

$$\langle t, s \rangle := \begin{cases} 1 & t = s, \\ 0 & \text{otherwise} \end{cases}$$

and extend it bilinearly to $R(X) \times R(X)$. Note that

$$\langle \partial s, t \rangle = \langle s, \delta t \rangle = \kappa(s,t).$$

Given $A \subset X$ we put

$$\begin{aligned} \operatorname{bd}_X A &:= \{\, t \in X \mid \ \kappa(s,t) \neq 0 \text{ for some } s \in A \,\}, \\ \operatorname{cbd}_X A &:= \{\, s \in X \mid \ \kappa(s,t) \neq 0 \text{ for some } t \in A \,\}. \end{aligned}$$

By the homology of an S-complex (X,κ) we mean the homology of the chain complex $(R(X), \partial^\kappa)$ and we denote it by $H(X,\kappa) := H(R(X), \partial^\kappa)$. The kernel and image of ∂^κ, i.e. the module of cycles and boundaries are denoted by $Z(X,\kappa)$ and $B(X,\kappa)$ respectively. We drop κ and write $H(X)$ and ∂ whenever κ is clear from the context. However, to emphasize the ring R used we often write $H(X,R)$ for $H(X,\kappa)$ with $\kappa : X \times X \to R$. The same convention applies to $Z(X,\kappa)$ and $B(X,\kappa)$. By $[z]_X \in H(X,\kappa)$ we mean the homology class of a cycle z and we write $[z]$ when X is clear from the context. For $R = \mathbb{Z}_2$ and a set $A \subset X$ we identify A with the chain $\sum_{a \in A} a$.

In the sequel we drop the braces in $\{\, s \,\}$, $\operatorname{bd}_X\{s\}$, $\operatorname{cbd}_X\{s\}$, $X \setminus \{\, s \,\}$, and $H(\{\, s \,\}, R)$ in the case of a singleton $\{s\} \subset X$ and write s, $\operatorname{bd}_X s$, $\operatorname{cbd}_X s$, $X \setminus s$, and $H(s,R)$ respectively. Note that when κ is given explicitly, for instance in the form of a matrix, then the S-complex is simply a chain complex with a fixed basis. However, in the context of an S-complex we assume that κ is given implicitly, via some coding of the elements of X. In particular, every simplicial complex and every cubical complex is an example of an S-complex (see [17]).

A subset X' of an S-complex X is called *regular* if for all $s, u \in X'$ and $t \in X$

$$t \in \operatorname{bd}_X s \text{ and } u \in \operatorname{bd}_X t \text{ implies } t \in X'.$$

Proposition 2.1. *(see* [17, Theorem 3.1]*) If X' is a regular subset of an S-complex X then (X', κ') where $\kappa' := \kappa|_{X' \times X'}$ is also an S-complex.*

If the map κ is clear from the context then by $\partial_{X'}$ we mean $\partial^{\kappa|_{X'\times X'}}$.

We say that $X' \subset X$ is *closed* in X if $\operatorname{bd}_X X' \subset X'$. We say that $X' \subset X$ is *open* in X if $X \setminus X'$ is closed in X.

Proposition 2.2. *(see* [17, Theorem 3.2]*) If $X' \subset X$ is closed in X, then X' and $X \setminus X'$ are regular.*

Given $A \subset X$ we define the *geometric boundary* of A by

$$\operatorname{gbd} A := \bigcup_{i=1}^{\infty} \operatorname{bd}^i A$$

where bd^i is i-th functional power of bd. It is straightforward to observe that the geometric boundary of any subset of an S-complex is closed.

As in [17] we say that a pair (a, b) of elements of X is an *elementary coreduction pair* or briefly a coreduction pair if $\kappa(b, a)$ is invertible in R and $\operatorname{bd}_X b = \{a\}$. From [17, Theorem 4.1] and [17, Corollary 3.6] we get the following proposition.

Proposition 2.3. *If (a, b) is a coreduction pair in an S-complex X then $X' := X \setminus \{a, b\}$ is a regular subset of X and $H(X)$ is isomorphic to $H(X')$.* □

3. Connected components of S-complexes

We say that two elements a, b of an S-complex X are *adjacent* if $\kappa(a, b) \neq 0$ or $\kappa(b, a) \neq 0$. This defines a symmetric relation on X. A *path* P joining $a, b \in X$ is a sequence $(p_1, p_2, \ldots, p_k)$ of elements in X such that p_i is adjacent to p_{i+1} for $i = 1, \ldots, k-1$ and $p_1 = a$, $p_k = b$. We say that such a path has *length* k. By the *dimension* of P we mean the maximum dimension of its elements.

The reflexive and transitive closure of the adjacency relation is an equivalence relation. The equivalence classes of this relation will be referred to as the *connected components* of X. We say that an S-complex X is *connected* if it is non-empty and has exactly one connected component.

Proposition 3.1. *A connected component of an S-complex X is a closed S-complex in X.*

Proof: It is straightforward to verify that a connected component of an S-complex is closed in X. The conclusion that it is an S-complex follows immediately from Proposition 2.1 and Proposition 2.2. $\square$

Lemma 3.2. *Assume $Y \subset X$ is a connected component of X. Then the inclusion $\iota : Y \to X$ induces the monomorphism*

$$\iota_* : H(Y) \to H(X).$$

Proof: From Proposition 3.1 we know that Y is closed in X. Therefore, by [17, Theorem 3.4] ι_* is a well defined homomorphism. It suffices to show $\iota_*[z]_Y = 0$ implies $[z]_Y = 0$. Let $z \in Z(Y)$ and assume $\iota_*[z]_Y = 0$. Then $[z]_X = 0$, so there exists a $c \in R(X)$ such that $\partial c = z$. We may write c as $c = c_{X \setminus Y} + c_Y$ where $c_{X \setminus Y} \in R(X \setminus Y)$ and $c_Y \in R(Y)$. Hence $\partial c = \partial c_{X \setminus Y} + \partial c_Y$ and consequently $\partial c_{X \setminus Y} = \partial c - \partial c_Y$. Since Y is a connected component of X, $X \setminus Y$ is a sum of connected components and by Proposition 3.1 the sets Y and $X \setminus Y$ are closed in X. Since $\partial c = z \in R(Y)$, we see that $\partial c - \partial c_Y \in R(Y)$. However, $\partial c_{X \setminus Y} \in R(X \setminus Y)$ which is possible only when $\partial c - \partial c_Y = 0$. Therefore, $\partial c = \partial c_Y = z$ and $[z]_Y = 0$. Thus, ι_* is a monomorphism. $\square$

Lemma 3.3. *If X^1 and X^2 are two different connected components of an S-complex X, $c \in R(X^1)$ and $x \in X^2$, then $\langle \partial c, x \rangle = 0$.*

Proof: Assume the conclusion does not hold. Since $c = \sum_{y \in X^1} \langle c, y \rangle y$, we see that

$$0 \neq \langle \partial c, x \rangle = \sum_{y \in X^1} \langle c, y \rangle \langle \partial y, x \rangle.$$

Therefore $\langle \partial y, x \rangle = \kappa(y, x) \neq 0$ for some $y \in X^1$. We get from Proposition 3.1 that $x \in X^1$, a contradiction. $\square$

The homology of an S-complex splits as the direct sum of the homologies of its connected components. More precisely, we have the following theorem.

Theorem 3.4. *Let X be an S-complex with connected components*

$$X^1, X^2, \dots, X^n.$$

Then

$$H(X) \cong \bigoplus_{i=1}^{n} H(X^i). \tag{1}$$

Proof: From Lemma 3.2 we get monomorphisms

$$\iota^i_* : H(X^i) \to H(X)$$

for $i = 1, \dots, n$. We will show that the required isomorphism is

$$\iota_* : \bigoplus_{i=1}^{n} H(X^i) \ni (\xi_i)_{i=1}^n \to \sum_{i=1}^{n} \iota^i_*(\xi_i) \in H(X).$$

It is straightforward to observe that ι_* is a monomorphism. To see that it is an epimorphism take $[z] \in H(X)$. Then $z = \sum_{i=1}^n z_i$ where z_i is a chain in X^i. We will show that z_i is a cycle in X^i. Indeed, if $\partial_{X^{i_0}} z_{i_0} \neq 0$ for some i_0, then $\langle \partial_{X^{i_0}} z_{i_0}, y\rangle \neq 0$ for some $y \in X^{i_0}$. However, by Lemma 3.3 $\langle \partial_{X^{i_k}} z_{i_k}, y\rangle = 0$ for $k \neq 0$, therefore

$$\langle \partial_X z, y\rangle = \langle \partial_X z_{i_0}, y\rangle = \langle \partial_{X^{i_0}} z_{i_0}, y\rangle \neq 0,$$

a contradiction. It follows that

$$\iota_*([z_i]_{X^i})_{i=1}^n = [z]_X.$$

□

We refer to an S-complex X as *r-faceless* if for all $q \leq r$ we have $X_q = \emptyset$. A 0-faceless S-complex X will be also referred to as *vertexless*.

Lemma 3.5. *Let $p > 0$ be a fixed integer. Let X be a p-dimensional, $(p-2)$-faceless S-complex and let $a \in X_{p-1}$. Then a and $X \setminus a$ are S-complexes, the map*

$$\bar\partial_p : H_p(X \setminus a, \mathbb{Z}_2) \ni [z] \to [\partial_X z] \in H_{p-1}(a, \mathbb{Z}_2)$$

is well defined, $H_k(X, \mathbb{Z}_2) = 0$ for $k \notin \{\, p-1, p \,\}$ and

$$H_p(X, \mathbb{Z}_2) \cong \begin{cases} H_p(X \setminus a, \mathbb{Z}_2) & \text{if } \bar\partial_p = 0, \\ \ker \bar\partial_p & \text{otherwise,} \end{cases} \tag{2}$$

$$H_{p-1}(X, \mathbb{Z}_2) \cong \begin{cases} H_{p-1}(a, \mathbb{Z}_2) \oplus H_{p-1}(X \setminus a, \mathbb{Z}_2) & \text{if } \bar\partial_p = 0, \\ H_{p-1}(X \setminus a, \mathbb{Z}_2) & \text{otherwise.} \end{cases} \tag{3}$$

Proof: To prove that $\bar\partial_p$ is well defined we first observe that a is closed in X, because X is $(p-2)$-faceless. Therefore a and $X \setminus a$ are S-complexes by Proposition 2.1 and Proposition 2.2. For $z, z' \in Z_p(X \setminus a, \mathbb{Z}_2)$ such that $[z]_{X \setminus a} = [z']_{X \setminus a}$ we will show that $\bar\partial_p[z]_{X\setminus a} = \bar\partial_p[z']_{X\setminus a}$. Since the homology classes of z and z' in $X \setminus a$ coincide, there exists a $\mathbb{Z}_2$-chain c in $X \setminus a$ such that $\partial_{X\setminus a} c = z - z'$. Therefore we get

$$\partial_X z - \partial_X z' = \partial_X \partial_{X\setminus a} c = \partial_{X\setminus a}\partial_{X\setminus a} c + \langle \partial_X \partial_{X\setminus a} c, a\rangle a = \langle \partial_X \partial_{X\setminus a} c, a\rangle a.$$

Hence $[\partial_X z]_a = [\partial_X z']_a$ and consequently $\bar\partial_p$ is well defined.

Now consider the long exact sequence (see [17, Theorem 3.4])

$$(4)\quad 0 \xrightarrow{\iota_p} H_p(X,\mathbb{Z}_2) \xrightarrow{\pi_p} H_p(X\setminus a,\mathbb{Z}_2) \xrightarrow{\bar\partial_p}$$
$$H_{p-1}(a,\mathbb{Z}_2) \xrightarrow{\iota_{p-1}} H_{p-1}(X,\mathbb{Z}_2) \xrightarrow{\pi_{p-1}} H_{p-1}(X\setminus a,\mathbb{Z}_2) \xrightarrow{\bar\partial_{p-1}} 0.$$

Obviously either $\operatorname{im}\bar\partial_p \cong 0$ or $\operatorname{im}\bar\partial_p \cong H_{p-1}(a,\mathbb{Z}_2)$. In both cases the exact sequence (4) splits into two short exact sequences. In the first case the sequences are

$$(5)\qquad 0 \xrightarrow{\iota_p} H_p(X,\mathbb{Z}_2) \xrightarrow{\pi_p} H_p(X\setminus a,\mathbb{Z}_2) \xrightarrow{\bar\partial_p} 0,$$

$$(6)\qquad 0 \to H_{p-1}(a,\mathbb{Z}_2) \xrightarrow{\iota_{p-1}} H_{p-1}(X,\mathbb{Z}_2) \xrightarrow{\pi_{p-1}} H_{p-1}(X\setminus a,\mathbb{Z}_2) \xrightarrow{\bar\partial_{p-1}} 0$$

and in the other case the sequences are

$$(7)\qquad 0 \xrightarrow{\iota_p} H_p(X,\mathbb{Z}_2) \xrightarrow{\pi_p} H_p(X\setminus a,\mathbb{Z}_2) \xrightarrow{\bar\partial_p} H_{p-1}(a,\mathbb{Z}_2) \xrightarrow{\iota_{p-1}} 0,$$

$$(8)\qquad 0 \to H_{p-1}(X,\mathbb{Z}_2) \xrightarrow{\pi_{p-1}} H_{p-1}(X\setminus a,\mathbb{Z}_2) \xrightarrow{\bar\partial_{p-1}} 0.$$

Now, we obtain (2) from (5) and (7), and (3) from (6) and (8). □

4. Weak p-pseudomanifolds.

Now we extend the concept of weak p-pseudomanifolds from CW complexes to S-complexes.

We say that an S-complex X is a *weak p-pseudomanifold* if $X_q = \emptyset$ for $q > p$ and for each $s \in X_{p-1}$ the cardinality of $\operatorname{cbd}_X s$ is exactly two.

Given an element $x \in X$ we denote by $\operatorname{cc}_X(x)$ the connected component of X to which x belongs.

Lemma 4.1. *Let X be a $(p-2)$-faceless weak p-pseudomanifold. If $a \in X_{p-1}$ is such that $\operatorname{cbd} a = \{ b_1, b_2 \}$ for some $b_1 \neq b_2$, then for the map $\bar\partial_p$ defined in Lemma 3.5 we have*

$$\bar\partial_p \neq 0 \text{ if and only if } \operatorname{cc}_{X\setminus a}(b_1) \neq \operatorname{cc}_{X\setminus a}(b_2).$$

Proof: Assume $\bar\partial_p \neq 0$. There exists a $\mathbb{Z}_2$-chain A in $X\setminus a$ such that $\bar\partial_p[A] \neq 0$. Therefore

$$0 \neq \langle \partial_X A, a\rangle = \langle A, \delta_X a\rangle = \langle A, b_1\rangle + \langle A, b_2\rangle.$$

It follows that exactly one of the two elements b_1, b_2 belongs to A, i.e.

$$\operatorname{cc}_{X\setminus a}(b_1) \neq \operatorname{cc}_{X\setminus a}(b_2).$$

The proof of the reverse implication is analogous. □

Recall that for $R = \mathbb{Z}_2$ and a set $A \subset X$ we identify A with the chain $c = \sum_{a\in A} a$, so $[A] = [c] \in H(X,\mathbb{Z}_2)$.

Theorem 4.2. *If X is a connected $(p-2)$-faceless weak p-pseudomanifold then*

$$H_p(X, \mathbb{Z}_2) = [X_p].$$

Proof: Let $X_p = \{ x_1, \ldots, x_n \}, X_{p-1} = \{ y_1, \ldots, y_m \}$ and $c = \sum_{i=1}^{n} \epsilon_i x_i$ for some $\epsilon_i \in \mathbb{Z}_2$. We will show that c is a nonzero cycle if and only if $\epsilon_i = 1$ for every $i \in \{ 1, 2, \ldots, n \}$. For this end observe that

$$\begin{aligned} \partial c &= \sum_i \epsilon_i \partial x_i \\ &= \sum_i \epsilon_i \sum_j \kappa(x_i, y_j) y_j \\ &= \sum_j \Big(\sum_i \epsilon_i \kappa(x_i, y_j) \Big) y_j \end{aligned}$$

and the latter is zero if and only if

$$\sum_i \epsilon_i \kappa(x_i, y_j) = 0 \tag{9}$$

for every $j \in \{ 1, 2, \ldots, m \}$.

Since X is a weak p-pseudomanifold, for every $j \in \{ 1, 2, \ldots, m \}$ there exist exactly two indices $i_0(j), i_1(j)$, such that

$$\kappa(x_{i_0(j)}, y_j) \neq 0 \text{ and } \kappa(x_{i_1(j)}, y_j) \neq 0$$

and consequently the equation (9) becomes

$$\epsilon_{i_0(j)} \kappa(x_{i_0(j)}, y_j) + \epsilon_{i_1(j)} \kappa(x_{i_1(j)}, y_j) = 0$$

or

$$\epsilon_{i_0(j)} + \epsilon_{i_1(j)} = 0. \tag{10}$$

Therefore, if $\epsilon_i = 1$ for all i, then equation (9) is obviously satisfied, because of the $\mathbb{Z}_2$ coefficients we use. To prove the opposite implication, assume by contrary that there exist two nonempty subsets I_0, I_1 of $I := \{ 1, \ldots, n \}$ such that $I_0 \cup I_1 = I$ and $\epsilon_i = q$ for $i \in I_q$, $q \in \{ 0, 1 \}$. Since X is connected, for some $i_0 \in I_0$ and $i_1 \in I_1$ there exists a path $P = (p_i)_{i=1}^{k} \subset X_p \cup X_{p-1}$ between x_{i_0} and x_{i_1}. Without loss of generality we may assume that P has length 3. Then $p_2 \in X_{p-1}$, in particular $p_2 = y_j$ for some $j \in \{ 1, 2, \ldots m \}$. Since X is a weak p-pseudomanifold, we get $i_0 = i_0(j)$ and $i_1 = i_1(j)$. It follows from (10) that $\epsilon_{i_0} + \epsilon_{i_1} = 0$. However, by the choice of I_0 and I_1, we have

$$\epsilon_{i_0} + \epsilon_{i_1} = 0 + 1 = 1,$$

and we get a contradiction. Therefore, X_p is the only nontrivial p-cycle in X and since there are no q-chains in X for $q > p$, the conclusion follows. □

We denote by $\mathcal{C}(X)$ the collection of connected components of X and we put $\mathcal{C}_p(X) := \{ A_p \mid A \in \mathcal{C}(X) \}$.

Corollary 4.3. *If X is a $(p-2)$-faceless weak p-pseudomanifold then*

$$H_p(X, \mathbb{Z}_2) \cong \bigoplus_{A \in \mathcal{C}_p(X)} [A].$$

Proof: The result follows immediately from Theorem 3.4 and Theorem 4.2. □

5. The gluing algorithm

In this section we present the gluing algorithm which computes homology generators in dimension p and $p-1$ for $(p-2)$-faceless p-pseudomanifold. The algorithm is based on the standard linked-list representation of disjoint sets (see [3, Chapter 21.2]) which maintains a collection $\mathtt{S} = \{ S_1, \dots, S_k \}$ of disjoint sets. Each set in $\mathtt{S}$ is identified by a representative, which is a list node. The following operations may be performed on the structure $\mathtt{S}$

- $\mathtt{S}.\,\mathrm{makeSet}(x)$ - creates a new set whose only member (and thus representative) is x,
- $\mathtt{S}.\,\mathrm{find}(x)$ - returns a pointer to the representative of the (unique) set containing x,
- $\mathtt{S}.\,\mathrm{union}(x, y)$ - unites the sets that contain x and y into a new set that is the union of these two sets,
- $\mathtt{S}.\,\mathrm{size}(x)$ - returns the size of a the set containing x.

Above operations may be implemented using a node for each element of the sets. Each node is an element of a linked list and contains an additional direct link to the first element in its list. The first element is a representative for a set. In this setting operations $\mathtt{S}.\,\mathrm{makeSet}(x)$ and $\mathtt{S}.\,\mathrm{find}(x)$, and $\mathtt{S}.\,\mathrm{size}(x)$ may be implemented in $O(1)$. The operation $\mathtt{S}.\,\mathrm{union}(x, y)$ may be implemented in $O(\min(\mathtt{S}.\,\mathrm{size}(x), \mathtt{S}.\,\mathrm{size}(y)))$ when the shorter list is attached at the end of the longer list and we update pointers to the set representative in the shorter list. More information about the data structure is available in [3, Chapter 21.2].

For a $(p-2)$-faceless weak p-pseudomanifold X we consider the graph

$$G = (X_p, \{ \{ b_1, b_2 \} \mid \{ b_1, b_2 \} = \mathrm{cbd}\, a \text{ for some } a \in X_{p-1} \}).$$

Algorithm 5.1. GetZ2Generators

```
function GetZ2Generators(S-complex X, integer p)
begin
   S := empty structure for linked-list disjoint sets;
   visited := an array of boolean indexed by elements of X;
   foreach b in X_p do
      S.makeSet(b);
   foreach b in X do
      visited[b] := false;
   foreach b in X_p do begin
      if visited[b] = true then continue;
      Q := a queue of elements of X_p;
      Q.push(b);
      visited[b] := true;
      while Q not empty do begin
         b_1 := Q.pop();
         foreach a ∈ bd b_1 do begin
            if visited[a] = true then continue;
            b_2 := the element of cbd a different from b_1;
new_a:      visited[a] := true;
            if visited[b_2] = false then
               Q.push(b_2);
            visited[b_2] := true;
            if S.find(b_1)=S.find(b_2) then
               S.makeSet(a);
            else
               S.union(b_1, b_2);
         end;
      end;
   end;
   return sets from S;
end;
```

Breadth-First Search (BFS) algorithm (see [3, Chapter 22.2]) for the graph G together with Lemma 3.5 lead to Algorithm 5.1 for computing homology groups of $(p-2)$-faceless weak p-pseudomanifolds.

Let $\mathtt{a}^i$ denote the contents of variable a on the ith pass through label new_a and let k be the number of times this label is passed. We know from analysis of BFS in [3, Chapter 22.2] that this number is finite.

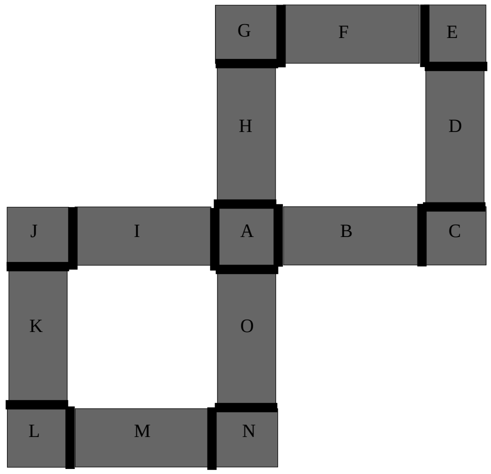

FIGURE 3. An example of cubical vertexless 2-pseudomanifold. The strips contain only cells of dimension 1 and 2.

Let $X^0 = X_p$ and for $i = 1, 2, \ldots k$ put

$$X^i := X_p \cup \{ \mathtt{a}^1, \mathtt{a}^2, \ldots, \mathtt{a}^i \}.$$

From [3, Theorem 22.5] we know that $X^k = X$. Let $\mathtt{S}^i$, $\mathtt{b}_1^i$, $\mathtt{b}_2^i$, and $\mathtt{visited}^i$ denote the contents of the variables $\mathtt{S}$, $\mathtt{b}_1$, $\mathtt{b}_2$, and $\mathtt{visited}$ respectively at the end of the third **foreach** loop after the ith pass of label new_a for $i > 0$ and before the first pass of label new_a for $i = 0$.

An example of a cubical vertexless 2-pseudomanifold T is presented in Figure 3. In this example T_2 consists of the set of grey rectangles and T_1 is the set of black bars. We label elements of T_2 with $A, B, \ldots, O$ in alphabetical order as in Figure 3 and we indicate elements of T_1 as pairs (α, β) where $\alpha, \beta \in T_2$, $\alpha < \beta$ and $\operatorname{bd}\alpha \cap \operatorname{bd}\beta \neq \emptyset$. Without loss of generality we may assume that $\mathtt{b}_1^0 = \mathtt{b}_1^1 = A$, $\mathtt{b}_2^0 = \mathtt{b}_2^1 = B$, $\mathtt{b}_2^2 = H$, $\mathtt{b}_2^3 = I$, $\mathtt{b}_2^4 = O$. When executing Algorithm 5.1 with T on input we obtain:

$$\begin{aligned}
\mathtt{S}^1 &= \{ \{ A, B \}, C, \ldots, O \}, \\
\mathtt{S}^2 &= \{ \{ A, B, H \}, C, \ldots, G, I, \ldots, O \}, \\
\mathtt{S}^3 &= \{ \{ A, B, H, I \}, C, \ldots, G, J, \ldots, O \}, \\
\mathtt{S}^4 &= \{ \{ A, B, H, I, O \}, C, \ldots, G, J, \ldots, N \}.
\end{aligned}$$

After a few iterations of the third **foreach** loop the algorithm performs $\mathtt{S}.\,\mathrm{makeSet}(.)$ for the first time. Then

$$\mathtt{S}^{14} = \{ \{ A, \ldots, K, M, N, O \}, L, (E, F) \}.$$

And the algorithm completes with

$$\mathtt{S} = \{ \{ A, \ldots, O \}, (E, F), (L, M) \}.$$

Lemma 5.1. *For $i = 0, \ldots, k$ the set X^i is an S-complex.*

Proof: We proceed by induction on i. For $i = k$ we have $X = X^k$, so X^k is an S-complex. Assume that the lemma is true for some $i \in \{ 1, 2, \ldots, k \}$. We will show that it holds also for $i - 1$. By the induction assumption X^i is an S-complex. Moreover, $\mathtt{a}^i$ is closed in X^i, hence X^{i-1} is open in X^i. By Proposition 2.1 and Proposition 2.2 the conclusion holds for $i - 1$. □

Lemma 5.2. *For $i = 0, 1, 2, \ldots, k$ and for all $S \in \mathtt{S}^i$ we have $S \subset X^i_p$ or $S \subset X^i_{p-1}$.*

Proof: We proceed by induction on i. Consider first the case $i = 0$. Before the first pass of label `new_a` no $\mathtt{S}.\operatorname{union}(\cdot, \cdot)$ operation is applied yet to the structure $\mathtt{S}$. Therefore, $\mathtt{S} = \{ \{ b \} \mid b \in X_p \}$ and the lemma holds true. Assume that it is true for $i - 1$. Let $S \in \mathtt{S}^i$. If $S \in \mathtt{S}^{i-1}$, then the conclusion holds by the induction assumption. If $S \notin \mathtt{S}^{i-1}$, then $S = \{ \mathtt{a}^i \}$ or $\operatorname{cbd} \mathtt{a}^i = \{ \mathtt{b}^i_1, \mathtt{b}^i_2 \} \subset S$. In the first case the lemma is obviously true. In the second case S is the union of $S_1 = \mathtt{S}^{i-1}.\operatorname{find}(\mathtt{b}^i_1)$ and $S_2 = \mathtt{S}^{i-1}.\operatorname{find}(\mathtt{b}^i_2)$, but by the induction assumption $S_1, S_2 \subset X^i_p$, therefore, $S = S_1 \cup S_2 \subset X^i_p$. □

Lemma 5.2 allows us to define the dimension $\dim S$ of $S \in \mathtt{S}^i$ as the common dimension of the elements of S. We put

$$\mathtt{S}^i_q := \{ S \in \mathtt{S}^i \mid \dim S = q \}.$$

Lemma 5.3. *For $i = 0, 1, 2, \ldots, k$ and for all $u, v \in X_p$*

$$\operatorname{cc}_{X^i}(u) = \operatorname{cc}_{X^i}(v) \tag{11}$$

if and only if

$$\mathtt{S}^i.\operatorname{find}(u) = \mathtt{S}^i.\operatorname{find}(v). \tag{12}$$

Proof: We proceed by induction on i. Before the first pass of label `new_a` no $\mathtt{S}.\operatorname{union}(\cdot, \cdot)$ operation is applied yet to the structure $\mathtt{S}$. Therefore $\mathtt{S} = \{ \{ b \} \mid b \in X_p \}$ and the lemma holds true for $i = 0$. Thus fix $i > 0$ and assume the conclusion holds true for $j < i$.

Let $u, v \in X_p$. First observe that properties (11) and (12) are monotone with respect to i in the sense that if the property holds for some i then it holds for $i + 1$, because the algorithm only glues sets.

Assume $\operatorname{cc}_{X^i}(u) = \operatorname{cc}_{X^i}(v)$. If $\operatorname{cc}_{X^{i-1}}(u) = \operatorname{cc}_{X^{i-1}}(v)$, the conclusion follows from the induction assumption and the monotonicity of (12). Otherwise $\operatorname{cc}_{X^{i-1}}(u) \neq \operatorname{cc}_{X^{i-1}}(v)$ and $\operatorname{cc}_{X^i}(u) = \operatorname{cc}_{X^i}(v)$, so without loss of generality we may conclude that $\mathtt{b}_1^i \in \mathrm{cc}_{X^{i-1}}(u)$ and $\mathtt{b}_2^i \in \mathrm{cc}_{X^{i-1}}(v)$. From the induction assumption

$$\mathtt{S}^{i-1}.\,\mathrm{find}(u) = \mathtt{S}^{i-1}.\,\mathrm{find}(\mathtt{b}_1^i) \neq \mathtt{S}^{i-1}.\,\mathrm{find}(\mathtt{b}_2^i) = \mathtt{S}^{i-1}.\,\mathrm{find}(v).$$

However, then the operation $\mathtt{S}.\,\mathrm{union}(\mathtt{b}_1^i, \mathtt{b}_2^i)$ occurs, therefore

$$\mathtt{S}^{i}.\,\mathrm{find}(u) = \mathtt{S}^{i}.\,\mathrm{find}(v).$$

This proves that (11) implies (12).

To prove the opposite implication assume that $\mathtt{S}^{i}.\,\mathrm{find}(u) = \mathtt{S}^{i}.\,\mathrm{find}(v)$. If $\mathtt{S}^{i-1}.\,\mathrm{find}(u) = \mathtt{S}^{i-1}.\,\mathrm{find}(v)$, then the conclusion follows from the induction assumption and the monotonicity of (11). Otherwise

$$\mathtt{S}^{i-1}.\,\mathrm{find}(u) \neq \mathtt{S}^{i-1}.\,\mathrm{find}(v) \text{ and } \mathtt{S}^{i}.\,\mathrm{find}(u) = \mathtt{S}^{i}.\,\mathrm{find}(v).$$

Without loss of generality we may assume that

$$\mathtt{S}^{i-1}.\,\mathrm{find}(u) = \mathtt{S}^{i-1}.\,\mathrm{find}(\mathtt{b}_1^i) \text{ and } \mathtt{S}^{i-1}.\,\mathrm{find}(v) = \mathtt{S}^{i-1}.\,\mathrm{find}(\mathtt{b}_2^i),$$

so from the induction assumption $u \in \mathrm{cc}_{X^{i-1}}(\mathtt{b}_1^i)$ and $v \in \mathrm{cc}_{X^{i-1}}(\mathtt{b}_2^i)$. Therefore:

$$\mathrm{cc}_{X^i}(u) = \mathrm{cc}_{X^i}(\mathtt{b}_1^i) = \mathrm{cc}_{X^i}(\mathtt{a}^i) = \mathrm{cc}_{X^i}(\mathtt{b}_2^i) = \mathrm{cc}_{X^i}(v)$$

which proves that (12) implies (11). □

Recall that for $R = \mathbb{Z}_2$ and a set $A \subset X$ we identify A with the chain $c = \sum_{a\in A} a$, so $[A] = [c] \in H(X, \mathbb{Z}_2)$.

Theorem 5.4. *Algorithm 5.1 called with a $(p-2)$-faceless weak p-pseudomanifold X returns a collection of sets* $\mathtt{S}$ *such that*

$$H(X, \mathbb{Z}_2) \cong \bigoplus_{S\in\mathtt{S}} [S]. \tag{13}$$

Proof: It is sufficient to prove that for $i = 0, 1, 2, \ldots, k$ and for $q \in \{\, p-1, p \,\}$

$$H_q(X^i, \mathbb{Z}_2) \cong \bigoplus_{S\in\mathtt{S}_q^i} [S]. \tag{14}$$

because we get (13) from (14) with $i = k$. Note that by Lemma 5.1 the homology $H_q(X^i, \mathbb{Z}_2)$ is well defined.

We proceed by induction on i. Consider first the case $i = 0$. Before the first pass of label `new_a` no $\mathtt{S}.\,\mathrm{union}(\cdot,\cdot)$ operation is applied yet to the structure $\mathtt{S}$. Therefore (14) follows immediately from Corollary 4.3 if $q = p$ while it is trivial if $q = p-1$, since X^0 does not contain elements of dimension $p-1$.

Fix $i > 0$ and assume (14) holds for $j < i$. We apply Lemma 3.5 and Lemma 4.1 with $X = X^{i-1}$ and $a = \mathtt{a}^i$. Observe that in this case $X \setminus a = X^i \setminus \mathtt{a}^i = X^{i-1}$. Let $b_1 = \mathtt{b}_1^i$ and $b_2 = \mathtt{b}_2^i$.

Consider first the case when $\mathtt{S}^{i-1}.\operatorname{find}(b_1) = \mathtt{S}^{i-1}.\operatorname{find}(b_2)$. Then, by Lemmas 4.1 and 5.3, $\bar{\partial}_p = 0$. Therefore, we get from Lemma 3.5 and the induction assumption

$$\begin{aligned} H_p(X^i, \mathbb{Z}_2) &\cong H_p(X^{i-1}, \mathbb{Z}_2) \\ &\cong \bigoplus_{S \in \mathtt{S}_p^{i-1}} [S] \cong \bigoplus_{S \in \mathtt{S}_p^{i}} [S]. \end{aligned} \tag{15}$$

By the same lemma we get

$$\begin{aligned} H_{p-1}(X^i, \mathbb{Z}_2) &\cong H_{p-1}(X^{i-1}, \mathbb{Z}_2) \oplus H_{p-1}(a, \mathbb{Z}_2) \\ &\cong \bigoplus_{S \in \mathtt{S}_{p-1}^{i-1}} [S] \oplus H_{p-1}(a, \mathbb{Z}_2). \end{aligned}$$

Since in the considered case the algorithm performs $\mathtt{S}.\operatorname{makeSet}(a)$, we see that

$$H_{p-1}(X^i, \mathbb{Z}_2) \cong \bigoplus_{S \in \mathtt{S}_{p-1}^{i}} [S]. \tag{16}$$

Consider now the case $\mathtt{S}^{i-1}.\operatorname{find}(b_1) \neq \mathtt{S}^{i-1}.\operatorname{find}(b_2)$. Then $\bar{\partial}_p \neq 0$, and consequently

$$\begin{aligned} H_{p-1}(X^i, \mathbb{Z}_2) &\cong H_{p-1}(X^{i-1}, \mathbb{Z}_2) \\ &\cong \bigoplus_{S \in \mathtt{S}_{p-1}^{i-1}} [S] \cong \bigoplus_{S \in \mathtt{S}_{p-1}^{i}} [S] \end{aligned} \tag{17}$$

by Lemma 3.5 and the induction assumption. There remains to prove that

$$H_p(X^i, \mathbb{Z}_2) \cong \bigoplus_{S \in \mathtt{S}_p^{i}} [S].$$

Let $Y_j := \operatorname{cc}_{X^{i-1}}(b_j)$ for $j = 1, 2$. Observe that by Lemma 5.3 the sets $(Y_1)_p, (Y_2)_p \in \mathtt{S}_p^{i-1}$. Let $Y := X^{i-1} \setminus (Y_1 \cup Y_2)$. Then

$$H_p(X^{i-1}) = H_p(Y) \oplus H_p(Y_1 \cup Y_2)$$

because of Theorem 3.4, and by Lemma 3.5

$$\begin{aligned} H_p(X^i, \mathbb{Z}_2) &\cong \ker \bar{\partial}_p \\ &\cong \ker \bar{\partial}_p|_{H_p(Y, \mathbb{Z}_2)} \oplus \ker \bar{\partial}_p|_{H_p(Y_1 \cup Y_2, \mathbb{Z}_2)} \\ &\cong H_p(Y, \mathbb{Z}_2) \oplus [(Y_1)_p \cup (Y_2)_p]. \end{aligned}$$

Therefore, by the induction assumption

$$\begin{aligned} H_p(X^i, \mathbb{Z}_2) &\cong \bigoplus_{S \in \mathtt{S}_p^{i-1} \setminus \{ (Y_1)_p, (Y_2)_p \}} [S] \oplus [(Y_1)_p \cup (Y_2)_p] \\ &\cong \bigoplus_{S \in \mathtt{S}_p^i} [S]. \end{aligned}$$

□

Lemma 5.5. *For $i = 0, 1, \ldots, k$ if $b \in X_p$ and $\mathtt{visited}^i[b] = \mathtt{false}$ then*

$$\mathtt{S}^i.\operatorname{size}(b) = 1.$$

Proof: Fix $i \in \{ 0, 1, \ldots, k \}$. Because the algorithm only glues sets and because of the first **foreach** loop $\mathtt{S}^i.\operatorname{size}(b) \geq 1$. Assume by contrary that $\mathtt{S}^i.\operatorname{size}(b) > 1$. Let $j \leq i$ be an integer such that $\mathtt{S}^{j-1}.\operatorname{size}(b) = 1$ and $\mathtt{S}^j.\operatorname{size}(b) > 1$. Then $b \in \{ \mathtt{b}_1^j, \mathtt{b}_2^j \}$. It means that $\mathtt{visited}^j[b] = \mathtt{true}$. The algorithm cannot change the value, so $\mathtt{visited}^i[b] = \mathtt{true}$ and we get a contradiction. □

Theorem 5.6. *Algorithm 5.1 runs in $O(n)$ time, where n denotes the cardinality of the S-complex on input.*

Proof: We call $\mathtt{S}.\operatorname{makeSet}(\cdot)$ at most $2n$ times. We call $\mathtt{S}.\operatorname{union}(\cdot,\cdot)$ and $\mathtt{S}.\operatorname{find}(\cdot)$ at most n times. By Lemma 5.5 each $\mathtt{S}.\operatorname{union}(\cdot,\cdot)$ operation is called only when one of the sets has size 1, so takes $O(1)$. By [3, Chapter 22.2] the total running time of BFS is $O(n)$ hence Algorithm 5.1 runs in $O(n)$ time. □

6. Coreduction

Algorithm 6.1 is a simple modification of the coreduction algorithm [17, Algorithm 6.1]. The modification consists, in particular, in collecting all coreduction pairs in a list. When the algorithm reduces a coreduction pair, then we add the pair to a list L.

Let $M > 0$ be a fixed integer. By $\mathcal{S}_M$ we denote the class of S-complexes X such that for each $a \in X$ the cardinalities of $\operatorname{bd} a$ and $\operatorname{cbd} a$ are bounded by M. Note that chain complexes of cubical sets (see [10]) embedded in $\mathbb{R}^d$ are elements of $\mathcal{S}_{2d}$.

Theorem 6.1. *Let $M > 0$ be a fixed integer. Algorithm 6.1 called with an S-complex $X \in \mathcal{S}_M$ and a generator $s \in X_0$ on input returns a pair (Y, L) such that $H(Y)$ is isomorphic to $H(X \setminus s)$ and L is a list of all reduction pairs removed from X by the algorithm. The algorithm runs in time $O(n)$, where n denotes the cardinality of X.*

Algorithm 6.1. Coreduction ([17, Algorithm 6.1])

```
function Coreduction (S-complex S, a generator s)
begin
   Q := empty queue of generators;
   L := empty list of coreduction pairs;
   enqueue(Q,s);
   while Q ≠ ∅ do begin
      s := dequeue(Q);
      if s ∉ S continue;
      if bd_S s contains exactly one element t then begin
         S := S \ {s};
         foreach u ∈ cbd_S t do
            if u ∉ Q then enqueue(Q,u);
         S := S \ {t};
         pushBack(L,(t,s));
      end
      else if bd_S s = ∅ then
         foreach u ∈ cbd_S s do
            if u ∉ Q then enqueue(Q,u);
   end;
   return (S,L);
end;
```

Proof: The fact that $H(Y)$ is isomorphic to $H(X \setminus s)$ follows immediately from Proposition 2.3. The fact that L is a list of all reduction pairs removed from X is obvious. The fact that the algorithm terminates, as well as the analysis of its complexity follow by the same argument as in the case of [17, Algorithm 6.1] presented in [17, Corollary 6.3]. □

Theorem 6.2. *If X on input of Algorithm 6.1 is a weak 2-pseudomanifold, then also Y returned by the algorithm is a weak 2-pseudomanifold.*

Proof: Let us assume that the Algorithm 6.1 reduces a sequence of elementary coreduction pairs $\{ (f_i, c_i) \}_{i=1}^{r}$ [17, Chapter 4]. We proceed by induction on r to show that $Y = X \setminus \bigcup_{i=0}^{r} \{ f_i, c_i \}$ is a weak 2-pseudomanifold.

For $r = 0$ we have $Y = X$ and the assertion is obvious. Therefore fix an $r > 0$ and assume $Y^j = X \setminus \bigcup_{i=0}^{j} \{ f_i, c_i \}$ is a weak 2-pseudomanifold for $j < r$. There are only three possibilities for $\{ f_r, c_r \}$:

(i) $f_r = \emptyset$ and $\dim c_r = 0$
(ii) $\dim(f_r) = 0$ and $\dim c_r = 1$
(iii) $\dim(f_r) = 1$ and $\dim c_r = 2$

We have to show that for all $e \in Y_1^r$ the cardinality of $\operatorname{cbd}_{Y^r} e$ is exactly two. In the cases (i) and (ii) $\operatorname{cbd}_{Y^r} e = \operatorname{cbd}_{Y^{r-1}} e$ for any $e \in Y_1^r$. In the third case $\operatorname{bd}_{Y^{r-1}} c_r = \{ f_r \}$, because it is a coreduction pair. Hence again $\operatorname{cbd}_{Y^r} e = \operatorname{cbd}_{Y^{r-1}} e$ for any $e \in Y_1^r$. It follows by the induction assumption that the cardinality of $\operatorname{cbd}_{Y^r} e$ is two for any $e \in Y_1^r$. Therefore $Y = Y^r$ is a weak 2-pseudomanifold. □

It is straightforward to give examples for which Algorithm 6.1 cannot reduce all input. For instance, the algorithm executed for a torus cannot reduce two strips which are shown in Figure 3. These strips have size proportional to the torus size, which means that for a big torus the remaining complex might still be too big for the Smith diagonalization algorithm. From numerical experiments we know that in most cases Algorithm 6.1 reduces its input only partially. In the sequel we present Algorithm 8.2 which can compute homology in that case much quicker than the Smith diagonalization algorithm.

7. Geometric S-complexes.

We say that an S-complex X is *geometric* if the following three conditions are satisfied:

(i) $X_q = \emptyset$ for $q < 0$,
(ii) for each $a \in X_1$ the set $\operatorname{bd} a$ consists of exactly two elements $a^-, a^+ \in X_0$ such that $\kappa(a, a^-) = -\kappa(a, a^+)$,
(iii) for each $p \geq 2$ and for each $b \in X_p$ the geometric boundary of b is connected.

It is straightforward to observe that an S-complex generated by a regular CW complex is geometric.

Theorem 7.1. *Algorithm 6.1 called with a geometric, connected S-complex X and a generator $s \in X_0$ on input returns a pair (Y, L) such that Y is a vertexless S-complex.*

Proof: The algorithm deletes the vertex s provided on input. Therefore, it is sufficient to prove that for any $u, v \in X_0$ if the coreduction algorithm deletes u, then it also deletes v. Since X is connected, there exists a path joining u and v. Let $P = (p_i)_{i=1}^k$ be such a path of minimal dimension and let the dimension be q. We claim that q is one. To see this, let $p_j \in X_q$ be a q-dimensional element of P. Then $p_{j-1}, p_{j+1} \in \operatorname{bd} p_j$ and if $q \geq 2$, then by the third property in the definition of a geometric complex there exists a path P' in $\operatorname{bd} p_j$ joining

p_{j-1} and p_{j+1}. Therefore, replacing p_j in P by P' we obtain a new path joining u and v of dimension $q-1$, a contradiction. Thus we may assume that P is of dimension one.

First consider the case $k = 3$. Since $p_1 = u$ is deleted, p_2 is placed in the queue `Q`. Suppose by contrary that $v = p_3$ is not deleted. There are two cases to consider. Either p_2 is deleted by the algorithm or it is not. The other case leads immediately to a contradiction, because then (p_2, p_3) constitutes a coreduction pair, which is removed from X when p_2 is removed from the queue `Q`. Thus, assume that p_2 is deleted. Then it is deleted in a coreduction together with its face or its coface. Since $p_2 \in X_1$ and X is geometric, the only face left for a coreduction is p_3, so in this case p_3 is deleted. Thus, assume p_2 is deleted together with its coface c. Let T denote the contents of S variable on entering the pass of the while loop on which the pair (p_2, c) is deleted by the algorithm. Since T is an S-complex and $\operatorname{bd}_T c = \{ p_2 \}$,

$$0 = \partial_T \partial_T c = \kappa(c, p_2) \partial_T p_2 = \kappa(c, p_2) \kappa(p_2, p_3) p_3 \neq 0,$$

a contradiction.

Now fix $k > 3$ and assume that the conclusion holds for all paths of length less than k. Let $P = (p_i)_{i=1}^k$ be a path of length k such that p_1 is deleted. Observe that $p_{k-1} \in X_1$ and $p_{k-2} \in X_0$. Using the induction assumption for path $P_0 = (p_i)_{i=1}^{k-2}$ of length $k-2$ and for path $P_1 = (p_i)_{i=k-2}^k$ of length 3 we conclude that p_k is deleted. □

Theorem 7.2. *If X is a geometric, connected S-complex, then $H_0(X)$ is isomorphic to R.*

Proof: First observe that since X is connected, it is nonempty and since it is geometric, $X_0 \neq \emptyset$. Let $v \in X_0$. Then v is closed in X, so we have the following exact sequence

$$0 \to H_1(X) \to H_1(X \setminus v) \xrightarrow{\bar{\partial}_1} H_0(v) \to H_0(X) \to H_0(X \setminus v) \to 0.$$

Let $z \in Z_1(X \setminus v)$. We have

$$\partial_X z = \partial_{X \setminus v} z + \alpha v$$

for some $\alpha \in R$. Since z is a cycle in $X \setminus v$, we get

$$\partial_X z = \alpha v.$$

Consider the augmentation map $\epsilon : R(X_0) \to R$ defined on generator $u \in X_0$ by $\epsilon(u) = 1$.

By assumption (ii) of a geometric S-complex we see that $\epsilon(\partial_X z) = 0$. Therefore

$$\alpha = \epsilon(\alpha v) = \epsilon(\partial_X z) = 0,$$

Algorithm 8.1. WeakPseudomanifoldBettiNumbers

```
function WeakPseudomanifoldBettiNumbers(S-complex X)
begin
   { X^1, ..., X^k } := ConnectedComponents(X);
   foreach i ∈ { 1, ..., k } do begin
      a^i := any vertex in X^i_0;
      (Y^i, L^i) := Coreduction(X^i, a^i);
   end;
   S := GetZ2Generators(⋃_{i=1}^k Y^i, 2);
   β_0 := k;
   β_1 := card S_1;
   β_2 := card S_2;
   return (β_0, β_1, β_2);
end;
```

which means that $\bar{\partial}_1 = 0$. By Theorem 6.1 and Theorem 7.1 the homology of $X \setminus v$ is isomorphic to the homology of a vertexless S-complex, so $H_0(X \setminus v)$ is zero. It follows that $H_0(X)$ is isomorphic to $H_0(v)$ and hence isomorphic to R. □

8. Weak 2-pseudomanifolds.

In this section we show how the results of the preceding section may be applied to computing homology of geometric weak 2-pseudomanifolds. The proposed algorithm is based on Algorithm 5.1 and Algorithm 6.1. We also use `ConnectedComponents` function which computes connected components of an S-complex. Note that the problem of finding the connected components of an S-complex is equivalent to finding the connected components of the graph $G = (X, E)$ where

$$E = \{ \{ x, y \} \in X \times X \mid x \text{ is adjacent to } y \}.$$

For the graph we may use `BFS` or `DFS` approach presented in [3, Chapter 22.3] and in both cases the complexity is linear.

Theorem 8.1. *Let $M > 0$ be a fixed integer. Algorithm 8.1 called with a geometric weak 2-pseudomanifold $X \in \mathcal{S}_M$ on input returns the Betti numbers of $H(X, \mathbb{Z}_2)$ in time $O(n)$, where n denotes the cardinality of X.*

Proof: By Theorem 3.4 and Theorem 7.2 the number β_0 returned by the algorithm is indeed the 0th Betti number of X. Theorems 6.2 and

Algorithm 8.2. WeakPseudomanifoldHomology

```
function WeakPseudomanifoldHomology(S-complex X)
begin
   { X^1, ..., X^k } := ConnectedComponents(X);
   L := empty list of coreduction pairs;
   foreach i ∈ { 1, ..., k } do begin
      a^i := any vertex in X_0^i;
      (Y^i, L^i) := Coreduction(X^i, a^i);
      L.append(L^i);
   end;
   S := GetZ2Generators(⋃_{i=1}^k Y^i, 2);
   G := ExtractCoreductionGenerators(S, L);
   return G;
end;
```

7.1 imply that the input of algorithm `GetZ2Generators` satisfies the assumptions of Theorem 5.4. Therefore, we get from Theorem 5.4 that β_1 and β_2 are the first and second Betti numbers of X. By [3, Chapter 22.3] the `ConnectedComponents` function may be computed in time $O(n)$. Since Theorem 6.1 implies that the `Coreduction` function calls have complexity $O(n)$, we get from Theorem 5.6 the total complexity of $O(n)$. □

We can also get the generators of $H(X, \mathbb{Z}_2)$ via a simple modification of Algorithm 8.1. For this end we need the function `Extract-CoreductionGenerators` which computes $\iota^\alpha(g)$ for all $g \in Y$ where

$$\iota^\alpha = \iota^{(a_1,b_1)} \circ \iota^{(a_2,b_2)} \circ \cdots \circ \iota^{(a_n,b_n)}$$

for $(a_i, b_i) \in L$ and

$$\iota^{(a,b)}(c) := \begin{cases} c - \frac{\langle \partial c, a\rangle}{\langle \partial b, a\rangle} b & \text{if } k = m, \\ c & \text{otherwise.} \end{cases}$$

(see [18]).

Theorem 8.2. *Algorithm 8.2 called with a geometric weak 2-pseudo-manifold $X \in \mathcal{S}_M$ on input returns the generators of $H(X, \mathbb{Z}_2)$ in time $O(nm)$, where n denotes the cardinality of the S-complex on input and m is the number of homology generators.*

Proof: By [3, Chapter 22.3] the `ConnectedComponents` function may be computed in time $O(n)$. By [17, Corollary 6.3] the `Coreduction` function calls have complexity $O(n)$. By Theorem 5.6 `GetZ2Generators` has complexity $O(n)$. `ExtractCoreductionGenerators` may be computed in $O(nm)$ (see [18, Theorem 3.1]) which results in the total complexity $O(nm)$. □

9. Final comments

An implementation by the second author of the coreduction homology algorithm, written in C++, is available from [15] and the web pages of the Computer Assisted Proofs in Dynamics Project [23, 24] and the Computational Homology Project [22]. An implementation by the first author of the adaptation of the coreduction homology algorithm to weak 2-pseudomanifolds presented in this paper is in preparation [9].

By using the elementary coreduction pairs together with elementary reduction pairs it is possible to extend the results of this paper to 2-dimensional S-complexes with the property that each edge has at most two elements in its coboundary. The details will be presented in [9].

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Mateusz Juda, Institute of Computer Science, Jagiellonian University, ul.St. Łojasiewicza 3, 30-348 Kraków, Poland

E-mail address: `Mateusz.Juda@ii.uj.edu.pl`

Marian Mrozek, Institute of Computer Science, Jagiellonian University, ul. St. Łojasiewicza 3, 30-348 Kraków, Poland and Division of Computational Mathematics, Graduate School of Business, ul. Zielona 27, 33-300 Nowy Sącz, Poland

E-mail address: `Marian.Mrozek@ii.uj.edu.pl`