The Euler-Poincaré characteristic of index maps*

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Abstract

We apply the concept of the Euler-Poincaré characteristic and the periodicity number to the index map of an isolated invariant set in order to obtain a new criterion for the existence of periodic points of a continuous map in a given set.

1 Introduction

Among the fundamental problems in discrete dynamical systems is the search of fixed and periodic points. The Lefschetz Fixed Point Theorem is a classical example how a topological invariant, the Lefschetz number, may be used to guarantee the existence of fixed points. In 1953 F. Fuller [7] showed that every homeomorphism of a connected polyhedron with Euler-Poincaré characteristic different from zero has a periodic point. This was generalized in 1969 by C. Bowszyc [2] who introduced the Euler number and the periodicity number of a continuous map of a compact polyhedron and showed that if one of these numbers is nonzero then the map has a periodic point. The aim of this note is to generalize these results to the case of an isolated invariant set.

Let X be a compact metric space and let $f : X \to X$ be a continuous map. An isolated invariant set S of f is a compact invariant set of f, which

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is maximal in some its open neighborhood. For such a set its cohomological Conely index (see e.g. [15]) takes the form of the pair (E, e), where E is a graded vector space and e is an automorphism acting on E, called the index automorphism. If X is an ANR (cf. [3]), for instance if X is a polyhedron, then E is of finite type, i.e. all E_k have finite dimension and almost all are zero (see [14]). For such an E one defines its Euler-Poincaré characteristic by

$$\chi(E) := \sum_{k=0}^{\infty} (-1)^k \dim E_k. \tag{1}$$

The main result of this paper is the following theorem.

Theorem 1 *Assume S is an isolated invariant set of a continuous map $f : X \to X$ acting on a compact ANR X. If the Euler-Poincaré characteristic of the Conley index of S is non-zero, then f has a periodic point in S.*

The Euler-Poincaré characteristic of the Conley index reduces to the Euler number of the map in the special case when X is a compact polyhedron and the isolated invariant set S coincides with the whole space X. It reduces to the Euler-Poincaré characteristic of the polyhedron if additionally one assumes that f is a homeomorphism. Theorem 1 is a straightforward consequence of a more general result based on an extension of the Bowszyc's concept of periodicity number to the index automorphism of the Conley index.

The results of Fuller and Bowszyc are global in the sense that no information is given about the existence of fixed points or periodic points of a map in a prescribed set. In 1968 C. Bowszyc [1] presented a relative version of the Lefschetz Fixed Point Theorem, which may be considered as a step towards localization of this theorem. In [11] and [13] index pairs were chosen as a tool for obtaining a Lefschetz Fixed Point Type Theorem working on a set. The essential difference when compared to the Bowszyc's result is that in the case of an index pair there is no requirement that the map maps the pair into itself.

Using the language of [6] one can say that the Bowszyc's relative result works in the filtration setting, whereas the index pairs may be viewed as a tool to reduce the problem to a filtration setting. However, an obstacle is the need to work with ANRs to ensure that the Euler-Poincar'e characteristic makes sense. The existence of index pairs consisting of ANRs is proved for isolated invariant sets in the Euclidean space [22] and in the Hilbert cube

[19] but for a general ANR X the answer is not known. Moreover, it is easy to give examples of isolated invariant sets in a compact ANR X for which exist index pairs consisting of sets which are not ANRs. The homology of such index pairs may not be of finite type despite the fact that the Conley index of S is (see [14]). To overcome this difficulty we use open index pairs introduced in [13], because open subsets of ANRs are always ANRs. Since it has not been proved so far that open index pairs may be used to compute the Conley index, we prove that in every neighborhood of an isolated invariant set there exists a pair of index pairs, one open and one closed, whose index automorphisms are conjugate. This does not prove that every open index pair may be used to compute the Conley index but it is sufficent to prove Theorem 1.

The referee suggested to us a different approach to the proof. When working with the Conley index, the easiest way to achieve the filtration setting is to quotient out the second set in the index pair. In order to guarantee that the quotient is an ANR it is enough to take an index pair consisting of compact ANRs, because the quotient of two compact ANRs is a compact ANR (see [9], chapter VI, Theorem 1.2). To overcome the problem with the existence of index pairs consisting of compact ANRs one can lift the map f to some open subset of the Hilbert cube, similarly to the proof of Theorem 6 in [19] and then project back the periodic point to the original space. However, the lifting requires non trivial results on approximating compact sets in normal linear spaces, so our feeling is that our approach is more elementary. Even more important is the fact that although both the process of taking the quotient and the lifting to an infinite dimensional space are convenient tools for proofs, they are hard on the algorithmic side [16]. On the other hand, taking into account the results of [16] and [17], the process of computing the relative homology of an index pair is easily algorithmizable, therefore the proof we present may be turned into an algorithm computing the Euler-Poincaré characteristic of the Conley index of an isolated invariant set. Since the computer assisted proofs in dynamics are more and more common [10], this may lead to broader scope of potential applications.

For the sake of simplicity we present the results for compact spaces. However, a generalization to locally compact spaces and even to maps of compact attraction on non compact metric ANRs along the lines in [18] is straightforward.

2 Preliminaries

In this section we briefly recall main definitions and results used in this note.

Throughout the paper $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ are used to denote the sets of all reals, integers and positive integers, respectively. For a topological space X and $A \subset X$ we denote the interior and the closure of A respectively by $\operatorname{int}_X(A)$ and $\operatorname{cl}_X(A)$. When the space X is clear from the context, we drop the subscript X in this notation.

Let X be a compact ANR (see [3]). Assume $f : X \to X$ is continuous. The function $\sigma : \mathbb{Z} \to X$ is called a *solution to f through x* if $f(\sigma(i)) = \sigma(i+1)$ for $i \in \mathbb{Z}$ and $\sigma(0) = x$. We define $\operatorname{Inv}(N, f)$, the *invariant part* of $N \subset X$, as the set of all $x \in N$ which admit a solution σ to f through x with $\sigma(\mathbb{Z}) \subset N$. The set N is called *invariant* (with respect to f) if $\operatorname{Inv}(N, f) = N$. A compact set $N \subset X$ is called an *isolating neighborhood* if $\operatorname{Inv}(N, f) \subset \operatorname{int}(N)$. A set S which admits an isolating neighborhood N such that $S = \operatorname{Inv}(N, f)$ is called an *isolated invariant set*.

Definition 1 (cf. [11, 13]) Let $S \subset X$ be an isolated invariant set. A pair $P = (P_1, P_2)$ of subsets of X will be called an *index pair* for S if the following conditions are satisfied:

$$P_2 \cap f^{-1}(P_1) \subset P_2, \tag{2}$$

$$P_1 \setminus f^{-1}(P_1) \subset P_2, \tag{3}$$

$$S = \operatorname{Inv}(P_1, f) \subset \operatorname{int}(P_1 \setminus P_2). \tag{4}$$

The index pair is called closed (open) if both P_1, P_2 are closed (open) in X. In the sequel every index pair will be assumed to be closed unless explicitly specified as an open index pair. The index pair (P_1, P_2) is called *regular* if

$$\operatorname{cl}(f(P_2) \setminus P_1) \cap \operatorname{cl}(P_1 \setminus P_2) = \emptyset \tag{5}$$

and there exists a set U open in P_1 such that

$$\operatorname{cl}_{P_1} P_2 \subset U \text{ and } f(U \setminus P_2) \subset P_2. \tag{6}$$

Every isolated invariant set admits an index pair and the Conley index captures the common information present in various index pairs. There are many ways to define the Conley index (see e.g. [20, 15, 21, 6]). We need an algebraic Conley index, so we will use the definition based on the Leray reduction (see [12]). Let $\varphi = \{\varphi_k\}$ be an endomorphism of a graded vector space $E = \{E_k\}$ over the field of rational numbers. The generalized kernel of φ is defined as $\operatorname{gker}(\varphi) := \bigcup\{\ker \varphi^n \mid n \in \mathbb{N}\}$. We call φ the *Leray endomorphism* provided the quotient space $E' := E/\operatorname{gker}(\varphi)$ is of a *finite type*. It is straightforward to verify that for every Leray endomorphism φ there is a well defined induced graded automorphism $\varphi' = \{\varphi'_k\}$ consisting of induced automorphisms $\varphi'_k : E'_k \to E'_k$. It is called the *Leray reduction* of φ.

The following proposition is an immediate consequence of the definition of the index pair.

Proposition 1 (cf. [11], Lemma 1 and [13], Proposition 4) *Assume $P = (P_1, P_2)$ is a regular index pair or a regular open index pair for S. Then f maps the pair (P_1, P_2) into $(P_1 \cup f(P_2), P_2 \cup f(P_2))$ and the inclusion*

$$i_P : (P_1, P_2) \to (P_1 \cup f(P_2), P_2 \cup f(P_2))$$

induces an isomorphism in singular homology with rational coefficients.

Let H_* denote the functor of singular homology with rational coefficients. The above proposition enables us to define the *index map of the index pair* $I_P : H_*(P_1, P_2) \to H_*(P_1, P_2)$ by $I_P := (i_P)_*^{-1} \circ (f_P)_*$, where f_P denotes the mapping f considered as a mapping of the pair (P_1, P_2) into $(P_1 \cup f(P_2), P_2 \cup f(P_2))$.

We have the following theorem, which follows from Theorem 6.2 in [15] and Theorem 3 in [14]

Theorem 2 *Assume X is a compact ANR and S is an isolated invariant set of a continuous map $f : X \to X$. If P and Q are two index pairs of S then I_P and I_Q are Leray endomorphisms and the Leray reductions of I_P and I_Q are conjugate.*

Since every isolated invariant set always admits at least one index pair (see [11], Theorem 2 and 3), the above theorem allows us to define I_S, the *index autmorphism* of S, as the conjugacy class of I_P for any index pair P of S.

3 Main results

Given $\alpha : E \to E$, an endomorphism of a vector space E, let $\mathcal{L}(\alpha)$ denote the set of all complex, nonzero eigenvalues of α and for $\lambda \in \mathcal{L}(\alpha)$ let $m(\lambda, \alpha)$

denote the multiplicity of λ as an eigenvalue of α. If E is finitely dimensional, then let $\operatorname{tr}(\alpha)$ denote the trace of α. We have for every integer $n \geq 1$ (cf. e.g. [4], Section 7, Proposition 1.2.)

$$\operatorname{tr}(\alpha^n) = \sum_{\lambda \in \mathcal{L}(\alpha)} m(\lambda, \alpha)\lambda^n. \tag{7}$$

Note that the above equality is also true for $n = 0$ under the assumption that $\alpha^0 = \operatorname{id}_E$.

Now let $\varphi = \{\varphi_k\}$ be a Leray endomorphism of a graded vector space $E = \{E_k\}$. We define the *Lefschetz number* of φ as the alternating sum of the traces of Leray reductions of φ_k

$$\Lambda(\varphi) := \sum_{k=0}^{\infty} (-1)^k \operatorname{tr}(\varphi_k').$$

Note that the sum is finite, because in the case of a Leray endomorphism $\varphi_k' = 0$ for almost all k.

Let $\mathcal{L}(\varphi) := \cup_k \mathcal{L}(\varphi_k)$ and for a $\lambda \in \mathcal{L}(\varphi)$ put

$$s(\lambda, \varphi) := \sum_{k=0}^{\infty} (-1)^k m(\lambda, \varphi_k).$$

Since obviously $\mathcal{L}(\varphi_k) = \mathcal{L}(\varphi_k')$, from (7) we obtain for every $n \in \mathbb{N}$

$$\Lambda(\varphi^n) := \sum_{k=0}^{\infty} \sum_{\lambda \in \mathcal{L}(\varphi_k)} (-1)^k m(\lambda, \varphi_k)\lambda^n = \sum_{\lambda \in \mathcal{L}(\varphi)} s(\lambda, \varphi)\lambda^n. \tag{8}$$

The number

$$\chi(\varphi) := \Lambda(\varphi^0)$$

is called the *Euler-Poincaré characteristic* of φ. Since $\varphi_k^0 = \operatorname{id}_{E_k}$, we get

$$\chi(\varphi) = \sum_{k=0}^{\infty} (-1)^k \dim E_k.$$

Therefore the Euler Poincaré characteristic of a map is compatible with (1) in the setting of the Conley index, i.e. when E is a Conley index and φ is the corresponding index automorphism. Moreoveor, by (8) we have

$$\chi(\varphi) = \sum_{\lambda \in \mathcal{L}(\varphi)} s(\lambda, \varphi). \tag{9}$$

We define the *periodicity number of* φ by

$$\tau(\varphi) := \operatorname{card}\{\, \lambda \in \mathcal{L}(\varphi) \mid s(\lambda, \varphi) \neq 0 \,\}.$$

From (9) we get

$$\chi(\varphi) \neq 0 \;\Rightarrow\; \tau(\varphi) \neq 0. \tag{10}$$

It is straightforward to verify that the Euler-Poincaré characteristic, Lefschetz number and periodicity number of an endomorphism φ of a graded vector space do not depend on the conjugacy class of φ, which allows us to extend these concepts to conjugacy classes of endomorphisms. In particular they make sense for the Conley, which is defined only up to a conjugacy class.

Property (10) shows that in order to prove Theorem 1 it suffices to prove the following theorem.

Theorem 3 *Let X be a metric* ANR. *Assume S is an isolated invariant set of a continuous map $f : X \to X$. If $\tau(I_S) \neq 0$ then f has a periodic point in S.*

We postpone the proof of this theorem to Section 5. The following example serves as an elementary illustration of Theorem 1.

Example 1 Consider the logistic map $f : [0,1] \to [0,1]$ given for $x \in [0,1]$ by

$$f(x) := 4x(1-x).$$

It is a lengthy but straightforward task to verify that $N = [\frac{1}{4}, \frac{1}{2}] \cup [\frac{13}{16}, \frac{15}{16}]$ is an isolating neighborhood for f isolating an isolated invariant set S whose Conley index is nontrivial only in dimension one with the index map

$$I_S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore $\chi(I_S)$=-2. It follows that f has a periodic point in S. Obviously, the existence of this periodic point may be easily established by a direct computation. However, it is straightforward to generalize this example to the case of the map $g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}^d$ given for $(x, y) \in \mathbb{R} \times \mathbb{R}^d$ by

$$g(x, y) := (f(x) + \mu(x, y), \nu(x, y)),$$

where $\mu, \nu : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ are some continuous functions. Let B denote the unit ball in $\mathbb{R}^d$ centered at the origin. It is straightforward to verify that for

μ, ν sufficiently small $N \times B$ is an isolating neighborhood for g with the same Conley index as the Conley index of S. It follows that g has a periodic point in $N \times B$.

Concrete applications of Theorems 1 and 3 are for isolating neighborhoods computed algorithmically by means of rigorous numerics (see [22, 16]). This is because the Conley indices obtained algorithmically may be very complicated and the Euler-Poincaré characteristic may serve as an elementary indicator, that an interesting dynamics is detected.

4 Auxiliary results and proofs

In the sequel $\mathrm{ind}(f, V)$ will denote the fixed point index of f in V (see [5]).

The following theorem is a special case of Theorem 3 in [13]

Theorem 4 (cf. [13], Theorem 3) *Let X be a compact* ANR *and let S be an isolated invariant set of a continuous map $f : X \to X$. If (P_1, P_2) is a regular open index pair for S then I_P is a Leray endomorphism. Moreover, for every $n \in \mathbb{N}$ there exists U, an open neighborhood of S, such that if $P_1 \subset U$, then*

$$\mathrm{ind}(f^n, \mathrm{int}(P_1 \setminus P_2)) = \Lambda(I_P^n). \tag{11}$$

Note that the localization of the index pair in a sufficiently small neighborhood of the isolated invariant set S is crucial for the validity of (11).

The following proposition follows directly from Lemma 2.1 in [8].

Proposition 2 *Assume ϕ, ψ are two endomorphisms of graded vector spaces such that $\phi = gh$, $\psi = hg$ for some morphisms $h : E \to F$, $g : F \to E$ (this is in particular satisfied if ϕ and ψ are conjugated). If one of them is a Leray endomorphism then so is the other and $\Lambda(\phi^k) = \Lambda(\psi^k)$ for all natural k.*

We have the following

Lemma 1 *Let $P_2 \subset P_1$ be open subsets of X and let $A \subset X$ satisfy $A \cap P_1 \subset P_2$ and $\mathrm{cl}(P_1 \setminus P_2) \cap A = \emptyset$. Then $i : (P_1, P_2) \hookrightarrow (P_1 \cup A, P_2 \cup A)$ is an excision, i.e. it induces an isomorphism in homology.*

Proof: Observe that the excised set is $(P_1 \cup A) \setminus P_1 = A \setminus P_1 = A \setminus P_2 = (P_2 \cup A) \setminus P_2$. Since $A \setminus P_1$ is closed in $P_1 \cup A$ and $P_2 \cup A$ is open in $P_1 \cup A$, we observe that $\mathrm{cl}_{P_1 \cup A}(A \setminus P_1) = A \setminus P_1 \subset P_2 \cup A = \mathrm{int}_{P_1 \cup A}(P_2 \cup A)$. Therefore the standard excision property of the singular homology implies that i is an excision. □

Let U be an open neighborhood of an isolated invariant set S of f. We say the function $\kappa : U \to [0, \infty)$ *increases* along trajectories of f on U if

$$\kappa(x) > 0, f(x) \in U \Rightarrow \kappa(f(x)) > \kappa(x).$$

and *decreases* along trajectories of f on U if

$$\kappa(x) > 0, f(x) \in U \Rightarrow \kappa(f(x)) < \kappa(x).$$

Consider functions $\phi, \gamma : U \to [0, \infty]$. We say (cf. [11]) that the pair (ϕ, γ) is a Lyapunov pair for S if the following conditions are satisfied

(i) ϕ decreases and γ increases along trajectories of f on U,

(ii) $S \subset \phi^{-1}(0) \cap \gamma^{-1}(0)$,

(iii) for every neighborhood W of K there exists an $\epsilon > 0$ such that the set

$$H(\epsilon, \phi, \gamma) := \{x \in U \mid \phi(x) < \epsilon, \gamma(x) < \epsilon\}$$

satisfies the condition

$$\mathrm{cl} H(\epsilon, \phi, \gamma) \subset W.$$

We say that (ϕ, γ) is a continuous Lyapunov pair for S if (ϕ, γ) is a Lyapunov pair for S and both ϕ an γ are continuous.

Theorem 5 *Let X be a compact* ANR. *Assume $f : X \to X$ is a continuous map and S is an isolated invariant set of f. For every neighborhood U of S there exist an open index pair $P = (P_1, P_2)$ and a closed index pair $Q = (Q_1, Q_2)$ such that $P \subset Q$, $Q_1 \subset U$, both I_Q and I_P are Leray endomorphisms and for every $n \in \mathbb{N}$*

$$\Lambda(I_Q^n) = \Lambda(I_P^n).$$

Proof: Let (φ, γ) be a continuous Lyapunov pair for S on some its neighborhood (the existence of such a pair follows from Theorem 1 and Theorem 2 in [11]). For small enough $\varepsilon > 0$ let $P = (P_1, P_2)$ and $Q = (Q_1, Q_2)$ be given by

$$P_1 := \varphi^{-1}([0,\varepsilon)) \cap \gamma^{-1}([0,\varepsilon)), \quad P_2 := \left\{ x \in P_1 : \gamma(f(x)) > \frac{\varepsilon}{2} \right\} \tag{12}$$

and

$$Q_1 := \varphi^{-1}([0,\varepsilon]) \cap \gamma^{-1}([0,\varepsilon]), \quad Q_2 := \left\{ x \in Q_1 : \gamma(f(x)) \geq \frac{\varepsilon}{2} \right\}. \tag{13}$$

It is proved respectively in [11] and [13] that P and Q are regular index pairs for S. Clearly Q is closed, P is open, $P \subset Q$ and $Q_1 \subset U$.

Consider $x \in Q_1$ with $f(x) \notin P_1$. Then $\varphi(x) \leq \varepsilon$ and $\gamma(x) \leq \varepsilon$. Since $\varphi(f(x)) < \varphi(x) \leq \varepsilon$ and $f(x) \notin P_1$, we have $\gamma(f(x)) \geq \varepsilon$ which means that $x \in Q_2$. Thus $f(Q_1) \subset P_1 \cup f(Q_2)$. Consequently f maps the pair (Q_1, Q_2) into $(P_1 \cup f(Q_2), P_2 \cup f(Q_2))$.

Next we will prove that

$$\mathrm{cl}(P_1 \setminus P_2) \cap f(Q_2) = \emptyset. \tag{14}$$

Assume the contrary and consider $x \in \mathrm{cl}(P_1 \setminus P_2) \cap f(Q_2)$. Let $\{x_n\} \subset P_1 \setminus P_2$ be such that $x_n \to x$. Observe that then $\gamma(f(x_n)) \leq \frac{\varepsilon}{2}$ and consequently $\gamma(x) < \gamma(f(x)) \leq \frac{\varepsilon}{2}$. On the other hand, taking into account $u \in Q_2$ with $x = f(u)$ we have $\gamma(x) = \gamma(f(u)) \geq \frac{\varepsilon}{2}$ which brings a contradiction.

Let us consider $x \in f(Q_2) \cap P_1$. Then $x = f(u)$ with some $u \in Q_2$ which implies $\gamma(x) = \gamma(f(u)) \geq \frac{\varepsilon}{2}$. Moreover, since $\gamma(f(x)) > \gamma(x)$, we have $x \in P_2$. This shows that

$$f(Q_2) \cap P_1 \subset P_2. \tag{15}$$

Consider now the following commutative diagram

$$\begin{array}{ccccc}
 & & (P_1 \cup f(P_2), P_2 \cup f(P_2)) & & \\
 & f \nearrow & \downarrow k_1 & \nwarrow i_1 & \\
(P_1, P_2) & \xrightarrow{f} & (P_1 \cup f(Q_2), P_2 \cup f(Q_2)) & \xhookleftarrow{i_2} & (P_1, P_2) \\
\downarrow i_4 & f \nearrow & \downarrow k_2 & & \downarrow i_4 \\
(Q_1, Q_2) & \xrightarrow{f} & (Q_1 \cup f(Q_2), Q_2 \cup f(Q_2)) & \xhookleftarrow{i_3} & (Q_1, Q_2)
\end{array}$$

in which mappings denoted by f are induced by f treated as a mapping of a suitable pair and $i_1, i_2, i_3, i_4, k_1, k_2$ are inclusions. Taking into account (14) and (15) and applying Lemma 1 to $f(Q_2)$ we see that $i_2 : (P_1, P_2) \hookrightarrow (P_1 \cup f(Q_2), P_2 \cup f(Q_2))$ induces an isomorphism in homology. Both $i_1 : (P_1, P_2) \hookrightarrow (P_1 \cup f(P_2), P_2 \cup f(P_2))$ and $i_3 : (Q_1, Q_2) \hookrightarrow (Q_1 \cup f(Q_2), Q_2 \cup f(Q_2))$ induce isomorphisms in homology as (Q_1, Q_2) and (P_1, P_2) are regular index pairs.

Finally, observe that I_P and I_Q satisfy assertions of Proposition 2. Moreover, by Theorem 4 I_P is a Leray endomorphism, therefore so is I_Q and for every $n \in \mathbb{N}$ we have $\Lambda(I_Q^n) = \Lambda(I_P^n)$. This completes the proof. □

5 Proof of Theorem 3

Proof of Theorem 3. By Theorem 2 the index map I_S is a Leray endomorphism. Let $\mathcal{L}(I_S) = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$. We will show that there exists $n \in \mathbb{N}$ such that $\Lambda(I_S^n) \neq 0$. Assume the contrary. Then by (8)

$$\sum_{i=1}^{p} s(\lambda_i, I_S)\lambda_i^n = 0 \text{ for } n \in \mathbb{N}.$$

This shows that the numbers $s(\lambda_i, I_S)$ for $i = 1, 2, \ldots, p$ satisfy the system of linear equations

$$\begin{array}{ccccccccc} \lambda_1 x_1 & + & \lambda_2 x_2 & + & \cdots & + & \lambda_p x_p & = & 0 \\ \lambda_1^2 x_1 & + & \lambda_2^2 x_2 & + & \cdots & + & \lambda_p^2 x_p & = & 0 \\ . & . & . & . & \cdots & . & . & . & . \\ \lambda_1^p x_1 & + & \lambda_2^p x_2 & + & \cdots & + & \lambda_p^p x_p & = & 0. \end{array}$$

One can verify that the determinant of this system is

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_p \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_p^2 \\ . & . & \cdots & . \\ \lambda_1^p & \lambda_2^p & \cdots & \lambda_p^p \end{vmatrix} = \prod_{i=1}^{p} \lambda_i \prod_{1 \le i < j \le p} (\lambda_j - \lambda_i)\,.$$

Since $\lambda \in \mathcal{L}(I_S)$ are distinct, the determinant is nonzero. This implies $s(\lambda, I_S) = 0$ for each $\lambda \in \mathcal{L}(I_S)$. Therefore $\tau(I_S) = 0$, a contradiction.

Thus there exists an $n \in \mathbb{N}$ such that $\Lambda(I_S^n) \neq 0$. For such an n select U, an open neighborhood of S, such that the conclusion of Theorem 4 is satisfied.

By Theorem 5 one can find an open index pair $P = (P_1, P_2)$ and a closed index pair $Q = (Q_1, Q_2)$ such that $P \subset Q$ and $Q_1 \subset U$ and

$$\Lambda(I_Q^n) = \Lambda(I_P^n).$$

Is is straightforward to verify that, taking a smaller U if necessary, we also have

$$\{\, x \in U \mid f^n(x) = x \,\} \subset S. \tag{16}$$

Since P is an index pair of S, by definition I_S is the conjugacy class of I_P, therefore

$$\Lambda(I_S^n) = \Lambda(I_Q^n).$$

Thus it follows from Theorem 4 that

$$\mathrm{ind}(f^n, \mathrm{int}(P_1 \setminus P_2)) = \Lambda(I_P^n) = \Lambda(I_Q^n) = \Lambda(I_S^n) \neq 0.$$

and the fundamental property of the fixed point index implies that f^n has a fixed point in $\mathrm{int}(P_1 \setminus P_2)$. By (16) the fixed point is in S, which completes the proof. □

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