INDEX PAIRS ALGORITHMS

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ABSTRACT. We introduce some modifications and extensions of the concept od index pair in the Conley index theory. We then show how these concepts may be used to overcome some difficulties in obtaining efficient algorithms computing the Conley index. We also present examples of applications to computer assisted proofs in dynamics.

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1. INTRODUCTION.

Index pairs and the associated index maps constitute the main building blocks of the Conley index theory [1, 5]. The Conley index is a topological invariant of dynamical systems used to investigate the existence and the internal structure of isolated invariant sets. In order to construct the Conley index one proves that every isolated invariant set admits at least one index pair and any two such pairs provide some common topological information which constitutes the index. There are various analytic techniques for computing the Conley index but in many concrete problems the complexity of analytic computations is too high for such an approach to be of practical value. In such situations the index may be found rigorously with the help of the computer by means of algorithmic computations. So far all algorithms computing the Conley index are based on the explicit construction of an index pair and an index map. This approach proved to be successful in many cases [4, 19, 7, 13, 2] and first general algorithms were presented in [9, 10, 19].

The efficiency of the algorithmic approach depends on the particular version of the definition of index pair. There are many different definitions of index pairs ranging from the classical concept of index

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pair introduced by Conley [1] to the most general definition given by Robbin and Salamon [14]. In general one observes the rule, that the more restrictive is the definition of index pair, the easier is the proof of the correctness of the construction of the Conley index but the less effective is the algorithmic construction of such an index pair.

From this point of view the most general definition of index pairs should be most appropriate for algorithmic computations. In particular, in the case of the dynamical systems induced by differential equations, the amount of numerical computations necessary to find the index significantly depends on the size of the exit set (second element of the index pair). The smaller the exit set, the less numerical computations are necessary. The definition of index pair introduced in [14] allows for the exit set to be a part of the boundary of the isolating neighborhood, which means that the exit set may be very thin. This makes this definition a good candidate for an effective algorithm. Unfortunately, for another reason this definition is practically useless from the algorithmic point of view. This is because it requires the verification of the continuity of a certain map, which is very difficult to achieve in finite computation.

In this paper we introduce and study the concept of weak index pair, which is a special case of the index pair in the sense of Robbin and Salamon. This concept was already announced and used in [3]. Weak index pairs have the nice feature that the exit set may be a part of the boundary of isolating neighborhood but, unlike the definition of Robbin and Salamon, it is easily algorithmizable. We also address systematically the question of the algorithmic construction of the index map. So far only ad hoc methods of finding index maps, applicable only to dimensions one and two, were used. In order to get a general method for computing index maps we introduce the concept of index quadruple and we show how one can obtain an index quadruple for various types of index pairs and how the index quadruple may be used to obtain the index map. All the constructions we present are algorithmizable and allow us to present concrete algorithms for finding index pairs and index quadruples.

The organization of the paper is as follows. We begin with recalling the quotient functor as presented in [8] and prove some characterization of continuity of certain quotient maps. In Section 3 we recall the basic concepts of the Conley index theory. In Section 4 we introduce and analyse the concept of weak index pairs. Index quadruples are presented and studied in Section 5. In the following section we recall the tools needed in raising the problem of algorithmic construction of index pairs and index maps to the combinatorial level treatable in finite computations. Section 7 presents the combinatorial analogue of the concept of index pair. The main results of the paper concerning the construction of weak index pairs and associated index quadruples via combinatorial index pairs are contained in Section 8. The next section discusses an important special case: the case of isolating blocks. We present here an approach which enables algorithmic computation of the Conley index by finding the bounds of the flow only on the boundary of an isolating block. This approach is essential in situations when finding bounds for the flow is computationally expensive. Sample algorithms based on the theory introduced in the paper are presented in Section 10. Finally in the last section we discuss two examples. The first example concerns the Hénon map and shows the results of algorithmic computations based on weak index pairs which yield the existence of a chaotic invariant set. A similar example, but without the necessary theoretical background introduced in this paper, was already presented in [3]. The other example shows the usefulness of the algorithms based on the concept of an isolating block. The example is discussed only briefly. All the details will be presented in [11].

Throughout the paper $\mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}^-, \mathbb{R}$ denote respectively the sets of integers, non-negative integers, non-positive integers and real numbers. Given a subset $A \subset \mathbb{R}^d$ by $\operatorname{cl} A, \operatorname{int} A, \operatorname{ext} A, \operatorname{bd} A$ we denote respectively the closure, the interior, the exterior and the boundary of A. For $A \subset X \subset \mathbb{R}^d$ the notation $\operatorname{bd}_X A$ stands for the boundary of A in X.

2. The category of Pairs and the Quotient Functor.

Let us recall the category Prs, introduced in [8] and defined as follows. The objects of Prs are pairs of compact topological spaces (P_1, P_2) such that $P_2 \subset P_1$. The set of morphisms from $P = (P_1, P_2)$ to $Q = (Q_1, Q_2)$ consists of all partial continuous maps $h : P_1 \nrightarrow Q_1$ such that

$$\operatorname{dom} h \text{ is closed in } P_1 \tag{1}$$

$$h(P_2) \subset Q_2 \tag{2}$$

$$h(\operatorname{bd}_{P_1}(\operatorname{dom} h)) \subset Q_2 \tag{3}$$

Observe that the identity map $\mathrm{id} : P_1 \to P_1$ is the identity morphism of $P = (P_1, P_2)$ in Prs.

Given $\alpha : P \to Q$ and $\beta : Q \to R$ morphisms in Prs, we take as the composition $\beta\alpha$ the mapping

$$\beta\alpha : \alpha^{-1}(\operatorname{dom} \beta) \ni x \to \beta(\alpha(x)) \in R_1$$

It is straightforward to verify that Prs constitutes a category. Given a topological space X we will denote by Prs_X the subcategory of pairs (P_1, P_2) satisfying $P_1 \subset P_2 \subset X$.

Let $*$ denote a point, which does not belong to X. Given a subset $A \subset X$ let A^* denote $A \cup \{*\}$.

Let $P \in \mathrm{Prs}_X$. The quotient space P_1/P_2 is defined as the set of equivalence classes of the smallest equivalence relation in P_1^* containing all the pairs $(x, *)$ for all $x \in P_2$. It is straightforward to verify that the equivalence class of $x \in P_1^*$ is

$$[x] = \begin{cases} \{x\} & \text{if } x \in P_1 \setminus P_2, \\ P_2^* = [*] & \text{otherwise.} \end{cases}$$

In the quotient space P_1/P_2 we introduce the strongest topology under which the map

$$q_P : P_1^* \ni x \to [x] \in P_1/P_2$$

is continuous.

The following proposition is straightforward.

Proposition 2.1. *A map $g : P_1/P_2 \to Z$ is continuous if and only if gq_P is continuous.*

For $P, Q \in \mathrm{Prs}$ and $h : P \to Q$, a morphism in Prs put

$$P^\wedge := (P_1/P_2, P_2^*)$$

$$h^\wedge([x]) := \begin{cases} q_Q(h(x)) & \text{for } x \in \operatorname{dom} h, \\ Q_2^* & \text{otherwise.} \end{cases}$$

Let Comp_* denote the category of compact pointed spaces. We define a functor $\mathrm{Quot} : \mathrm{Prs} \to \mathrm{Comp}_*$ by

$$\begin{aligned} \mathrm{Quot}(P) &:= P^\wedge \text{ for } P \in \mathrm{Prs} \\ \mathrm{Quot}(h) &:= h^\wedge \text{ for } h \in \mathrm{Prs}(P, Q). \end{aligned}$$

Proposition 2.2. *(*[8, Corollary 5.3]*)* Quot : Prs $\to$ Comp_* *is a well defined covariant functor.*

Let $f : P \to Q$ be a continuous map of pairs. We say that f is *excisive* if f is closed and f restricted to $P_1 \setminus P_2$ is a homeomorphism onto $Q_1 \setminus Q_2$. One can easily verify the following characterization of excisive inclusions.

Proposition 2.3. *Assume $P, Q \in \mathrm{Prs}$ are such that $P \subset Q$. The inclusion $i : P \hookrightarrow Q$ is excisive if and only if one of the following conditions is satisfied*

$$P_1 \setminus P_2 = Q_1 \setminus Q_2$$

or

$$Q_1 \setminus P_1 = Q_2 \setminus P_2.$$

In the sequel excisive inclusions will be briefly called *excisions*.

Let $T : \mathrm{Prs} \to \mathcal{E}$ be a functor. We say that T is *excisive* if $T(f)$ is an isomorphism for every excisive map f.

Theorem 2.4. *(*[8, Theorem 5.5]*) The functor* $\mathrm{Quot} : \mathrm{Prs} \to \mathrm{Comp}_*$ *is excisive.*

Assume $f : X \to X$ is a continuous map. Let $P = (P_1, P_2) \in \mathrm{Prs}$ and define $f_P : P_1/P_2 \to P_1/P_2$ by

$$f_P([x]) := \begin{cases} [f(x)] & \text{if } x \in P_1 \cap f^{-1}(P_1) \\ P_2^* & \text{otherwise.} \end{cases} \tag{4}$$

Lemma 2.5. *The map f_P is well defined if and only if*

$$f(P_2) \cap P_1 \subset P_2. \tag{5}$$

Proof: If f_P is well defined, then for any $y \in f(P_2) \cap P_1$ there exists an $x \in P_2$ such that

$$y = f(x) \in [f(x)] = f_P([x]) = f_P([*]) = P_2^* = P_2 \cup \{*\}.$$

But $y \neq *$, therefore (5) is proved.

To prove the opposite implication it is enough to show that $f_P([x]) = P_2^*$ for $x \in P_2$, which is obvious if $f(x) \notin P_1$ and follows from (5) if $f(x) \in P_1$. □

Given a set $A \subset X$, define its f*-boundary* by

$$\mathrm{bd}_f\, A := \mathrm{cl}\, A \cap \mathrm{cl}(f(A) \setminus A).$$

Lemma 2.6. *Assume $f : X \to X$ is continuous and $N \subset X$ is compact. Then*

$$f(\mathrm{bd}_N(N \setminus f^{-1}(N))) \subset \mathrm{bd}_f\, N.$$

Proof: Let $x \in \mathrm{bd}_N(N \setminus f^{-1}(N))$. Since

$$\mathrm{bd}_N(N \setminus f^{-1}(N)) = \mathrm{cl}(N \setminus f^{-1}(N)) \cap N \cap f^{-1}(N),$$

it follows that $f(x) \in N$. On the other hand there exists a sequence $\{x_n\} \subset N \setminus f^{-1}(N)$ such that $x_n \to x$. Therefore $f(x) = \lim f(x_n) \in \mathrm{cl}(f(N) \setminus N)$ and consequently $f(x) \in \mathrm{bd}_f\, N$. □

Lemma 2.7. *The map f_P is continuous if and only if*

$$\mathrm{bd}_f\, P_1 \subset P_2. \tag{6}$$

Proof: Take a $y \in \operatorname{bd}_f P_1$. Then we can choose a sequence $\{x_n\} \subset P_1 \setminus f^{-1}(P_1)$, convergent to some $x \in P_1$ and such that $y = \lim f(x_n) = f(x)$. If f_P is continuous, then

$$[y] = [f(x)] = f_P([x]) = \lim f_P([x_n]) = \lim\, [*] = P_2^*.$$

Thus $y \in P_2$, which proves (6).

Assume in turn that (6) is satisfied. Then

$$\operatorname{bd}_{P_1}(P_1 \cap f^{-1}(P_1)) \subset f^{-1}(P_2). \tag{7}$$

Indeed, if $x \in \operatorname{bd}_{P_1}(P_1 \cap f^{-1}(P_1)) = \operatorname{bd}_{P_1}(P_1 \setminus f^{-1}(P_1))$, then by Lemma 2.6 we get $f(x) \in \operatorname{bd}_f P_1 \subset P_2$.

Define maps

$$\tilde{f}_1 : P_1 \cap f^{-1}(P_1) \ni x \to [f(x)] \in P_1/P_2,$$
$$\tilde{f}_2 : \operatorname{cl}(P_1 \setminus f^{-1}(P_1)) \ni x \to P_2^* \in P_1/P_2.$$

Obviously these maps are continuous and

$$\operatorname{dom} \tilde{f}_1 \cap \operatorname{dom} \tilde{f}_2 = \operatorname{bd}_{P_1}(P_1 \cap f^{-1}(P_1)).$$

It follows from (7) that $\tilde{f}_1(x) = \tilde{f}_2(x)$ for $x \in \operatorname{bd}_{P_1}(P_1 \cap f^{-1}(P_1))$. Therefore we have a well defined and continuous map $\tilde{f} : P_1^* \to P_1/P_2$ given by

$$\tilde{f}(x) := \begin{cases} \tilde{f}_1(x) & \text{if } x \in P_1 \cap f^{-1}(P_1), \\ \tilde{f}_2(x) & \text{if } x \in \operatorname{cl}(P_1 \setminus f^{-1}(P_1)), \\ P_2^* & \text{if } x \in \{*\}. \end{cases}$$

It is straightforward to verify that $\tilde{f} = f_P q_P$, therefore the conclusion follows from Proposition 2.1. $\square$

3. Isolating neighborhoods and the Conley index.

Let X denote a fixed, locally compact, metrizable space. By a *local discrete semidynamical system* on X we mean a continuous map

$$f : U \to X$$

defined on some open subset U of X. We say that the function $\sigma : \mathbb{Z} \to X$ is a *solution* to f through x in $N \subset X$ if $f(\sigma(i)) = \sigma(i+1)$ for all $i \in \mathbb{Z}$, $\sigma(0) = x$ and $\sigma(i) \in N$ for all $i \in \mathbb{Z}$. The *invariant part* of $N \subset X$ with respect to f is defined as the set of all $x \in N$ which admit a solution to f through x in N. It will be denoted by $\operatorname{Inv}(N, f)$. The set S is said to be *invariant* if $f(S) = S$. This is easily seen to be equivalent to $S = \operatorname{Inv}(S, f)$. The set S is called *isolated invariant*, if it admits a compact neighborhood N such that $S = \operatorname{Inv}(N, f)$. The neighborhood N is then called an *isolating neighborhood* of S.

Let $N \subset X$ be an isolating neighborhood for f. The following definition of index pair for discrete dynamical systems is modelled on the classical definition of Conley [1].

Definition 3.1. A pair of compact sets $P = (P_1, P_2)$, where $P_2 \subset P_1 \subset N$ is called an *index pair for f in N* if the following three properties are satisfied.

(i) $f(P_i) \cap N \subset P_i$,
(ii) $P_1 \setminus f^{-1}(P_1) \subset P_2$,
(iii) $\mathrm{Inv}(N, f) \subset \mathrm{int}(P_1 \setminus P_2)$.

One can prove that every isolating neighborhood admits at least one index pair [8].

For P, an index pair for f, we have an associated object

$$(P^{\wedge}, f_P) \in \mathrm{Endo}(\mathrm{Quot}(\mathrm{Prs})),$$

which we will denote by P_f. This object carries all the information needed to define the Conley index. To do so we need to recall first some concepts. Let $\mathcal{E}$ be a category. By $\mathrm{Endo}(\mathcal{E})$ we mean the category whose objects are the endomorphisms of $\mathcal{E}$ and morphisms are the morphisms of $\mathcal{E}$ which commute with the endomorphisms. By $\mathrm{Auto}(\mathcal{E})$ we mean the restriction of $\mathrm{Endo}(\mathcal{E})$ to automorphisms (see [8] for the detailed definitions).

Let $T : \mathrm{Comp}_* \to \mathcal{E}$ be a homotopy invariant functor. In the sequel, in order to fix the notation, we assume that T is covariant. However, it is straightforward to obtain analogous results for contravariant functors. The functor T extends in a natural way to a functor $T : \mathrm{Endo}(\mathrm{Comp}_*) \to \mathrm{Endo}(\mathcal{E})$ denoted by the same letter. Assume also that $\mathcal{C} \subset \mathrm{Endo}(\mathrm{Comp}_*)$ is a subcategory such that $T(\mathrm{Endo}(\mathrm{Comp}_*)) \subset \mathcal{C}$. Let $L : \mathcal{C} \to \mathrm{Auto}(\mathcal{E})$ be a normal functor as defined in [8]. Let $LT := L \circ T : \mathrm{Endo}(\mathrm{Comp}_*) \to \mathrm{Auto}(\mathcal{E})$ denote the composite functor.

Theorem 3.2. [8, Theorem 1.7] *Assume S is an isolated invariant set with respect to f. Then $LT(P_f)$ and $LT(Q_f)$ are isomorphic objects in $\mathrm{Auto}(\mathcal{E})$ for any isolating neighborhoods N, M of S and index pairs P in N and Q in M.*

The common value $LT(P_f)$ for all index pairs P of S is called the (L,T)-*Conley index* of S and denoted by $C_{L,T}(S, f)$.

The Conley index is used to detect the existence of invariant sets and to study their internal structure. As observed by Szymczak [18], when the isolating neighborhood decomposes into a finite union of compact sets, one can refine the information carried by the Conley index by taking into account the decomposition in the construction of the index. We briefly recall this construction.

Given a finite set A and a category $\mathcal{E}$ let $\mathcal{E}_{(A)}$ denote the category whose objects are the objects of $\mathcal{E}$ and morphisms are collections of morphisms in $\mathcal{E}$ indexed by finite sequences of elements of A (see [18] for the details).

Let K be a compact set. A finite collection $\{K_j\}_{j\in J}$ of pairwise disjoint compact subsets of K is called a *decomposition* of K if $K = \bigcup\{K_j \mid j \in J\}$.

Assume we are given a fixed decomposition $\mathcal{S} := \{S_j\}_{j\in J}$ of an isolated invariant set S. Let $P \in \mathrm{Prs}$. We say that P is $\mathcal{S}$*-compatible* if $\mathrm{cl}(P_1 \setminus P_2)$ admits a decomposition $\{K_j\}_{j\in J}$ such that $S_j = S \cap K_j$ for every $j \in J$. Given an $\mathcal{S}$-compatible pair P it is straightforward to verify that for $j \in J$ the formula

$$r_j(x) = x \text{ for } x \in K_j. \tag{8}$$

defines a morphism $r_j : P \multimap P$ in the category Prs. Assume that f_P is well defined and continuous. Then

$$f_{P,J} := \{f_{P,j}\}_{j\in J},$$

where

$$f_{P,j} := f_P r_j \text{ for } j \in J,$$

is a morphism in $\mathrm{Prs}_{(J)}$, which gives rise to an object $LT(P^\wedge, f^\wedge_{P,J})$ of $\mathrm{Auto}(\mathcal{E}_{(J)})$. Note that the morphisms r_j and consequently also $f_{P,J}$ may depend on a particular decomposition of $\mathrm{cl}(P_1 \setminus P_2)$ compatible with the decomposition of S.

We have now the following refinement of Theorem 3.2.

Theorem 3.3. [18, Theorem 3.1] *Assume S is an isolated invariant set with respect to f and $\mathcal{S} = \{S_j\}_{j\in J}$ is a fixed decomposition of S. Then* $LT(P^\wedge, f^\wedge_{P,J})$ and $LT(Q^\wedge, f^\wedge_{Q,J})$ *are isomorphic objects in* $\mathrm{Auto}(\mathcal{E}_{(J)})$ *for any isolating neighborhoods N, M of S, $\mathcal{S}$-compatible index pairs P in N and Q in M, and respective decompositions of* $\mathrm{cl}(P_1 \setminus P_2)$ *and* $\mathrm{cl}(Q_1 \setminus Q_2)$.

The common value $LT(P^\wedge, f^\wedge_{P,J})$ for all $\mathcal{S}$- compatible index pairs P is called the (L,T)-*Conley index for decompositions* of S and denoted by $C_{L,T}(\mathcal{S}, f)$.

Given a finite sequence $\theta = (\theta_1, \theta_2, \dots \theta_k)$ of elements of A we define the morphism $f_{P,\theta} : P \multimap P$ as the composition

$$f_{P,\theta} := f_{P,\theta_k} \circ f_{P,\theta_{k-1}} \circ \cdots \circ f_{P,\theta_1}.$$

The usefulness of this map comes from the fact that if T is a homology or cohomology functor and the space X is a compact ANR then a nonzero Lefschetz number of $T(f_{P,\theta})$ implies the existence of a periodic point x in S following the itinerary given by θ (see [18, Theorem 4.5]).

4. Weak index pairs.

Sometimes it is convenient to consider the concept of index pair independently of a particular isolating neighborhood. For this reason we recall the following definition.

Definition 4.1. An *index pair* for f is a pair of compact sets $P = (P_1, P_2)$, where $P_2 \subset P_1$, satisfying the following three properties:

(i) $f(P_2) \cap P_1 \subset P_2$,
(ii) $P_1 \setminus f^{-1}(P_1) \subset P_2$,
(iii) $\mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_2), f) \subset \mathrm{int}(P_1 \setminus P_2)$.

It is straightforward to observe that if P is an index pair for f then $\mathrm{cl}(P_1 \setminus P_2)$ is an isolating neighborhood for f.

Note that condition (ii) of Definition 4.1 forces the set P_2 in an index pair (P_1, P_2) to be relatively thick. It must be thick enough to catch every trajectory which leaves P_1. As a consequence, an algorithmic construction of such an index pair may be computationally expensive. To overcome this difficulty we first recall the definition of index pair introduced in [14]. The pair $P = (P_1, P_2)$ is called an *index pair in the sense of* Robbin and Salamon if the map

$$P_1/P_2 \ni [x] \to \begin{cases} [f(x)] & \text{if } x \in (P_1 \setminus P_2) \cap f^{-1}(P_1 \setminus P_2) \\ P_2^* & \text{otherwise.} \end{cases} \tag{9}$$

is continuous. The advantage of this type of index pairs is the fact that P_2 may be only a part of the boundary of P_1. The disadvantage lies in the difficulty of verifying algorithmically the continuity condition. For this reason we introduce the concept of weak index pair, which shares the nice features of Robbin-Salamon index pairs but at the same time, as we show in the sequel, may be constructed algorithmically.

Definition 4.2. A *weak index pair* for f is a pair of compact sets $P = (P_1, P_2) \in \mathrm{Prs}$ satisfying conditions (i), (iii) of Definition 4.1 together with

(ii') $\mathrm{bd}_f\, P_1 \subset P_2$.

This definition of weak index pair is motivated by [15]. By Lemma 2.5 and Lemma 2.7 the map f_P in the case of a weak index pair is well defined and continuous. In particular the object P_f is also well defined.

By Definition 4.1(i) f_P coincides with the map defined by (9). Therefore weak index pairs constitute a special case of the index pairs in the sense of Robbin and Salamon.

Let S be an isolated invariant set for f. We say that P is a *(weak) index pair for f and S* if P is a (weak) index pair for f and $S = \mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_2), f)$.

Proposition 4.3. *Every index pair for f and S is a weak index pair for f and S.*

Proof: Take a $y \in \mathrm{bd}_f P_1$. Then we can find a sequence $\{x_n\} \subset P_1 \setminus f^{-1}(P_1)$ convergent to some $x \in P_1$ and such that $y = \lim f(x_n) = f(x)$. By Definition 4.1(ii) we have $\{x_n\} \subset P_2$. Hence $x \in P_2$ and by Definition 4.1(i) we get $y = f(x) \in P_2$. $\square$

Another concept of index pair was introduced by Szymczak. A pair of compact sets $P = (P_1, P_2)$, where $P_2 \subset P_1$ is an *index pair in the sense of* Szymczak [19] if it satisfies conditions (i), (iii) of Definition 4.1 and

(ii") $f(P_1 \setminus P_2) \cap (P_1 \setminus P_2) \subset \mathrm{int}\, P_1$.

Theorem 4.4. *Every index pair in the sense of Szymczak is a weak index pair.*

Proof: Let (P_1, P_2) be an index pair for f in the sense of Szymczak. We need to verify (ii'). Let $y \in \mathrm{bd}_f P_1$. Then $y \in P_1$ and there exists a sequence $\{x_n\} \subset P_1$ converging to $x \in P_1$ such that $y = \lim f(x_n) = f(x)$. If $x \in P_2$ then $y \in f(P_2) \cap P_1 \subset P_2$. Otherwise $y \in f(P_1 \setminus P_2)$ and from (ii") we also conclude that $y \in P_2$. $\square$

The converse of Theorem 4.4 is not true as the following two examples show.

Example 4.5. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 6x - 2 & \text{if } x \leq 1/2, \\ -6x + 4 & \text{if } x \geq 1/2,. \end{cases}$$

Let $P_1 = [0, 1]$ and $P_2 = \{0\}$. Then $P := (P_1, P_2)$ is a weak index pair for f but P is not an index pair in the sense of Szymczak, because $1 \in f(P_1 \setminus P_2) \cap (P_1 \setminus P_2)$ but $1 \notin \mathrm{int}[0, 1]$.

Example 4.6. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := x - 1$. Let $P_1 := [0, 1] \cup \{2\}$ and $P_1 := \{0\}$. Then (P_1, P_2) is a weak index pair for f but it is not an index pair in the sense of Szymczak, because $1 \in f(P_1 \setminus P_2) \cap (P_1 \setminus P_2)$ but $1 \notin \mathrm{int}\, P_1$.

Assume $P = (P_1, P_2)$ and $Q = (Q_1, Q_2)$ are weak index pairs for f. Define the partial map $f_{PQ} : P_1 \nrightarrow Q_1$ by

$$\operatorname{dom} f_{PQ} := P_1 \cap f^{-1}(Q_1)$$

and

$$f_{PQ}(x) := f(x) \text{ for } x \in \operatorname{dom} f_{PQ}.$$

We have the following

Proposition 4.7. *If $P \subset Q$ are weak index pairs for f such that*

$$P_2 \cap f^{-1}(Q_1) \subset f^{-1}(Q_2),$$

then f_{PQ} is a morphism in $\operatorname{Prs}(P, Q)$.

Proof: Since obviously $\operatorname{dom} f_{PQ}$ is closed in P_1, we need to verify only properties (2) and (3). We have

$$f_{PQ}(P_2) = f(P_2 \cap f^{-1}(Q_1)) \subset f(f^{-1}(Q_2)) \subset Q_2$$

which proves (2). By Lemma 2.6 and Definition 4.1(ii')

$$\begin{gathered} f(\operatorname{bd}_{P_1}(\operatorname{dom} f_{PQ})) = f(\operatorname{bd}_{P_1}(P_1 \cap f^{-1}(Q_1))) = \\ f(P_1 \cap f^{-1}(Q_1) \cap \operatorname{cl}(P_1 \setminus f^{-1}(Q_1)) \subset \\ f(Q_1 \cap f^{-1}(Q_1) \cap \operatorname{cl}(Q_1 \setminus f^{-1}(Q_1)) = \\ f(\operatorname{bd}_{Q_1}(Q_1 \setminus f^{-1}(Q_1))) = \operatorname{bd}_f Q_1 \subset Q_2 \end{gathered}$$

which proves (3). □

Proposition 4.8. *If P is a weak index pair for f then f_{PP} is a morphism in $\operatorname{Prs}(P, P)$. Moreover, $f_{PP}^{\wedge} = f_P$.*

Proof: Proposition 4.7 implies that f_{PP} is a morphism in $\operatorname{Prs}(P, P)$. The fact that $f_{PP}^{\wedge} = f_P$ follows from the definition of the quotient functor and (4). □

The following two lemmas show how a weak index pair may be reconstructed to an index pair and vice versa.

Lemma 4.9. *Assume $P = (P_1, P_2)$ is a weak index pair for f and S. Then $P^+ := (P_1, P_2^+)$, where*

$$P_2^+ := \operatorname{cl}(P_1 \setminus f^{-1}(P_1)) \cup P_2,$$

is an index pair for f and S and

$$LT(P_f) \cong LT(P_f^+). \tag{10}$$

Moreover, if $\operatorname{cl}(P_1 \setminus P_2) = P_1$, then P_1 is an isolating neighborhood for f isolating S and P^+ is an index pair in P_1.

Proof: First we will show that P^+ is an index pair for f. To prove (i) take $y \in f(P_2^+) \cap P_1$. Then $y = f(x)$ for some $x \in P_2^+$. If $x \in P_2$, then by the relative positive invariance of P_2 we get $y \in P_2 \subset P_2^+$. Thus assume $x \notin P_2$. Then $x \in \mathrm{cl}(P_1 \setminus f^{-1}(P_1))$ and since $x \in f^{-1}(P_1)$, we see that $x \in \mathrm{bd}_{P_1}(P_1 \setminus f^{-1}(P_1))$. It follows from Lemma 2.6 that $f(x) \in \mathrm{bd}_f P_1 \subset P_2 \subset P_2^+$. Thus (i) is proved.

Property (ii) is immediate. Before we prove (iii) let us observe that

$$\mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_2^+), f) \subset \mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_2), f) \tag{11}$$

and by (iii) applied to P

$$\mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_2), f) \subset \mathrm{int}(P_1 \setminus P_2) = \mathrm{int}\, P_1 \setminus P_2. \tag{12}$$

We will prove that

$$\mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_2), f) \subset \mathrm{int}(P_1 \setminus P_2^+). \tag{13}$$

Assume the contrary. Then there exists an $x \in \mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_2), f)$ such that $x \notin \mathrm{int}(P_1 \setminus P_2^+) = \mathrm{int}\, P_1 \setminus P_2^+$. It follows from (12) that $x \in P_2^+ \setminus P_2$, therefore $x \in \mathrm{cl}(P_1 \setminus f^{-1}(P_1))$. Since $x \in P_1 \cap f^{-1}(P_1)$ we see that $x \in \mathrm{bd}_{P_1}(P_1 \setminus f^{-1}(P_1))$. Thus $f(x) \in \mathrm{bd}_f(P_1) \subset P_2$ by Lemma 2.6. Again by (12) we get $f(x) \notin \mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_2), f)$, which contradicts $x \in \mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_2), f)$ and proves (13). Properties (11) and (13) imply (iii) and

$$\mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_2^+), f) = \mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_2), f).$$

This in particular shows that P^+ is an index pair for f and S.

By Proposition 4.7 there are well defined morphisms f_{PP}, f_{PP^+} and $f_{P^+P^+}$. Let $\iota : P \hookrightarrow P^+$ denote the inclusion map. We have the following commutative diagram of morphisms in Prs.

$$\begin{tikzcd}
P^+ \arrow[r, "f_{P^+P^+}"] & P^+ \\
P \arrow[u, "\iota"] \arrow[ur, "f_{PP^+}"] \arrow[r, "f_{PP}"] & P \arrow[u, "\iota"']
\end{tikzcd}$$

Applying [8, Theorem 1.4] we conclude (10).

Now, since $\mathrm{cl}(P_1 \setminus P_2)$ is an isolating neighborhood for f, the assumption $\mathrm{cl}(P_1 \setminus P_2) = P_1$ implies that P_1 is an isolating neighborhood for f. It is straightforward to verify that in this case P^+ is an index pair in P_1. $\square$

Lemma 4.10. *Assume* $P = (P_1, P_2)$ *is an index pair for* f *and* S. *Then* $P^- := (P_1^-, P_2^-)$, *where*

$$P_1^- := \mathrm{cl}(P_1 \setminus P_2),$$
$$P_2^- := P_2 \cap P_1^-,$$

is a weak index pair for f *and* S *such that*

$$\mathrm{cl}(P_1^- \setminus P_2^-) = P_1^- \tag{14}$$

and

$$LT(P_f) \cong LT(P_f^-). \tag{15}$$

Proof: First we will show that P^- is a weak index pair. We begin with verifying (i). Let $x \in P_2^-$ be such that $f(x) \in P_1^-$. Then $x \in P_2$ and $f(x) \in P_1$. Therefore $f(x) \in P_2$ and consequently $f(x) \in P_2 \cap P_1^- = P_2^-$.

In order to prove (ii') take $x \in \mathrm{bd}_f(P_1^-)$. Then $x \in P_1^-$ and we can select a sequence $\{u_n\} \subset P_1^-$ such that $f(u_n) \to x$ and $f(u_n) \notin P_1^-$, which implies $f(u_n) \notin P_1$ or $f(u_n) \in P_2$. Passing to a subsequence, if necessary, we may assume that u_n converges to some $u \in P_1^-$ and either $f(u_n) \notin P_1$ for all n or $f(u_n) \in P_2$ for all n. In the first case we get from (ii) applied to P that $u_n \in P_2$ for all n. It follows that $u \in P_2$, and by (i) $x = f(u) \in P_2$. In the other case we get immediately that $x = f(u) \in P_2$. Since $x \in P_1^-$, we see that $x \in P_2 \cap P_1^- = P_2^-$ and (ii') is proved. Now observe that

$$P_1^- \setminus P_2^- = P_1^- \setminus P_2 = \mathrm{cl}(P_1 \setminus P_2) \setminus P_2 = P_1 \setminus P_2. \tag{16}$$

Thus we get $\mathrm{cl}(P_1^- \setminus P_2^-) = \mathrm{cl}(P_1 \setminus P_2) = P_1^-$, which proves (14). Moreover,

$$\mathrm{Inv}(\mathrm{cl}(P_1^- \setminus P_2^-), f) = \mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_2), f) \subset$$
$$\mathrm{int}(P_1 \setminus P_2) = \mathrm{int}(P_1^- \setminus P_2^-),$$

which proves (iii). From (16) we conclude that P^- is an index pair for f and S.

It remains to prove (15). By Proposition 4.7 there are well defined morphisms f_{PP}, f_{P^-P} and $f_{P^-P^-}$. Let $\iota : P^- \hookrightarrow P$ denote the inclusion map. We have the following commutative diagram of morphisms in Prs.

$$\begin{tikzcd}
P^- \arrow[r, "f_{P^-P^-}"] \arrow[d, "\iota"'] \arrow[dr, "f_{P^-P}"] & P^- \arrow[d, "\iota"] \\
P \arrow[r, "f_{PP}"] & P
\end{tikzcd}$$

Applying [8, Theorem 1.4] we conclude (15). □

Now we are able to prove a theorem which justifies the use of weak index pairs in the computation of the Conley index.

Theorem 4.11. *Assume S is an isolated invariant set for f and $P = (P_1, P_2)$ is a weak index pair for f such that* $\mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_2), f) = S$. *Then*

$$\mathrm{Con}_{L,T}(S, f) = LT(P_f). \tag{17}$$

Proof: Put $P' := P^+$ and $P'' := (P')^-$ and $P''' := (P'')^+$. By Lemmas 4.9 and 4.10

$$LT(P_f) \cong LT(P'_f) \cong LT(P''_f) \cong LT(P'''_f).$$

Moreover, $\mathrm{cl}(P''_1 \setminus P''_2) = P''_1$ by Lemma 4.10. Thus Lemma 4.9 implies that P''' is an index pair in P''_1, which is an isolating neighborhood for S. Therefore the conclusion follows from [8, Theorem 1.7 and Definition 1.8] □

The above theorem may be easily extended to the case of decompositions of isolated invariant sets. More precisely, we have the following theorem, whose proof is analogous to the proof of Theorem 4.11 and we leave it to the reader.

Theorem 4.12. *Assume S is an isolated invariant set for f, $\mathcal{S} := \{S_j\}_{j\in J}$ is a decomposition of S and $P = (P_1, P_2)$ is an $\mathcal{S}$- compatible weak index pair for f and S. Then*

$$\mathrm{Con}_{L,T}(\mathcal{S}, f) = LT(P^\wedge, f_{P,J}). \tag{18}$$

□

5. Index quadruples.

From the computational point of view the process of taking the quotient is not a useful tool. Therefore, when passing with an index pair to the algebraic level by means of a (co)homology functor, it is convenient to replace the quotient by the relative (co)homology. However, this raises the question if it is possible to change the definition of the index map in this setting in a way, which would enable the computation of the Conley index. For the same reason we would like to eliminate the quotients of the partial maps r_j in the definition of the Conley index of decompositions of isolated invariant sets. In order to answer these questions let us introduce the concept of an index quadruple.

Definition 5.1. *A pair of pairs $(P, \bar{P})$ is a (weak) index quadruple for f and S if P is a (weak) index pair for f and S, $P \subset \bar{P}$, the inclusion map $\iota : P \hookrightarrow \bar{P}$ is an excision and $f(P) \subset \bar{P}$.*

Theorem 5.2. *Assume T is an excisive covariant functor. Let $(P, \bar{P})$ be a weak index quadruple for f and S. Then*

$$\mathrm{Con}_{L,T}(S, f) \cong L(T(P), T(f_{P\bar{P}})T(\iota)^{-1}).$$

Proof: First observe that for every $x \in P_1^*$ we have

$$\iota^\wedge f^\wedge_{PP}([x]) = f^\wedge_{P\bar{P}}([x]).$$

Indeed, this is obvious if $x \notin P_1 \setminus f^{-1}(P_1)$ and for $x \in P_1 \setminus f^{-1}(P_1)$ we have $f(x) \in \bar{P}_1 \setminus P_1 = \bar{P}_2 \setminus P_2 \subset \bar{P}_2$, which implies $f^\wedge_{P\bar{P}}([x]) = \bar{P}_2^\wedge = \iota^\wedge f^\wedge_{PP}([x])$. Therefore we have the following commutative diagram in Prs.

$$\begin{tikzcd}
P \arrow[r, "f_{P\bar{P}}"] \arrow[dd, "q"'] & \bar{P} \arrow[d, "\bar{q}"] & P \arrow[l, "\iota"'] \arrow[dd, "q"] \\
 & \bar{P}^\wedge & \\
P^\wedge \arrow[ur, "f^\wedge_{P\bar{P}}"] \arrow[rr, "f^\wedge_{PP}"] & & P^\wedge \arrow[ul, "\iota^\wedge"']
\end{tikzcd}$$

Applying functor T we get the following commutative diagram

$$\begin{tikzcd}
T(P) \arrow[r, "T(f_{P\bar{P}})"] \arrow[dd, "T(q)"'] & T(\bar{P}) \arrow[d, "T(\bar{q})"] & T(P) \arrow[l, "T(\iota)"'] \arrow[dd, "T(q)"] \\
 & T(\bar{P}^\wedge) & \\
T(P^\wedge) \arrow[ur, "T(f^\wedge_{P\bar{P}})"'] \arrow[rr, "T(f^\wedge_{PP})"] & & T(P^\wedge) \arrow[ul, "T(\iota^\wedge)"']
\end{tikzcd}$$

Notice that since T is excisive and the projection q is an excisions, the morphisms $T(f_{P\bar{P}})T(\iota)^{-1}$ and $T(f^\wedge_{PP})$ are conjugate. The conclusion follows now from [8, Theorem 1.4] $\square$

The above theorem allows us to replace quotients by relative homology or cohomology in the process of computing the Conley index. Consider in turn the question how to eliminate the quotients of the partial maps r_j in the case of decompositions of isolated invariant sets. To answer this question assume $\mathcal{S} = \{S_j\}_{j\in J}$ is a decomposition of an isolated invariant set S and $\mathcal{P}$ is an $\mathcal{S}$-compatible pair with a decomposition $\{K_j\}_{j\in J}$ of $\mathrm{cl}(P_1 \setminus P_2)$. For $j \in J$ put

$$K_j^* := \bigcup_{i\neq j} K_i$$

and consider the following inclusions:

$$\begin{aligned} \alpha_j &: P \hookrightarrow (P_1, K_j^* \cup P_2) \\ \kappa_j &: (K_j, K_j \cap P_2) \hookrightarrow (P_1, K_j^* \cup P_2) \\ \iota_j &: (K_j, K_j \cap P_2) \hookrightarrow P \end{aligned}$$

It is straightforward to verify that κ_j is an excision, therefore $T(\kappa_j)$ is an isomorphism. We leave to the reader the proof of the following easy proposition, which shows again that the quotient may be avoided.

Proposition 5.3. *The maps $T(r_j^\wedge)$ and $T(\iota_j)T(\kappa_j)^{-1}T(\alpha_j)$ are conjugate.*

□

There remains the question how to construct the index quadruples.

Let S be an isolated invariant set for f. The following two propositions follow immediately from the definition of the index pair.

Proposition 5.4. *Assume X is compact, N is an isolating neighborhood for f and $P = (P_1, P_2)$ is an index pair for f in S. Let*

$$\bar{P} := (P_1 \cup X \setminus \mathrm{int}\, N, P_2 \cup X \setminus \mathrm{int}\, N).$$

Then $(P, \bar{P})$ is an index quadruple.

□

Proposition 5.5. *Assume $P = (P_1, P_2)$ is an index pair for f and S. Let*

$$\bar{P} := (P_1 \cup f(P_2), P_2 \cup f(P_2)).$$

Then $(P, \bar{P})$ is an index quadruple.

□

The next proposition shows how one can construct a weak index quadruple.

Proposition 5.6. *Assume $P = (P_1, P_2)$ is a weak index pair for f. Let*

$$\bar{P} := (P_2 \cup f(P_1), P_2 \cup (f(P_1) \setminus P_1)).$$

Then $(P, \bar{P})$ is a weak index quadruple.

Proof: Obviously $\bar{P}_1$ is a compact set. To show that $\bar{P}_2$ is compact, it is enough to prove that

$$P_2 \cup (f(P_1) \setminus P_1) = P_2 \cup \mathrm{cl}(f(P_1) \setminus P_1). \tag{19}$$

Obviously the left-hand side of (19) is contained in the-right hand side. To prove the opposite inclusion take an $x \in P_2 \cup \mathrm{cl}(f(P_1) \setminus P_1)$. If $x \in P_2$, then x belongs to the left-hand side. Thus assume $x \in \mathrm{cl}(f(P_1) \setminus P_1) \setminus P_2$. If $x \notin f(P_1) \setminus P_1$, then $x \in P_1$, that is $x \in \mathrm{bd}_f\, P_1 \subset P_2$. This is a contradiction, which proves (19). Obviously $f(P_1) \subset \bar{P}_1$. To show that $f(P_2) \subset \bar{P}_2$, assume the contrary. Then there exists an $x \in P_2$ such that $y := f(x) \notin \bar{P}_2$. It follows that $y \notin P_2$ and $y \in f(P_2) \subset f(P_1)$. Since $y \notin f(P_1) \setminus P_1$, we get $y \in P_1$ and from the relative positive invariance of P_2 in P_1 we conclude that $y \in P_2$. This is a contradiction, which proves that $f(P_2) \subset \bar{P}_2$.

We also have

$$\bar{P}_1 \setminus P_1 = (P_2 \cup f(P_1)) \setminus P_1 = f(P_1) \setminus P_1 = f(P_1) \setminus P_1 \setminus P_2 = \\ (P_2 \cup (f(P_1) \setminus P_1)) \setminus P_2 = \bar{P}_2 \setminus P_2,$$

which proves excision. □

6. Combinatorial Enclosures.

In the last two sections we reformulated the main concepts of the Conley index theory in order to meet the needs of effective algorithmic constructions. Now we want to present such algorithmic constructions of weak index pairs and index quadruples. Since sets constitute the expected outcome of these algorithms, we need a countable class of sets which admit a convenient finite representation. A general framework for such an approach is presented in [10]. For the sake of simplicity we present here a more concise approach, based on cubes of size one. This approach was also used in [3]. Obviously, in concrete applications families of cubes of arbitrary size are necessary but this is only the question of a suitable rescalling.

An *elementary interval* is an interval $[k, l] \subset \mathbb{R}$ such that $k, l \in \mathbb{Z}$ and $l = k + 1$ or $l = k$. In the latter case we say that the interval is *degenerate*. Otherwise it is *nondegenerate*. By an *elementary cube* Q in $\mathbb{R}^d$ we mean a finite product

$$I_1 \times I_2 \times \cdots \times I_d \subset \mathbb{R}^d,$$

of elementary intervals. The number of nondegenerate intervals in this product is called the *dimension* of Q and the number of degenerate intervals is called the *codimension* of Q. We denote the set of all elementary cubes in $\mathbb{R}^d$ by $\mathcal{K}$ and the set of elementary cubes of dimension

k by $\mathcal{K}_k$. An elementary cube is *full* if its dimension is d. Obviously the set of all full elementary cubes in $\mathbb{R}^d$ is $\mathcal{K}_d$. Note that the family $\mathcal{K}_d$ is an example of a *grid* [10, Definition 2.5].

With every elementary cube $Q = I_1 \times I_2 \times \cdots \times I_d$ we associate the *cell* $\overset{\circ}{Q}$ given by

$$\overset{\circ}{Q} := \overset{\circ}{I}_1 \times \overset{\circ}{I}_2 \times \cdots \times \overset{\circ}{I}_d,$$

where

$$\overset{\circ}{[k,l]} := \begin{cases} (k,l) & \text{if } k < l, \\ [k,k] & \text{otherwise.} \end{cases}$$

Note that the family of all cells in $\mathbb{R}^d$ coincides with the family of elementary representable sets over the grid $\mathcal{K}_d$ in the sense of [10].

Given a family $\mathcal{A} \subset \mathcal{K}$ of elementary cubes we will use the notation

$$|\mathcal{A}| := \bigcup \{ A \mid A \in \mathcal{A} \}.$$

Let $A \subset \mathbb{R}^d$ be an arbitrary set and let $\mathcal{X} \subset \mathcal{K}$. Define

$$\begin{aligned} \mathcal{K}(A) &:= \{ Q \in \mathcal{K} \mid Q \subset A \}, \\ \mathcal{K}_{\mathcal{X}}(A) &:= \{ Q \in \mathcal{X} \mid Q \subset A \}, \\ o(A) &:= \{ Q \in \mathcal{K} \mid Q \cap A \neq \emptyset \}, \\ o_d(A) &:= \{ Q \in \mathcal{K}_d \mid Q \cap A \neq \emptyset \}, \\ o_{\mathcal{X}}(A) &:= \{ Q \in \mathcal{X} \mid Q \cap A \neq \emptyset \}. \end{aligned}$$

The set $A \subset \mathbb{R}^d$ is called *cubical* if there exists a finite family $\mathcal{A} \subset \mathcal{K}$ such that $A = |\mathcal{A}|$. Then the family $\mathcal{A}$ is referred to as a *representation* of A. Obviously A may have many representations but one can easily check that there is a unique minimal representation, which will be denoted by $\mathcal{K}_{\min}(A)$ and called the *minimal representation* of A.

A cubical set is called a *full cubical set* if its minimal representation consists only of full elementary cubes. A family $\mathcal{A} \subset \mathcal{K}$ is called *semifull* if $|\mathcal{A}|$ is a full cubical set. Note that a semifull family may contain some elementary cubes which are not full

Proposition 6.1. *Assume X is a cubical set. Then*

(i) $\operatorname{int} X = \{ x \in \mathbb{R}^d \mid o(x) \subset \mathcal{K}(X) \} =$
$\{ x \in \mathbb{R}^d \mid o_d(x) \subset \mathcal{K}_{min}(X) \}$.

Additionally, if X is a full cubical set, then

(ii) *Every $x \in \operatorname{bd} X$ belongs to a full cube in $\mathcal{K}(X)$ and a full cube not in $\mathcal{K}(X)$.*

(iii) *Every $Q \in \mathcal{K}_{min}(\operatorname{bd} X)$ is the intersection of a unique full cube in $\mathcal{K}(X)$ and a unique full cube not in $\mathcal{K}(X)$.*

(iv) $\operatorname{bd} X$ *is a cubical set whose minimal representation consists of elementary cubes of dimension* $d-1$.

Proof: To prove (i) we need to show three inclusions. First take $x \in \operatorname{int} X$ and assume that there is a $Q \in o(x) \setminus \mathcal{K}(X)$. Then we can choose a sequence (x_n) in $\overset{\circ}{Q}$ such that $x_n \to x$. By [10, Theorem 3.3] we have $\overset{\circ}{Q} \cap X = \emptyset$. Therefore $x_n \notin X$ and consequently $x \notin \operatorname{int} X$. This is a contradiction, which proves that the first set in (i) is contained in the second. To prove that the second set is contained in the third take $x \in \mathbb{R}^d$ such that $o(x) \subset \mathcal{K}(X)$. Then obviously $o_d(x) \subset o(x) \subset \mathcal{K}(X)$. Since each full cube contained in X belongs to $\mathcal{K}_{\min}(X)$, we get $o_d(x) \subset \mathcal{K}_{\min}(X)$. There remains to be proved that the third set is contained in the first. For this end assume $x \in \mathbb{R}^d$ is such that $o_d(x) \subset \mathcal{K}_{\min}(X)$. Then by [10, Lemma 3.8]

$$x \in \operatorname{int} |o(x)| = \operatorname{int} |o_d(x)| \subset \operatorname{int} X$$

and (i) is proved.

Now take $x \in \operatorname{bd} X$. Since $\operatorname{bd} X \subset X$ and X is a full cubical set, we can find an $R \in o_d(x) \cap \mathcal{K}(X)$. Since $x \notin \operatorname{int} X$, by (i) we can find an $\bar{R} \in o_d(x) \setminus \mathcal{K}(X)$. This proves (ii).

To prove (iii) take $x \in \overset{\circ}{Q}$ and select an $R \in o_d(x) \cap \mathcal{K}(X)$ and an $\bar{R} \in o_d(x) \setminus \mathcal{K}(X)$. Let $R = I_1 \times I_2 \times \cdots \times I_d$ and let $\bar{R} = \bar{I}_1 \times \bar{I}_2 \times \cdots \times \bar{I}_d$. Put

$$R_k := \bar{I}_1 \times \bar{I}_2 \times \cdots \times \bar{I}_k \times I_{k+1} \times \cdots \times I_d.$$

Then $R_0 = R \in \mathcal{K}(X)$ and $R_d = \bar{R} \notin \mathcal{K}(X)$. It follows that there exists a k such that $R_k \in \mathcal{K}(X)$ and $R_{k+1} \notin \mathcal{K}(X)$. Obviously $R_k \cap R_{k+1} \subset \operatorname{bd} X$. By [10, Theorem 3.3] we get $Q \subset R_k \cap R_{k+1}$, which implies $Q = R_k \cap R_{k+1}$, because otherwise $Q \notin \mathcal{K}_{\min}(\operatorname{bd} X)$. The uniqueness is obvious.

Property (iv) is an immediate consequence of (iii). □

In the sequel we assume that X is a fixed full cubical set and $\mathcal{X}$ is a fixed representation of X. We do not assume that $\mathcal{X}$ is the minimal representation, that is we allow for the situation when $\mathcal{X}$ contains more elements than $\mathcal{K}_{\min}(X)$. As will be seen in the sequel, we do this to enable exact representations of boundaries of full cubical sets, which is important in some situations to ensure effectiveness of algorithms.

We say that a family $\mathcal{A} \subset \mathcal{X}$ is $\mathcal{X}$-*complete* if

$$A \in \mathcal{A}, Q \in \mathcal{X}, Q \subset A \;\Rightarrow\; Q \in \mathcal{A}.$$

The following two propositions are easy to prove.

Proposition 6.2. *A family $\mathcal{A} \subset \mathcal{X}$ is $\mathcal{X}$-complete if and only if*

$$\mathcal{K}_{\mathcal{X}}(|\mathcal{A}|) = \mathcal{A}. \tag{20}$$

□

Proposition 6.3. *If $\mathcal{A} \subset \mathcal{X}$ is $\mathcal{X}$-complete and for some $A \subset \mathbb{R}^d$ we have $o_d(A) \subset \mathcal{A}$, then $o_{\mathcal{X}}(A) \subset \mathcal{A}$.*

□

Let $N \subset X$ be a full cubical set. It is straightforward to verify that $\mathcal{K}_{\mathcal{X}}(N)$ is a representation of N, although it needn't be the minimal representation. Put $\mathcal{N} := \mathcal{K}_{\mathcal{X}}(N)$.

Proposition 6.4. *Assume $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$. If $\mathcal{B}$ is $\mathcal{X}$-complete or $|\mathcal{A}|$ is full, then*

$$|\mathcal{A}| \cap \operatorname{int} |\mathcal{B}| \subset |\mathcal{A} \cap \mathcal{B}|.$$

Proof: Take $x \in |\mathcal{A}| \cap \operatorname{int} |\mathcal{B}|$. Let $A \in \mathcal{A}$ be such that $x \in A$. By Proposition 6.1(i) we have $o(x) \subset \mathcal{K}(|\mathcal{B}|)$, hence $A \subset |\mathcal{B}|$. Therefore $A \subset B$ for some $B \in \mathcal{B}$. If $\mathcal{B}$ is $\mathcal{X}$-complete, then $A \in \mathcal{B}$. If $|\mathcal{A}|$ is full then A may be chosen to be a full cube. However in that case $A = B$, so $A \in \mathcal{B}$ too. Therefore $A \in \mathcal{A} \cap \mathcal{B}$ and $x \in |\mathcal{A} \cap \mathcal{B}|$. □

For $Q \in \mathcal{K}_{\min}(\operatorname{bd} N)$ let $b_{\mathcal{X}}(Q)$ denote Q if $Q \in \mathcal{X}$ and the unique full elementary cube R in $\mathcal{N}$ such that $Q \subset R$ otherwise. Obviously $Q \subset b_{\mathcal{X}}(Q)$ for every $Q \in \mathcal{K}_{\min}(\operatorname{bd} N)$. We will use the following notation

$$\operatorname{bd}_{\mathcal{X}} \mathcal{N} := \{ b_{\mathcal{X}}(Q) \mid Q \in \mathcal{K}(\operatorname{bd} \mathcal{N}) \}$$
$$\operatorname{int}_{\mathcal{X}} \mathcal{N} := \mathcal{N} \setminus \operatorname{bd}_{\mathcal{X}}(\mathcal{N}).$$

The following proposition is straightforward.

Proposition 6.5. *If N is a full cubical set and $\mathcal{N} := \mathcal{K}_{\mathcal{X}}(N)$, then*

$$\operatorname{bd} N \subset |\operatorname{bd}_{\mathcal{X}} \mathcal{N}|. \tag{21}$$

Proposition 6.6. *If $\mathcal{N} \subset \mathcal{X} \subset \mathcal{K}_d$ then*

$$|\operatorname{int}_{\mathcal{X}} \mathcal{N}| \subset \operatorname{int} |\mathcal{N}|. \tag{22}$$

Proof: Since we assume that $\mathcal{X}$ consists only of full cubes, we have $\operatorname{bd}_{\mathcal{X}} \mathcal{N} \subset \mathcal{N} \subset \mathcal{K}_d$. It follows that

$$\operatorname{bd}_{\mathcal{X}} \mathcal{N} = \{ Q \in \mathcal{N} \mid Q \cap \operatorname{bd} |\mathcal{N}| \neq \emptyset \}$$

and consequently

$$\operatorname{int}_{\mathcal{X}} \mathcal{N} = \{ Q \in \mathcal{N} \mid Q \cap \operatorname{bd} |\mathcal{N}| = \emptyset \}. \tag{23}$$

To prove (22), assume the contrary. Then there exists an $x \in |\operatorname{int}_{\mathcal{X}} \mathcal{N}| \setminus \operatorname{int} |\mathcal{N}|$. It follows that $x \in \operatorname{bd} |\mathcal{N}|$. On the other hand $x \in Q$ for some $Q \in \operatorname{int}_{\mathcal{X}} \mathcal{N}$. By (23) we get $x \notin \operatorname{bd} |\mathcal{N}|$, a contradiction. □

The next thing to do is to discuss a suitable finite representation for dynamics. There is no practical way to have an exact finite representation of interesting dynamics, but using multivalued maps defined on families of elementary cubes we can obtain rigorous, and as we will see later, useful bounds. Of course, to get appropriate bounds, the family of elementary cubes must be first rescaled to a suitable size.

Definition 6.7. By a *combinatorial multivalued map* on $\mathcal{X}$ we mean a multivalued map $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ such that the following two conditions are satisfied

(i) for every $Q \in \mathcal{X}$ the set $|\mathcal{F}(Q)|$ is full and $\mathcal{F}(Q)$ is $\mathcal{X}$-complete
(ii) $\mathcal{F}$ is *monotone* i.e. $Q, R \in \mathcal{X}, Q \subset R \Rightarrow \mathcal{F}(Q) \subset \mathcal{F}(R)$.

Let $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ be a multivalued combinatorial map. Let $\mathcal{A} \subset \mathcal{X}$. The *image of* A is defined by

$$\mathcal{F}(\mathcal{A}) := \bigcup_{Q \in \mathcal{A}} \mathcal{F}(Q).$$

The *inverse* of $\mathcal{F}$ is the combinatorial multivalued map $\mathcal{F}^{-1} : \mathcal{X} \rightrightarrows \mathcal{X}$ defined by

$$\mathcal{F}^{-1}(R) := \{ Q \in \mathcal{X} \mid R \in \mathcal{F}(Q) \}.$$

It is straightforward to verify that for $\mathcal{A} \subset \mathcal{X}$

$$\mathcal{F}^{-1}(\mathcal{A}) = \{Q \in \mathcal{X} \mid \mathcal{F}(Q) \cap \mathcal{A} \neq \emptyset\}.$$

The following definition is a variant of [19, definition (2.1)].

Definition 6.8. A combinatorial multivalued map $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ is a *combinatorial enclosure* of $f : X \to X$ if for every $Q \in \mathcal{X}$

$$o_d(f(Q)) \subset \mathcal{F}(Q). \tag{24}$$

In this case we say that f is a *selector* of $\mathcal{F}$.

For a discussion of algorithms constructing combinatorial enclosures of a given $f : X \to X$ we refer the reader to [12].

Proposition 6.9. *Assume $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is a combinatorial enclosure of $f : X \to X$. Then for any $Q \in \mathcal{X}$*

$$o_{\mathcal{X}}(f(Q)) \subset \mathcal{F}(Q).$$

Proof: Let $Q \in \mathcal{X}$ and $R \in o_{\mathcal{X}}(f(Q))$. Then $R \subset P$ for some $P \in \mathcal{K}_d$ and consequently $P \in o_d(f(Q)) \subset \mathcal{F}(Q)$. It follows from Definition 6.7(i) and Proposition 6.3 that $R \in \mathcal{F}(Q)$. □

The following proposition is an immediate consequence of Definition 6.8 and Proposition 6.1(i).

Proposition 6.10. *Assume $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is a combinatorial enclosure of $f : X \to X$. Then for every $Q \in \mathcal{X}$*

$$f(Q) \subset \operatorname{int} |\mathcal{F}(Q)|.$$

The way we want to obtain an algorithmic construction of weak index pairs and index quadruples is to find some combinatorial counterparts of these concepts for the combinatorial enclosures and then to prove that the counterparts, after some natural identification, fulfil the respective definitions for the original dynamics. For this end we introduce the following definitions.

Definition 6.11. Let I be an interval in $\mathbb{Z}$ containing 0. A *solution* through $Q \in \mathcal{K}$ under $\mathcal{F}$ is a function $\Gamma : I \to \mathcal{K}$ satisfying the following two properties:

(1) $\Gamma(0) = Q$,
(2) $\Gamma(n+1) \in \mathcal{F}(\Gamma(n))$ for all n such that $n, n+1 \in I$.

Definition 6.12. Assume $\mathcal{N} \subset \mathcal{K}$ is finite. The *invariant part* of $\mathcal{N}$ under $\mathcal{F}$, denoted $\operatorname{Inv}(\mathcal{N}, \mathcal{F})$, consists of $Q \in \mathcal{N}$ such that there exists a full solution $\Gamma : \mathbb{Z} \to \mathcal{N}$ through Q under $\mathcal{F}$. Similarly we define the *positively invariant part*, $\operatorname{Inv}^+(\mathcal{N}, \mathcal{F})$ and the *negatively invariant part*, $\operatorname{Inv}^-(\mathcal{N}, \mathcal{F})$, of $\mathcal{N}$ under $\mathcal{F}$ by replacing $\mathbb{Z}$ by $\mathbb{Z}^+$ and $\mathbb{Z}^-$ respectively.

Let $\mathcal{F}_{\mathcal{N}} : \mathcal{N} \rightrightarrows \mathcal{N}$ denote the map given by

$$\mathcal{F}_{\mathcal{N}}(Q) := \mathcal{F}(Q) \cap \mathcal{N}. \tag{25}$$

Theorem 6.13. *The following formulas hold for all $p > \operatorname{card} \mathcal{N}$*

$$\operatorname{Inv}^+(\mathcal{N}, \mathcal{F}) = \bigcap_{i=0}^{\infty} \mathcal{F}_{\mathcal{N}}^{-i}(\mathcal{N}) = \mathcal{F}_{\mathcal{N}}^{-p}(\mathcal{N}) \tag{26}$$

$$\operatorname{Inv}^-(\mathcal{N}, \mathcal{F}) = \bigcap_{i=0}^{\infty} \mathcal{F}_{\mathcal{N}}^{i}(\mathcal{N}) = \mathcal{F}_{\mathcal{N}}^{p}(\mathcal{N}) \tag{27}$$

$$\operatorname{Inv}(\mathcal{N}, \mathcal{F}) = \operatorname{Inv}^-(\mathcal{N}, \mathcal{F}) \cap \operatorname{Inv}^+(\mathcal{N}, \mathcal{F}). \tag{28}$$

Proof: To prove (26) first take a $Q \in \operatorname{Inv}^+(\mathcal{N}, \mathcal{F})$ and $\Gamma : \mathbb{Z}^+ \to \mathcal{N}$, a solution through Q under $\mathcal{F}$. Then $\Gamma(n) \in \mathcal{F}_{\mathcal{N}}^{n}(Q)$ for any $n \in \mathbb{Z}^+$, therefore $Q \in \mathcal{F}_{\mathcal{N}}^{-n}(\mathcal{N})$, which shows that the first set of

(26) is contained in the second set. To prove that the second set is contained in the first set observe that since the sequence $\mathcal{F}_{\mathcal{N}}^{-n}(\mathcal{N})$ is descending, it becomes constant after at most $\operatorname{card}\mathcal{N}+1$ steps. Finally, if $Q \in \mathcal{F}_{\mathcal{N}}^{-p}(\mathcal{N})$, then it is straightforward to construct a solution $\Gamma : \{0, 1, \dots, p\} \to \mathcal{N}$ under $\mathcal{F}$ through Q. If $p > \operatorname{card}\mathcal{N}$, then $\Gamma(p) = \Gamma(q)$ for some $q < p$ and consequently Γ may be extended to a solution $\bar{\Gamma} : \mathbb{Z}^+ \to \mathcal{N}$. Thus $Q \in \operatorname{Inv}^+(\mathcal{N}, \mathcal{F})$.

The proof of (27) is similar and (28) is obvious. $\square$

7. Combinatorial Index Pairs.

In this section we introduce the combinatorial counterparts of the concepts of isolating neighborhood and index pair.

We say that a subset $\mathcal{N}$ of $\mathcal{X}$ is a *combinatorial isolating neighborhood* for $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ and $\mathcal{S}$ if $\mathcal{N}$ is finite, $\mathcal{X}$-complete and

$$\mathcal{S} = \operatorname{Inv}(\mathcal{N}, \mathcal{F}) \subset \operatorname{int}_{\mathcal{X}} \mathcal{N}. \tag{29}$$

The proof of the following easy proposition is left to the reader.

Proposition 7.1. *If $\mathcal{X}$ consists only of full elementary cubes and $\mathcal{N}$ is a combinatorial isolating neighborhood for $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ and $\mathcal{S}$, then $o_d(\mathcal{S})$ is also a combinatorial isolating neighborhood for $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ and $\mathcal{S}$.*

Definition 7.2. We say that $(\mathcal{P}_1, \mathcal{P}_2)$ is a *combinatorial index pair* for $\mathcal{F}$ in $\mathcal{N}$ if $\mathcal{P}_2 \subset \mathcal{P}_1$ are $\mathcal{N}$-complete subfamilies of $\mathcal{N}$ and the following three conditions are satisfied.

(i) $\mathcal{F}(\mathcal{P}_i) \cap \mathcal{N} \subset \mathcal{P}_i$ for $i = 1, 2$,
(ii) $\mathcal{F}(\mathcal{P}_1) \cap \operatorname{bd}_{\mathcal{X}} \mathcal{N} \subset \mathcal{P}_2$,
(iii) $\operatorname{Inv}(\mathcal{N}, \mathcal{F}) \subset \mathcal{P}_1 \setminus \mathcal{P}_2$.

Theorem 7.3. *Assume $\mathcal{N} \subset \mathcal{X}$ is an isolating neighborhood for $\mathcal{F}$. Let*

$$\begin{aligned} \mathcal{P}_1 &:= \operatorname{Inv}^-(\mathcal{N}, \mathcal{F}), \\ \mathcal{P}_2 &:= \operatorname{Inv}^-(\mathcal{N}, \mathcal{F}) \setminus \operatorname{Inv}^+(\mathcal{N}, \mathcal{F}). \end{aligned}$$

Then $(\mathcal{P}_1, \mathcal{P}_2)$ is a combinatorial index pair for $\mathcal{F}$ in $\mathcal{N}$. Moreover, if $\mathcal{X}$ consists only of full cubes then

$$|\mathcal{P}_1| \setminus |\mathcal{P}_2| \subset \operatorname{int} |\mathcal{N}|.$$

Proof: To show that $\mathcal{P}_1$ is $\mathcal{N}$-complete take a $Q \in \mathcal{N}$ and an $R \in \mathcal{P}_1$ such that $Q \subset R$. Let $\Gamma : \mathbb{Z}^- \to \mathcal{N}$ be a solution through R. Since $Q \subset R = \Gamma(0) \in \mathcal{F}(\Gamma(-1))$ and $\mathcal{F}(\Gamma(-1))$ is $\mathcal{X}$-complete, we get

$Q \in \mathcal{F}(\Gamma(-1))$. Therefore replacing $\Gamma(0)$ by Q we obtain a solution on $\mathbb{Z}^-$ through Q, which shows that $Q \in \mathcal{P}_1$.

Assume in turn that $Q \subset P \in \mathcal{P}_2$. If $Q \notin \mathcal{P}_2$, then $Q \in \operatorname{Inv}^+(\mathcal{N}, \mathcal{F})$. Let $\Lambda : \mathbb{Z}^+ \to \mathcal{N}$ be a solution through Q. Since $\mathcal{F}$ is monotone, we get $\Lambda(1) \in \mathcal{F}(\Lambda(0)) = \mathcal{F}(Q) \subset \mathcal{F}(P)$. Therefore replacing $\Lambda(0)$ by P we get a solution on $\mathbb{Z}^+$ through P. This contradicts $P \in \mathcal{P}_2$ and shows that $\mathcal{P}_2$ is $\mathcal{N}$-complete.

Next we will verify properties (i)-(iii) of Definition 7.2. To verify property (i) for $i = 1$, take $Q \in \mathcal{F}(\mathcal{P}_1) \cap \mathcal{N}$. Then there exists a solution $\Gamma : \mathbb{Z}^- \to \mathcal{N}$, such that $Q \in \mathcal{F}(\Gamma(0))$. Putting

$$\bar{\Gamma}(n) := \begin{cases} Q & \text{for } n = 0, \\ \Gamma(n+1) & \text{for } n < 0. \end{cases}$$

we obtain a solution $\bar{\Gamma} : \mathbb{Z}^- \to \mathcal{N}$ through Q. Thus $Q \in \operatorname{Inv}^-(\mathcal{N}, \mathcal{F}) = \mathcal{P}_1$. Assume in turn that $Q \in \mathcal{F}(\mathcal{P}_2) \cap \mathcal{N}$. Then $Q \in \mathcal{F}(R)$ for some $R \in \mathcal{P}_2$. To prove that $Q \in \mathcal{P}_2$, assume the contrary. Then $Q \in \operatorname{Inv}(\mathcal{N}, \mathcal{F})$, i.e. we can take $\Gamma : \mathbb{Z} \to \mathcal{N}$, a solution to $\mathcal{F}$ through Q in $\mathcal{N}$. Define $\bar{\Gamma} : \mathbb{Z}^+ \to \mathcal{N}$ by

$$\bar{\Gamma}(n) := \begin{cases} R & \text{for } n = 0, \\ \Gamma(n-1) & \text{for } n > 0. \end{cases}$$

It is straightforward to verify that $\bar{\Gamma} : \mathbb{Z}^+ \to \mathcal{N}$ is a solution through R, therefore $R \in \operatorname{Inv}^+(\mathcal{N}, \mathcal{F})$ and $R \notin \mathcal{P}_2$, a contradiction. Thus (i) is proved.

Assume in turn that $Q \in \mathcal{F}(\mathcal{P}_1) \cap \operatorname{bd}_{\mathcal{X}} \mathcal{N}$. Then $Q \in \mathcal{P}_1$. If $Q \notin \mathcal{P}_2$, then $Q \in \operatorname{Inv}(\mathcal{N}, \mathcal{F})$, which contradicts $Q \in \operatorname{bd}_{\mathcal{X}} \mathcal{N}$. Therefore $Q \in \mathcal{P}_2$.

To prove (iii) observe that

$$\operatorname{Inv}(\mathcal{N}, \mathcal{F}) = \operatorname{Inv}^-(\mathcal{N}, \mathcal{F}) \cap \operatorname{Inv}^+(\mathcal{N}, \mathcal{F}) = \mathcal{P}_1 \setminus \mathcal{P}_2.$$

In particular, we also get

$$\mathcal{P}_1 \setminus \mathcal{P}_2 = \operatorname{Inv}(\mathcal{N}, \mathcal{F}) \subset \operatorname{int}_{\mathcal{X}} \mathcal{N}.$$

Therefore, if $\mathcal{X} \subset \mathcal{K}_d$, then by Proposition 6.6

$$|\mathcal{P}_1| \setminus |\mathcal{P}_2| \subset |\mathcal{P}_1 \setminus \mathcal{P}_2| \subset |\operatorname{int}_{\mathcal{X}} \mathcal{N}| \subset \operatorname{int} |\mathcal{N}|.$$

□

8. Weak Index Pairs from Combinatorial Index Pairs.

Now we will show how the combinatorial index pair for $\mathcal{F}$ may be used to obtain a weak index quadruple for any selector f of $\mathcal{F}$.

For $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ define

$$\mathcal{A} \bar{\cap} \mathcal{B} := \{ P \cap Q \mid P \in \mathcal{A}, Q \in \mathcal{B} \}$$

and let $\mathcal{A} \bar{\subset} \mathcal{B}$ denote that for every $A \in \mathcal{A}$ there exists a $B \in \mathcal{B}$ such that $A \subset B$.

Theorem 8.1. *Assume $\mathcal{N}$ is an isolating neighborhood for $\mathcal{F}$ and $(\mathcal{P}_1, \mathcal{P}_2)$ is a combinatorial index pair for $\mathcal{F}$ in $\mathcal{N}$. Then $(|\mathcal{P}_1|, |\mathcal{P}_2|)$ is a weak index pair for any selector f of $\mathcal{F}$.*

Moreover, if $\mathcal{M} \subset \mathcal{X}$ is such that

$$\mathcal{F}(\mathcal{P}_1) \subset \mathcal{M} \tag{30}$$

$$\mathcal{M} \cap \mathcal{N} \subset \mathcal{P}_1 \tag{31}$$

and one of the following conditions is satisfied

$$|\mathcal{P}_1| \setminus |\mathcal{P}_2| \subset \operatorname{int} |\mathcal{N}| \tag{32}$$

or

$$\mathcal{M} \bar{\cap} \operatorname{bd}_{\mathcal{X}} \mathcal{N} \bar{\subset} \mathcal{P}_2, \tag{33}$$

then $(|\mathcal{P}_1|, |\mathcal{P}_2|, |\mathcal{P}_1 \cup \mathcal{M}|, |\mathcal{P}_2 \cup \mathcal{M} \setminus \mathcal{P}_1|)$ is a weak index quadruple for any selector f of $\mathcal{F}$.

Proof: We need to prove properties (i),(ii'),(iii) of Definition 4.1. Let $P_i := |\mathcal{P}_i|$ for $i = 1, 2$. In order to prove (i) fix $x \in P_2$ such that $f(x) \in P_1$. Then $x \in Q$ for some $Q \in \mathcal{P}_2$ and $f(x) \in R$ for some $R \in \mathcal{P}_1$. Thus, by Proposition 6.9

$$R \in o_{\mathcal{X}}(f(x)) \subset o_{\mathcal{X}}(f(Q)) \subset \mathcal{F}(Q) \subset \mathcal{F}(\mathcal{P}_2)$$

and by Definition 7.2(i) we get $R \in \mathcal{P}_2$. It follows that $f(x) \in P_2$.

Put $N := |\mathcal{N}|$. For proving (ii), we will show first that

$$f(P_1) \cap N \subset P_1.$$

Indeed, $f(P_1) = f(|\mathcal{P}_1|) \subset \operatorname{int} |\mathcal{F}(\mathcal{P}_1)|$. Since $\mathcal{F}(\mathcal{P}_1)$ is $\mathcal{X}$-complete as the union of $\mathcal{X}$-complete sets, we get from Proposition 6.4

$$f(P_1) \cap N \subset \operatorname{int} |\mathcal{F}(\mathcal{P}_1)| \cap |\mathcal{N}| \subset |\mathcal{F}(\mathcal{P}_1) \cap \mathcal{N}| \subset |\mathcal{P}_1| = P_1.$$

In particular we obtain $f(P_1) \setminus P_1 \subset \mathbb{R}^d \setminus N$, therefore

$$\operatorname{bd}_f P_1 = \operatorname{cl}(f(P_1) \setminus P_1) \cap P_1 \subset \operatorname{cl}(\mathbb{R}^d \setminus N) \cap N = \operatorname{bd} N.$$

Thus we get from Proposition 6.10, Proposition 6.5, Proposition 6.4 and Definition 7.2(ii)

$$\operatorname{bd}_f P_1 \subset f(P_1) \cap \operatorname{bd} N \subset \operatorname{int} |\mathcal{F}(\mathcal{P}_1)| \cap |\operatorname{bd}_{\mathcal{X}} \mathcal{N}| \subset$$
$$|\mathcal{F}(\mathcal{P}_1) \cap \operatorname{bd}_{\mathcal{X}} \mathcal{N}| \subset |\mathcal{P}_2| = P_2.$$

which proves (ii). Before we prove property (iii) let us show that

$$\operatorname{cl}(Q \setminus P_2) = Q \text{ for } Q \in \mathcal{P}_1 \setminus \mathcal{P}_2 \tag{34}$$

and

$$\operatorname{cl}(P_1 \setminus P_2) = |\mathcal{P}_1 \setminus \mathcal{P}_2|. \tag{35}$$

To prove (34) take $Q \in \mathcal{P}_1 \setminus \mathcal{P}_2$. We have $\mathring{Q} \cap P_2 = \emptyset$, because otherwise $Q \subset P_2$ and $Q \in \mathcal{P}_2$ by the $\mathcal{N}$-completeness of $\mathcal{P}_2$. Therefore

$$Q = \operatorname{cl} \mathring{Q} = \operatorname{cl}(\mathring{Q} \setminus P_2) \subset \operatorname{cl}(Q \setminus P_2) \subset \operatorname{cl} Q = Q,$$

which shows (34). Now we have

$$P_1 \setminus P_2 = |\mathcal{P}_1| \setminus P_2 = \bigcup_{Q \in \mathcal{P}_1} Q \setminus P_2 = \bigcup_{Q \in \mathcal{P}_1 \setminus \mathcal{P}_2} Q \setminus P_2$$

Therefore by (34)

$$\operatorname{cl}(P_1 \setminus P_2) = \bigcup_{Q \in \mathcal{P}_1 \setminus \mathcal{P}_2} \operatorname{cl}(Q \setminus P_2) = \bigcup_{Q \in \mathcal{P}_1 \setminus \mathcal{P}_2} Q = |\mathcal{P}_1 \setminus \mathcal{P}_2|.$$

Thus (35) is proved.

Now put $M := \operatorname{cl}(P_1 \setminus P_2)$ and assume property (iii) is not satisfied. Then there exists an $x \in \operatorname{Inv}(M, f) \subset P_1$ such that $x \notin \operatorname{int}(P_1 \setminus P_2)$. We will show that

$$o(x) \cap \big((\mathcal{N} \setminus \mathcal{P}_1) \cup \operatorname{bd}_{\mathcal{X}} \mathcal{N} \cup \mathcal{P}_2\big) \neq \emptyset. \tag{36}$$

Since $\operatorname{int}(P_1 \setminus P_2) = \operatorname{int} P_1 \setminus P_2$, either $x \in P_2$ or $x \in \operatorname{bd} P_1$. If $x \in P_2$, then there exists a $Q \in o(x)$ such that $Q \in \mathcal{P}_2$, therefore (36) is satisfied. Thus consider the case $x \in \operatorname{bd} P_1$. If $x \notin \operatorname{int} N$, then $x \in \operatorname{bd} N$ and we get from Proposition 6.5 that $x \in |\operatorname{bd}_{\mathcal{X}} \mathcal{N}|$, which implies (36). Thus assume $x \in \operatorname{int} N$. Then, by Proposition 6.1(i)

$$o_d(x) \subset \mathcal{K}_{\min}(N) \subset \mathcal{N}. \tag{37}$$

Since $x \in \operatorname{bd} P_1$ implies $x \notin \operatorname{int} P_1$, by Proposition 6.1(i) we can select an $R \in o_d(x) \setminus \mathcal{P}_1$. From (37) we get $R \in o_d(x) \setminus \mathcal{P}_1 \subset o(x) \cap (\mathcal{N} \setminus \mathcal{P}_1)$ and (36) is proved.

Now, let $\gamma : \mathbb{Z} \to M$ be a solution through x under f. Since $\gamma(n) \in |\mathcal{P}_1 \setminus \mathcal{P}_2| \subset |\mathcal{N}|$, for every $n \in \mathbb{Z}$ we can choose $\Gamma(n) \in \mathcal{N}$ such that $\gamma(n) \in \Gamma(n)$. Moreover, by (36)

$$\Gamma(0) \in (\mathcal{N} \setminus \mathcal{P}_1) \cup \operatorname{bd}_{\mathcal{X}} \mathcal{N} \cup \mathcal{P}_2. \tag{38}$$

By Proposition 6.9

$$\Gamma(n+1) \in o_{\mathcal{X}}(\gamma(n+1)) = o_{\mathcal{X}}(f(\gamma(n))) \subset o_{\mathcal{X}}(f(\Gamma(n))) \subset \mathcal{F}(\Gamma(n)),$$

which shows that $\Gamma : \mathbb{Z} \to \mathcal{N}$ is a full solution through Q under $\mathcal{F}$. Therefore $\Gamma(n) \in \operatorname{Inv}(\mathcal{N}, \mathcal{F}) \subset \mathcal{P}_1 \setminus \mathcal{P}_2$ for every $n \in \mathbb{Z}$. In particular $\Gamma(0) \in \mathcal{F}(\Gamma(-1)) \subset \mathcal{F}(\mathcal{P}_1)$. On the other hand, by (38) and Definition 7.2(i-ii)

$$\Gamma(0) \in \mathcal{F}(\mathcal{P}_1) \cap \big((\mathcal{N} \setminus \mathcal{P}_1) \cup \operatorname{bd}_{\mathcal{X}} \mathcal{N} \cup \mathcal{P}_2\big) \subset \mathcal{P}_2,$$

a contradiction. Thus (iii) is proved.

Put $\bar{\mathcal{P}}_1 := \mathcal{P}_1 \cup \mathcal{M}$, $\bar{\mathcal{P}}_2 := \mathcal{P}_2 \cup \mathcal{M} \setminus \mathcal{P}_1$ and $\bar{P}_i := |\bar{\mathcal{P}}_i|$ for $i = 1, 2$. To complete the proof we need to show that

$$f : (P_1, P_2) \ni x \to f(x) \in (\bar{P}_1, \bar{P}_2),$$
$$\iota : (P_1, P_2) \ni x \to x \in (\bar{P}_1, \bar{P}_2)$$

are well defined maps of pairs and ι is an excision. For this end we will first prove that

$$\mathcal{F}(\mathcal{P}_i) \subset \bar{\mathcal{P}}_i \text{ for } i = 1, 2. \tag{39}$$

The property is obvious for $i = 1$. To prove it for $i = 2$ let $Q \in \mathcal{F}(\mathcal{P}_2)$. If $Q \in \mathcal{F}(\mathcal{P}_1) \setminus \mathcal{P}_1$, the conclusion is obvious. Thus assume that $Q \notin \mathcal{F}(\mathcal{P}_1) \setminus \mathcal{P}_1$. Then $Q \in \mathcal{P}_1$. Therefore by Definition 7.2(i) we get $Q \in \mathcal{P}_2$ and (39) is proved. It follows now from Proposition 6.10 that $f(|\mathcal{P}_i|) \subset |\bar{\mathcal{P}}_i|$. It remains to be proved that ι is an excision. To achieve this we will first show that

$$Q \in \mathcal{M} \setminus \mathcal{P}_1 \;\Rightarrow\; Q \setminus P_2 = Q \setminus P_1. \tag{40}$$

Let $Q \in \mathcal{M} \setminus \mathcal{P}_1$. Obviously $Q \setminus P_1 \subset Q \setminus P_2$. Assume the opposite inclusion does not hold. Then there exists an $x \in Q \setminus P_2$ such that $x \notin Q \setminus P_1$. It follows that $x \in P_1 \setminus P_2$. If (32) is satisfied, then $x \in \operatorname{int} |\mathcal{N}|$ and by Proposition 6.1(i) and Proposition 6.3 we get $o_{\mathcal{X}}(x) \subset \mathcal{N}$. In particular $Q \in \mathcal{N}$ and since $Q \in \mathcal{M}$, we get from (31) that $Q \in \mathcal{P}_1$, a contradiction. Thus (30) is proved, when (32) is satisfied. Hence consider the case when (33) holds. If $Q \in \mathcal{N}$, we can proceed as in the previous case. If $Q \notin \mathcal{N}$, then $x \in \operatorname{bd} |\mathcal{N}|$ and by Proposition 6.5 we get $x \in |\operatorname{bd}_{\mathcal{X}} \mathcal{N}|$. Select an $R \in \operatorname{bd}_{\mathcal{X}} \mathcal{N}$ such that $x \in R$. Then $Q \cap R \in \mathcal{M} \bar{\cap} \operatorname{bd}_{\mathcal{X}} \mathcal{N}$ and by (33) the intersection $Q \cap R$ is contained in an element of $\mathcal{P}_2$. Since $x \in Q \cap R$, it follows that $x \in P_2$, a contradiction. Thus (40) is proved.

Now observe that

$$\bar{P}_1 \setminus P_1 = \bigcup_{Q \in \bar{\mathcal{P}}_1} Q \setminus P_1 = \bigcup_{Q \in \bar{\mathcal{P}}_1 \setminus \mathcal{P}_1} Q \setminus P_1$$

and since $\bar{\mathcal{P}}_2 = \mathcal{P}_2 \cup (\bar{\mathcal{P}}_1 \setminus \mathcal{P}_1)$, we also have

$$\bar{P}_2 \setminus P_2 = \bigcup_{Q \in \bar{\mathcal{P}}_2} Q \setminus P_2 = \bigcup_{Q \in \mathcal{P}_2 \cup (\bar{\mathcal{P}}_1 \setminus \mathcal{P}_1)} Q \setminus P_2 = \bigcup_{Q \in \bar{\mathcal{P}}_1 \setminus \mathcal{P}_1} Q \setminus P_2.$$

Therefore we conclude from (30) that $\bar{P}_1 \setminus P_1 = \bar{P}_2 \setminus P_2$ and by Proposition 2.3 the inclusion ι is an excision. $\square$

The following straightforward proposition provides the simplest choice of $\mathcal{M}$ in Theorem 8.1.

Proposition 8.2. *The conditions (30) and (31) of Theorem 8.1 are satisfied for $\mathcal{M} := \mathcal{F}(\mathcal{P}_1)$.*

9. Isolating blocks.

As we shall see in the sequel, Theorem 7.3 may be used to obtain an algorithm finding combinatorial index pairs. However, the efficiency of such an algorithm will crucially depend on the amount of computations needed to find a good combinatorial enclosure $\mathcal{F}$ of $f|_N$. In some situations, especially when the dynamical system is induced by a differential equation, this may be a serious drawback. For this reason we discuss in this section an alternative approach, based on the concept of isolating block.

Definition 9.1. An $\mathcal{X}$-complete subset $\mathcal{N}$ of $\mathcal{X}$ is an *isolating block* for $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ if

$$\mathcal{F}^{-1}(\mathcal{N}) \cap \mathcal{N} \cap \mathcal{F}(\mathcal{N}) \subset \operatorname{int}_{\mathcal{X}} \mathcal{N}. \tag{41}$$

Given $\mathcal{N} \subset \mathcal{X}$ we define its *exit set* $\mathcal{N}^-$ by

$$\mathcal{N}^- := \{\, Q \in \operatorname{bd}_{\mathcal{X}} \mathcal{N} \mid \mathcal{F}(Q) \cap \mathcal{N} = \emptyset \,\}.$$

Theorem 9.2. *The set $\mathcal{N}$ is an isolating block for $\mathcal{F}$ if and only if*

$$\mathcal{F}(\mathcal{N}) \cap \operatorname{bd}_{\mathcal{X}} \mathcal{N} \subset \mathcal{N}^-. \tag{42}$$

Proof: Assume $\mathcal{N}$ is an isolating block for $\mathcal{F}$. Let $Q \in \mathcal{F}(\mathcal{N}) \cap \operatorname{bd}_{\mathcal{X}} \mathcal{N}$. Then $Q \notin \operatorname{int}_{\mathcal{X}} \mathcal{N}$ and from (41) we see that $Q \notin \mathcal{F}^{-1}(\mathcal{N})$, which means that $Q \in \mathcal{N}^-$. Thus (42) is satisfied. Assume in turn that (42) is satisfied and $\mathcal{N}$ is not an isolating block. Then we can find a $Q \in \mathcal{F}^{-1}(\mathcal{N}) \cap \mathcal{N} \cap \mathcal{F}(\mathcal{N}) \cap \operatorname{bd}_{\mathcal{X}} \mathcal{N}$. It follows from (42) that $Q \in \mathcal{N}^-$, i.e. $\mathcal{F}(Q) \cap \mathcal{N} = \emptyset$, which contradicts $Q \in \mathcal{F}^{-1}(\mathcal{N})$. $\square$

Theorem 9.3. *Assume $\mathcal{N}$ is an isolating block for $\mathcal{F}$. Then $\mathcal{N}$ is an isolating neighborhood for $\mathcal{F}$ and $(\mathcal{N}, \mathcal{N}^-)$ is a combinatorial index pair for $\mathcal{F}$ in $\mathcal{N}$.*

Proof: Obviously

$$\operatorname{Inv}(\mathcal{N},\mathcal{F}) = \operatorname{Inv}^-(\mathcal{N},\mathcal{F}) \cap \operatorname{Inv}^+(\mathcal{N},\mathcal{F}) \subset \mathcal{F}^{-1}(\mathcal{N}) \cap \mathcal{N} \cap \mathcal{F}(\mathcal{N}) \subset \operatorname{int}_{\mathcal{X}} \mathcal{N}.$$

Thus $\mathcal{N}$ is an isolating neighborhood. The proof that $(\mathcal{N},\mathcal{N}^-)$ is a combinatorial index pair for $\mathcal{F}$ is straightforward. □

The above theorem shows how to obtain a combinatorial index pair from an isolating block but as in the previous construction it requires the knowledge of a combinatorial enclosure $\mathcal{F}$ of $f|_N$. Our goal now is to present a construction, which requires only an enclosure of $f|_{\operatorname{bd} N}$. For this end we need first some lemmas.

Assume $X \subset \mathbb{R}^d$ is bounded. Then $\mathbb{R}^d \setminus X$ has exactly one unbounded connected component. Denote it by $\operatorname{ucc}(X)$ and denote the union of all bounded components of $\mathbb{R}^d \setminus X$ by $\operatorname{bcc}(X)$.

Lemma 9.4.

(i) *If* $Y \subset X \subset \mathbb{R}^d$ *are bounded, then* $\operatorname{ucc}(X) \subset \operatorname{ucc}(Y)$
(ii) *If* $X \subset \mathbb{R}^d$ *is compact, then* $\operatorname{ucc}(\operatorname{bd} X) = \operatorname{ucc}(X)$.

Proof: Obviously $\operatorname{ucc}(X) \subset \mathbb{R}^d \setminus X \subset \mathbb{R}^d \setminus Y$. Therefore, as a connected set, $\operatorname{ucc}(X)$ is contained in a connected component W of $\mathbb{R}^d \setminus Y$. Since $\operatorname{ucc}(X)$ is unbounded, so is W. It follows that $\operatorname{ucc}(X) \subset W = \operatorname{ucc}(Y)$, which proves (i). To prove (ii) observe that

$$\operatorname{ucc}(\operatorname{bd} X) \subset \mathbb{R}^d \setminus \operatorname{bd} X = \operatorname{int} X \cup \operatorname{ext} X$$

Since $\operatorname{int} X$, $\operatorname{ext} X$ are disjoint, open sets and $\operatorname{ucc}(\operatorname{bd} X)$ is connected, we see that either $\operatorname{ucc}(\operatorname{bd} X) \subset \operatorname{int} X$ or $\operatorname{ucc}(\operatorname{bd} X) \subset \operatorname{ext} X$. The first case is impossible, because $\operatorname{ucc}(\operatorname{bd} X)$ is unbounded. Therefore

$$\operatorname{ucc}(\operatorname{bd} X) \subset \operatorname{ext} X = \mathbb{R}^d \setminus X.$$

Hence, as a connected, unbounded set, $\operatorname{ucc}(\operatorname{bd} X)$ must be contained in the connected, unbounded component of $\mathbb{R}^d \setminus X$, i.e. $\operatorname{ucc}(\operatorname{bd} X) \subset \operatorname{ucc}(X)$. The opposite inclusion follows from (i). □

For a bounded set X we will now consider the set $\operatorname{bcf}(X) := \mathbb{R}^d \setminus \operatorname{ucc}(X)$, called the *bounded complement component filling of* X.

Lemma 9.5. *Assume* Y, X *are bounded subsets of* $\mathbb{R}^d$. *Then*

(i) $X \subset \operatorname{bcf}(X)$,
(ii) $Y \subset X \;\Rightarrow\; \operatorname{bcf}(Y) \subset \operatorname{bcf}(X)$,
(iii) $\operatorname{bcf}(\operatorname{int} X) \subset \operatorname{int} \operatorname{bcf}(X)$.
(iv) *If* $X \subset \mathbb{R}^d$ *is compact, then* $\operatorname{bcf}(X)$ *is compact*
(v) *If* $X \subset \mathbb{R}^d$ *is full cubical, then* $\operatorname{bcf}(X)$ *is full cubical.*

Proof: Properties (i) and (ii) are obvious. To prove property (iii) first observe that $\operatorname{ucc}(\operatorname{int} X)$, as a connected component of a closed set in $\mathbb{R}^d$ is closed. Therefore $\operatorname{bcf}(\operatorname{int} X)$, as a complement of a closed set is open. Since by (ii) $\operatorname{bcf}(\operatorname{int} X) \subset \operatorname{bcf}(X)$, we get $\operatorname{bcf}(\operatorname{int} X) \subset \operatorname{int} \operatorname{bcf}(X)$, which proves (iii).

Now assume that X is compact. Since $\mathbb{R}^d$ is a locally connected space, the set $\operatorname{ucc}(X)$ is open as a connected component of an open set. Therefore $\operatorname{bcf}(X)$ is closed. Let B be a closed ball such that $X \subset B$. Then $\mathbb{R}^d \setminus B \subset \operatorname{ucc}(X)$. Therefore $\operatorname{bcf}(X) = \mathbb{R}^d \setminus \operatorname{ucc}(X) \subset B$, i.e. $\operatorname{bcf}(X)$ is bounded. Thus $\operatorname{bcf}(X)$ is compact.

Finally assume that X is full cubical. In order to show that $\operatorname{bcf}(X)$ is full cubical it is enough to show that

$$\mathcal{X}' := \{\, Q \in \mathcal{K}_d \mid Q \cap \operatorname{ucc}(X) = \emptyset \,\} \tag{43}$$

is finite and

$$\operatorname{bcf}(X) = \bigcup \mathcal{X}'. \tag{44}$$

To show that $\mathcal{X}'$ is finite take a closed ball B such that $X \subset B$. Then by Lemma 9.4(i) $\operatorname{ucc}(B) \subset \operatorname{ucc}(X)$, therefore

$$\mathcal{X}' \subset \{\, Q \in \mathcal{K}_d \mid Q \cap \operatorname{ucc}(B) = \emptyset \,\}$$

However, since B is a closed ball, $\operatorname{ucc}(B) = \mathbb{R}^d \setminus B$, which implies that $\mathcal{X}' \subset \mathcal{K}_d(B)$ and proves that $\mathcal{X}'$ is finite.

To prove (44) first observe that obviously $\bigcup \mathcal{X}' \subset \operatorname{bcf}(X)$. Thus take $x \in \operatorname{bcf}(X)$. If $x \in X$, then $x \in Q$ for some $Q \in \mathcal{K}_d(X) \subset \mathcal{X}'$, so that $x \in \bigcup \mathcal{X}'$. Therefore assume that $x \notin X$, i.e. $x \in \operatorname{bcc}(X)$. Fix a $Q \in o_d(x)$. Then $\emptyset \neq Q \cap \operatorname{bcc}(X) = \operatorname{cl} \mathring{Q} \cap \operatorname{bcc}(X)$, and since $\operatorname{bcc}(X)$ is open we get $\mathring{Q} \cap \operatorname{bcc}(X) \neq \emptyset$. We will show that

$$Q \cap \operatorname{ucc}(X) = \emptyset. \tag{45}$$

Indeed, if (45) is not true, then a similar argument shows that $\mathring{Q} \cap \operatorname{ucc}(X) \neq \emptyset$. Since $\mathring{Q}$ is a connected set, we conclude that $\mathring{Q} \cap X \neq \emptyset$, which implies $Q \subset X$, a contradiction. Thus (45) shows that $Q \in \mathcal{X}'$, i.e. $x \in \bigcup \mathcal{X}'$ $\square$

The following proposition is straightforward.

Proposition 9.6. *If $\mathcal{A} \subset \mathcal{X}$, then $\mathcal{A} \subset \mathcal{K}_{\mathcal{X}}(\operatorname{bcf}(|\mathcal{A}|))$*

$\square$

Assume $U \subset \mathbb{R}^d$ is open. We will now consider a map $f : U \to \mathbb{R}^d$ which is a homeomorphism onto $f(U)$.

Theorem 9.7. *If $X \subset U$ is compact, then*

$$f(X) \subset \operatorname{bcf}(f(\operatorname{bd} X)). \tag{46}$$

Proof: Since f is a homeomorphism, we get from Lemma 9.4(ii)

$$\operatorname{ucc}(f(\operatorname{bd} X)) = \operatorname{ucc}(\operatorname{bd} f(X)) = \operatorname{ucc}(f(X))$$

and from Lemma 9.5(i)

$$f(X) \subset \operatorname{bcf}(f(X)) = \mathbb{R}^d \setminus \operatorname{ucc}(f(X)) =$$
$$\mathbb{R}^d \setminus \operatorname{ucc}(f(\operatorname{bd} X)) = \operatorname{bcf}(f(\operatorname{bd} X)).$$

□

Now we are ready to present a theorem, which may be used to construct index quadruples on the basis of a combinatorial enclosure of $f|_{\operatorname{bd} N}$.

Theorem 9.8. *Assume $\mathcal{N} \subset \mathcal{X}$ is $\mathcal{X}$-complete, $N := |\mathcal{N}|$ and $\mathcal{G} : \operatorname{bd}_{\mathcal{X}} \mathcal{N} : \rightrightarrows \mathcal{X}$ is a combinatorial enclosure of $f|_{\operatorname{bd} N}$. If $\mathcal{M} \subset \mathcal{X}$ is such that*

$$\mathcal{K}_{\mathcal{X}}(\operatorname{bcf}(|\mathcal{G}(\operatorname{bd}_{\mathcal{X}} \mathcal{N})|)) \subset \mathcal{M} \tag{47}$$

and

$$\mathcal{M} \bar{\cap} \operatorname{bd}_{\mathcal{X}} \mathcal{N} \bar{\subset} \mathcal{N}^-, \tag{48}$$

then N is an isolating block for f and $(|\mathcal{N}|, |\mathcal{N}^-|, |\mathcal{N} \cup \mathcal{M}|, |\mathcal{N}^- \cup (\mathcal{M} \setminus \mathcal{N})|$ is a weak index quadruple for f in N.

Proof: Let us define the map $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ by putting

$$\mathcal{F}(Q) := o_{\mathcal{X}}(f(Q)) \cup \bigcup \{\, \mathcal{G}(R) \mid R \in \operatorname{bd}_{\mathcal{X}} \mathcal{N},\ R \subset Q \,\}.$$

It is a lengthy but straightforward task to verify that $\mathcal{F}$ is a well defined combinatorial enclosure of f and $\mathcal{F}|_{\operatorname{bd}_{\mathcal{X}} \mathcal{N}} = \mathcal{G}$.

We will show that

$$\mathcal{F}(\mathcal{N}) \subset \mathcal{M}. \tag{49}$$

Let $P \in \mathcal{F}(\mathcal{N})$. If $P \in \mathcal{G}(R)$ for some $R \in \operatorname{bd}_{\mathcal{X}} \mathcal{N}$, then $P \in \mathcal{G}(\operatorname{bd}_{\mathcal{X}} \mathcal{N}) \subset \mathcal{M}$ by Proposition 9.6 and (47). Otherwise $P \subset P' \in o_d(f(Q))$ for some $Q \in \mathcal{N}$. We have from Theorem 9.7, Proposition 6.10 and Lemma 9.5(ii-iii) that

$$f(N) \subset \operatorname{bcf}(f(\operatorname{bd} N)) \subset \operatorname{bcf}(\operatorname{int} |\mathcal{F}(\operatorname{bd}_{\mathcal{X}} \mathcal{N}|) \subset \operatorname{int} \operatorname{bcf}(|\mathcal{F}(\operatorname{bd}_{\mathcal{X}} \mathcal{N}|)$$

Since $P' \cap f(Q) \neq \emptyset$, we get

$$P' \cap \operatorname{int} \operatorname{bcf}(|\mathcal{F}(\operatorname{bd}_{\mathcal{X}} \mathcal{N}|) \neq \emptyset.$$

However, by Lemma 9.5(v) the set $\operatorname{bcf}(|\mathcal{F}(\operatorname{bd}_{\mathcal{X}}\mathcal{N}|)$ is full cubical, so that by Proposition 6.1(i) we get

$$P' \subset \operatorname{bcf}(|\mathcal{F}(\operatorname{bd}_{\mathcal{X}}\mathcal{N}|).$$

It follows that $P \subset \operatorname{bcf}(|\mathcal{F}(\operatorname{bd}_{\mathcal{X}}\mathcal{N}|)$, i.e. $P \in \mathcal{K}_{\mathcal{X}}(\operatorname{bcf}(|\mathcal{G}(\operatorname{bd}_{\mathcal{X}}\mathcal{N})|))$ and by (47) $P \in \mathcal{M}$, which proves (49).

We will show that

$$\mathcal{M} \cap \operatorname{bd}_{\mathcal{X}}\mathcal{N} \subset \mathcal{N}^-. \tag{50}$$

Let $Q \in \mathcal{M} \cap \operatorname{bd}_{\mathcal{X}}\mathcal{N}$. Then $Q \in \mathcal{M} \bar{\cap} \operatorname{bd}_{\mathcal{X}}\mathcal{N}$ and by (48) we have $Q \subset Q'$ for some $Q' \in \mathcal{N}^-$. By Definition 6.7(ii) we get $Q \in \mathcal{N}^-$.

Now properties (49) and (50) imply that $\mathcal{F}(\mathcal{N}) \cap \operatorname{bd}_{\mathcal{X}}\mathcal{N} \subset \mathcal{N}^-$. Therefore $\mathcal{N}$ is an isolating block for $\mathcal{F}$ by Theorem 9.2 and $(\mathcal{N}, \mathcal{N}^-)$ is a combinatorial index pair for $\mathcal{F}$ by Theorem 9.3. Since

$$\mathcal{F}(\mathcal{N}) \bar{\cap} \operatorname{bd}_{\mathcal{X}}\mathcal{N} \subset \mathcal{M} \bar{\cap} \operatorname{bd}_{\mathcal{X}}\mathcal{N} \mathrel{\bar{\subset}} \mathcal{N}^-,$$

the conclusion follows now from Theorem 8.1 □

10. Algorithms.

Theorems presented in the previous sections lead to easy to implement algorithms. In this section we present some examples. We assume that there are given data structures `set` and `combinatorialMap` which allow us to store respectively families of elementary cubes and combinatorial multivalued maps.

Algorithm 10.1. Negative Invariant Part

```
function negativeInvariantPart(set N, combinatorialMap F)
F := F_N;
S := N;
repeat
    S' := S;
    S := F(S);
until (S = S');
return S;
```

Proposition 10.2. *Assume Algorithm 10.1 is called with* `N` *representing a collection of cubes* $\mathcal{N}$ *and* `F` *representing a combinatorial multivalued map* $\mathcal{F}$*. Then it always stops and returns the positive invariant part of* $\mathcal{F}$ *in* $\mathcal{N}$*.*

Proof: The algorithm stops, because the sequence containing the consecutive values of variable `S` is decreasing. The conclusion follows now from Theorem 6.13. □

Algorithm 10.3. Index Quadruple

function $\mathtt{indexQuadruple}(\mathtt{set}\ \mathtt{N}, \mathtt{combinatorialMap}\ \mathtt{F})$
$\mathtt{S}^- := \mathtt{negativeInvariantPart}(\mathtt{N}, \mathtt{F})$;
$\mathtt{S}^+ := \mathtt{negativeInvariantPart}(\mathtt{N}, \mathtt{F}^{-1})$;
if $\mathtt{S}^- \cap \mathtt{S}^+ \subset \mathtt{int}(\mathtt{N})$ **then**
$\mathtt{P}_1 := \mathtt{S}^-$; $\mathtt{P}_2 := \mathtt{S}^- \setminus \mathtt{S}^+$;
$\bar{\mathtt{P}}_1 := \mathtt{P}_1 \cup \mathtt{F}(\mathtt{P}_1)$; $\bar{\mathtt{P}}_2 := \mathtt{P}_2 \cup \mathtt{F}(\mathtt{P}_1) \setminus \mathtt{P}_1$;
return $(\mathtt{P}_1, \mathtt{P}_2, \bar{\mathtt{P}}_1, \bar{\mathtt{P}}_2)$;
else
return "*Failure*";
endif;

Theorem 10.4. *Let $f : U \to \mathbb{R}^d$ be a continuous map defined on an open subset of $\mathbb{R}^d$ and let $N \subset U$ be a full cubical set. Assume Algorithm 10.3 is called with $\mathtt{N}$ representing a collection of cubes $\mathcal{N}$ such that $|\mathcal{N}| = N$ and $\mathtt{F}$ representing a combinatorial enclosure $\mathcal{F}$ of f. If it does not fail, then it returns a weak index quadruple for f and N.*

Proof: The conclusion follows immediately from Theorem 7.3, Theorem 8.1 and Proposition 8.2. □

Similarly to Algorithm 10.3 one can obtain an algorithm for weak index quadruples based on isolating blocks Theorem 9.8. Details are left to the reader.

11. Examples

Example 11.1. Consider the Hénon map $h : \mathbb{R}^2 \to \mathbb{R}^2$ given by the formula

$$h(x, y) := (1 + y/5 - ax^2, 5bx)$$

at the parameter values $a = 1.4$ and $b = 0.2$. Figure 1 presents a family $\mathcal{M}$ of full cubes (dark and light gray) and its invariant part $\mathcal{S} := \operatorname{Inv}(\mathcal{M}, \mathcal{H})$ (light gray) for a combinatorial enclosure $\mathcal{H}$ of h.

Clearly, $\mathcal{M}$ is a combinatorial isolating neighborhood of $\mathcal{H}$, so that we can now apply Algorithm 10.3 to $\mathcal{M}$. However, by Proposition 7.1 the family $\mathcal{N} := o_d(\mathcal{S})$ is also an isolating neighborhood isolating $\mathcal{S}$, so, to avoid excessive computations, it is worth to apply Algorithm 10.3 to $\mathcal{N}$. The resulting combinatorial index quadruple $(\mathcal{P}_1, \mathcal{P}_2, \bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2)$ is indicated in Figure 2. Theorem 10.4 implies that

$$(P_1, P_2, \bar{P}_1, \bar{P}_2) := (|\mathcal{P}_1|, |\mathcal{P}_2|, |\bar{\mathcal{P}}_1|, |\bar{\mathcal{P}}_2|)$$

is an index quadruple for h. The figure also presents the decomposition of $\operatorname{cl}(P_1 \setminus P_2)$ into eight compact connected components consisting of

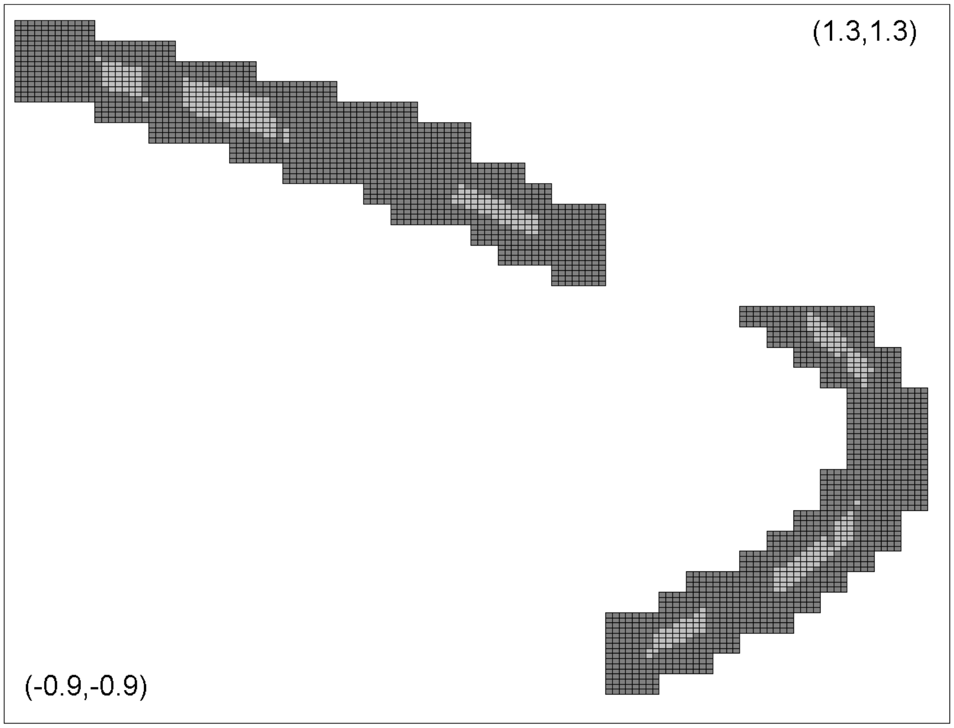

FIGURE 1. An isolating neighborhood $\mathcal{M}$ for a multivalued enclosure of the Hénon map with the invariant part $\mathcal{S}$ indicated in light gray.

full cubical sets $K_i = |\mathcal{K}_i|$ for $i = 1, 2, \ldots 8$. Computing the transition matrix $A = (a_{ij})$, where $a_{ij} := \operatorname{sgn}\operatorname{card}\mathcal{K}_j \cap \mathcal{H}(\mathcal{K}_i)$, we obtain

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The zeros in this matrix indicate which itineraries of h through the decomposition $\bigcup_{j=1}^{8} K_j$ are excluded. To show that all the others are allowed we first compute the matrices of the homology of the inclusion map $\iota : (P_1, P_2) \hookrightarrow (\bar{P}_1, \bar{P}_2)$ and using the combinatorial multivalued map $\mathcal{H}$ also the matrices of the homology of the induced map $h_{P\bar{P}} : (P_1, P_2) \to (\bar{P}_1, \bar{P}_2)$. We do so using the algorithm described in [6]. The computation results in all matrices being zero except the matrices in dimension one.

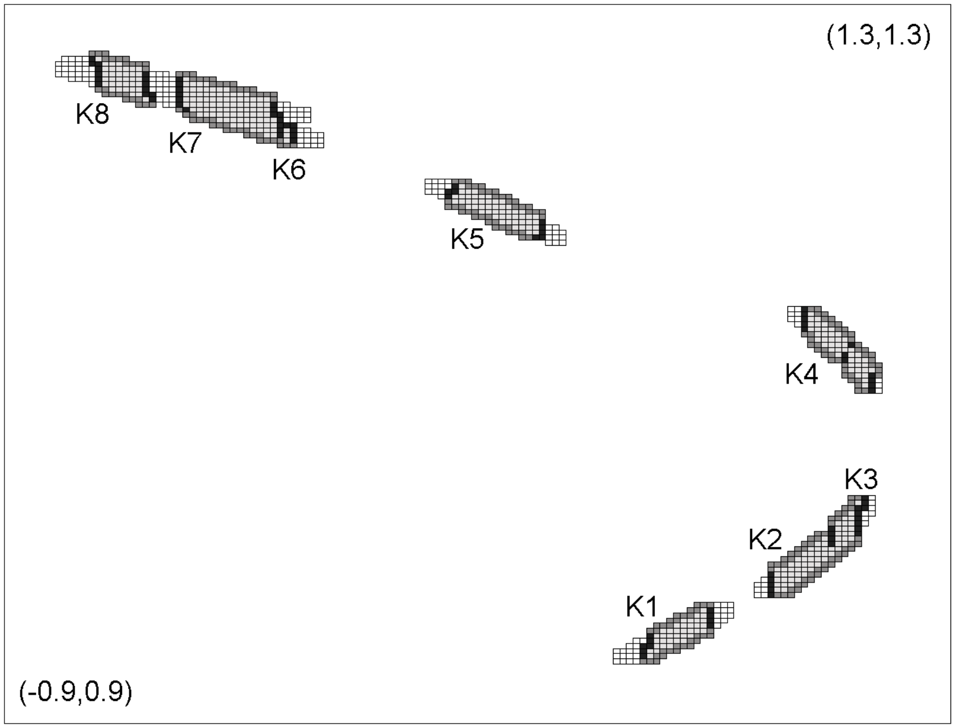

FIGURE 2. An index quadruple $(\mathcal{P}_1, \mathcal{P}_2, \bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2)$ for a combinatorial enclosure of the Hénon map. The set $\mathcal{P}_1$ consists of all light gray and black cubes, the set $\mathcal{P}_2$ consists of all black cubes, the set $\bar{\mathcal{P}}_1$ consists of all white, light gray and black cubes and the set $\bar{\mathcal{P}}_2$ consists of all white and black cubes. All cubes except the whites cubes constitute the combinatorial isolating neighborhood $\mathcal{N} = o(\mathcal{S})$.

The matrix A shows that the following three periodic itineraries are not excluded

$$\theta_1 := (2, 7),\ \theta_2 := (4, 5, 1, 8),\ \theta_3 := (4, 5, 1, 8, 2, 7)$$

To prove that h indeed possesses such periodic orbits we use [18, Theorem 4.5]. For this end it is enough to verify that the Lefschetz numbers of the maps Θ_i induced in homology by h_{P,θ_i} are non zero. Given the computed matrices of the maps induced in homology by $\iota : (P_1, P_2) \hookrightarrow (\bar{P}_1, \bar{P}_2)$ and $h_{P\bar{P}} : (P_1, P_2) \to (\bar{P}_1, \bar{P}_2)$ it is a straightforward computational task to find that these Lefschetz numbers are indeed non zero. Moreover, one can also verify that the composition of any two of the three maps Θ_i is, up to a sign, one of these maps again.

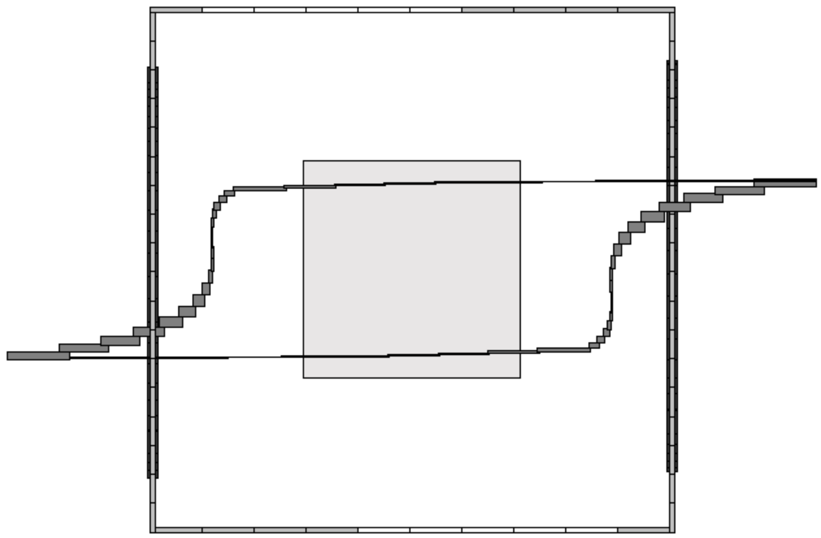

FIGURE 3. Projections onto the complex plane of two sample components of an isolating block for a Poincaré map of the dynamical system induced in $\mathbb{R}^3$ by the equation (51). The picture is artificially rescaled for better visualization.

This implies that for any concatenation of the sequences θ_i there exists a periodic point in S following this concatenation as its itinerary. Therefore we have proved the following theorem.

Theorem 11.2. *Let S be the invariant part of $\bigcup_{j=1}^{8} K_j$ under the Hénon map at parameter values $a = 1.4$ and $b = 0.2$. Then there exists a semiconjugacy $\rho : S \to \Sigma_A$ onto the set of biinfinite sequences on 8 symbols admissible under the matrix A such that for each periodic sequence $\theta \in \Sigma_A$ with period p, $\rho^{-1}(\theta)$ contains a periodic orbit with period p.*

Example 11.3. Consider the following differential equation in the complex plane

$$z' = (1 + e^{i\varphi t}|z|^2)\bar{z}. \tag{51}$$

The system induces a semidynamical system in $\mathbb{R}^3$ (with t as the third variable). It can be proved analytically [16] that for small positive φ the $2\pi/\varphi$-translation map exhibits chaotic dynamics. A similar fact for $\theta = 1$ may be obtained via a computer assisted proof. The proof requires non standard computational techniques, because the quadratic term on the right hand side of the equations causes that practically all the trajectories of the system escape to infinity in a very short time. To make things worse, for the short time segment when the trajectories can be enclosed numerically an extremely strong expansion is observed. The techniques introduced in this paper, in particular Theorem 9.8, constitute one of the two tools needed in overcoming this difficulty. The big square in Figure 3 is the projection onto the complex plane of a sample component $\mathcal{N}_1$ of a multicomponent combinatorial isolating block $\mathcal{N}$ constructed in rigorous numerical computation for a combinatorial multivalued representation $\mathcal{F}$ of some Poincaré map of the dynamical system induced in $\mathbb{R}^3$ by the equation (51). The combinatorial boundary of $\mathcal{N}_1$ is represented by a cubical set consisting of elementary cubes of dimension one (in the picture the cubes are inflated to dimension two for better visualization). The gray part of the boundary of $\mathcal{N}_1$ is the restriction of $\mathcal{N}^-$ to $\mathcal{N}_1$. The restriction of $\mathcal{F}(\mathcal{N}) \cap \operatorname{bd}\mathcal{N}$ to $\mathcal{N}_1$ is marked in black. The picture indicates that for $\mathcal{N}_1$ the condition (42) is satisfied. The smaller square is the projection of another component $\mathcal{N}_2$. It is the only component of $\mathcal{N}$ intersecting $\mathcal{F}(\mathcal{N}_1)$. The small rectangles aligned along a closed curve intersecting $\mathcal{N}_2$ in two pieces constitute the values of $\mathcal{F}$ on the one dimensional elementary intervals covering the boundary of $\mathcal{N}_1$. The values grow very rapidly and the computations break down when the cubes covering the boundary are taken to be two dimensional instead of one dimensional. The number of components of $\mathcal{N}$ needed in the actual computation was of order 100.

The proof with all the details will be presented in [11].

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