THE METHOD OF TOPOLOGICAL SECTIONS IN THE RIGOROUS NUMERICS OF DYNAMICAL SYSTEMS

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1. Introduction.

Computer assisted proofs in topological dymamics consist of three steps: the construction of a sufficiently good multivalued representation of the dynamical system based on rigorous numerics, the combinatorial computations of topological invariants which provide some information about the dynamical system and finally applying the relevant theory which allows to draw conclusions of interest from the computed topological invariants.

In order to obtain a multivalued representation of the dynamical system it is necessary to integrate it in a rigorous way to obtain bounds for the trajectories. This is achieved by applying some kind of Taylor expansion, for instance the Lohner method performed in interval arithmetic to include bounds for rounding errors (comp. [9]). The associated wrapping effect and exponential growth of error bounds causes that this process is computationally very expensive and in some cases impossible to carry out. One of the methods used to overcome the problems is the introduction of intermediate sections but even this method in some situations fails.

In this note we present a modification of the method of intermediate sections, which can significantly extend the range of problems in dynamical systems solvable by means of computer assisted proofs based on topological tools.

Throughout the paper $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote respectively the sets of positive integers, integers, rational numbers, real numbers and complex numbers. For $n \in \mathbb{N}$ we denote by $\mathbb{Z}_n$ the additive group of integers modulo n. Given a subset A of a topological set X by $\operatorname{cl}_X A$, $\operatorname{int}_X A$ and $\operatorname{bd}_X A$ we denote respectively the closure, the interior and the boundary of A in X. When X is clear from the context, we drop the subscript X.

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2. The Model Equation.

As our model equation we study the Poincaré map F of the nonautonomous periodic differential equation on the complex plane $\mathbb{C}$

$$z' = (1 + e^{i\eta t}|z|^2)\bar{z}. \tag{1}$$

considered by R. Srzednicki and K. Wójcik in [11]. Recall that the Poincaré map in this case is the T-translation along the solutions of the equation with $T := 2\pi/\eta$ being the period of the right hand side. The domain of F consists of all points in the complex plane whose trajectories do not escape to infinity in the time shorter than T.

The equation exhibits extremly strong expansion and actually most of its trajectories escape to infinity in a very short time, which causes that there are not many bounded trajectories of the equation and the domain of the Poincaré map is small. In particular, the classical methods for obtaining useful, rigorous bounds for this equation break down very quickly. This means that reasonable bounds for the solutions of the equation may be obtained only for short periods of time and there is no hope to obtain useful bounds for the whole period T, especially when η is large.

Despite the fact that there are few bounded trajectories, their dynamics is complicated. Actually, using topological methods combined with classical analytic estimates Srzednicki and Wójcik proved the following theorem

Theorem 2.1. *(see* [11]*) For $\eta \in (0, 1/288]$ the Poincaré map of (1) admits a chaotic invariant set, which is semiconjugate to symbolic dynamics on two symbols.*

The proof is classical. It is based on some topological characterization of the fixed points of the Poincaré map, computable by means of the isolating segment, the concept developed by Srzednicki in [10].

Three years later P. Zgliczyński and K. Wójcik [12] got a more general conclusion that the chaotic dynamics exists for $\eta \in (0, 495/1000]$. When comparing the two proofs it becomes clear that the theorem most likely extends for η beyond the interval $(0, 495/1000]$, but the analytic technicalities involved in the proof grow as η grows.

Therefore a natural question arises: Is it possible to replace the analytic constructions by an algorithmic approach and provide a computer assisted proof of the theorem for instance for $\eta = 1$? The answer is yes, but the strong expansion present in the system makes the problem challenging.

To solve the problem, we propose a method to translate the short term bounds obtained by means of the classical rigorous numerics to some topological data in such a way that the topological information about the Poincaré map necessary to draw conclusions about the chaotic behaviour may be glued together from the topological data obtained from the short term bounds.

3. Catching the escaping trajectories.

Adding the equation

$$t' = 1$$

to (1) we obtain an autonomous ODE which induces a flow on $\mathbb{C} \times \mathbb{R}$. Therefore, for every $p \in \mathbb{N}$ there is an induced flow φ on $X := \mathbb{C} \times S^1_{pT}$, where $S^1_{pT} := [0, pT]/\sim$ with $\sim$ denoting the relation identifying 0 and pT. The Poincaré map F will be studied on the Poincaré section $Z := \mathbb{C} \times \{0\}$. Since most of the trajectories escape to infinity in a short time, the Poincaré map is only partially defined. We can go around this problem in the following way. Given a compact set $W \subset \mathbb{C}$ define $\sigma_W : W \times \mathbb{R} \to \mathbb{R}$ by

$$\sigma_W(z, t) := \sup \{ s \geq 0 \mid \varphi((z, t), [0, s]) \subset W \times \mathbb{R} \}.$$

As in the case of the Ważewski Theorem (see [1], Th.II.2.2) one can prove that if W has no internal tangencies on $\operatorname{bd}_{\mathbb{C}} W \times [0, T]$, then σ_W is continuous.

In our case we just take $W := [-K, K] \times [-K, K]$ for some $K > 0$. Assume that σ_W is continuous. Let $p : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ denote the projection onto the first coordinate. Define a flow $\bar{\varphi}$ on $W \times \mathbb{R}$ by

$$\bar{\varphi}((z, t), s) := \begin{cases} \varphi((z, t), s) & \text{if } s \leq \sigma_W(z, t), \\ (p(\varphi((z, t), \sigma_W(z, t))), s - \sigma_W(z, t)) & \text{otherwise.} \end{cases}$$

Then $\bar{\varphi}$ is well defined and every trajectory of $\bar{\varphi}$ contained in $\operatorname{int}_X W \times \mathbb{R}$ is also a trajectory of φ.

Every face of $W \times [0, T]$ is contained in a plane given as $V^{-1}(0)$ for some affine function $V : \mathbb{C} \times \mathbb{R} \to \mathbb{R}$ which is positive outside of $W \times [0, T]$. Therefore, to verify that no internal tangency of φ is possible on that face, it is enough to prove that $\alpha : s \to V(\varphi((z, t), s))$ has no local maxima on the face. This may be achieved by dividing the face into boxes such that in every box either the derivative of alpha is negative (entrance points) or it is positive (exit points) or the sign of the drivative is unsettled but the second derivative is positive (this may mean an entrance point or an exit point or an external tangency but the internal tangency is excluded). Since we are interested only in verifying strong inequalities, the computations may be performed rigorously by means of interval arithmetic (see [3]). We successfully

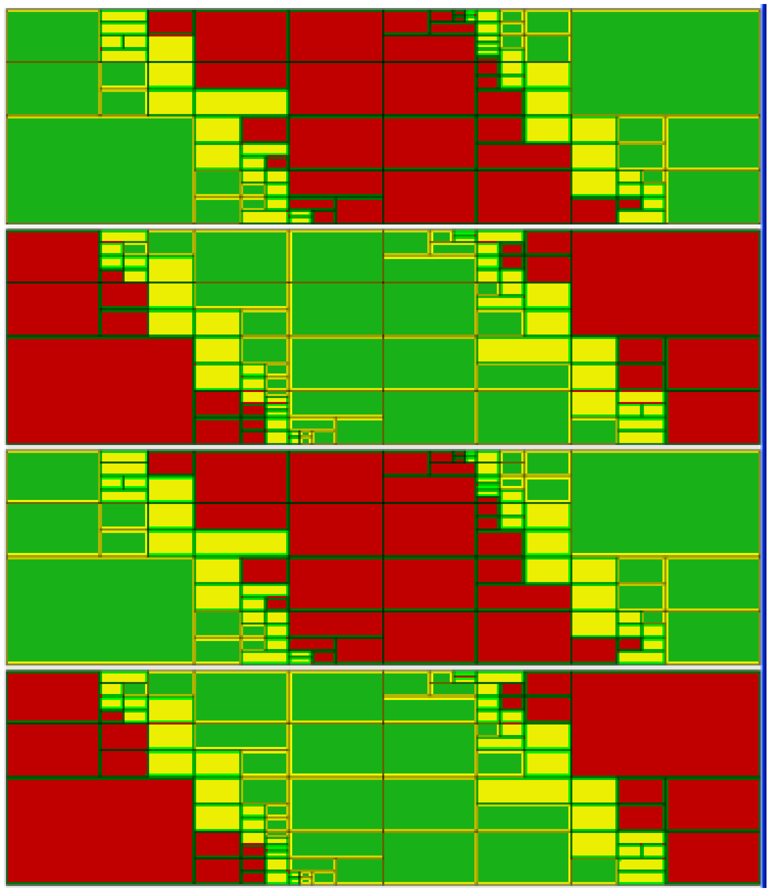

FIGURE 1. The four sides of $[-K, K] \times [-K, K] \times [0, T]$ for $K = 3.0$ with entrance points marked black, exit points marked dark gray and points where internal tangency has been excluded marked in light gray.

performed such calculations for $K = 3.0$. Figure 1 and Figure 2 show a graphical visualization of the outcome of the computations for $W = [-K, K] \times [-K, K] \times [0, T]$ with $K = 3.0$. Note that $\mathrm{bd}_{\mathbb{C}} W \times \mathbb{R}$ is invariant for the modified flow $\bar{\varphi}$.

Observe that if α has no local maxima on the faces of $W \times [0, T]$ for some $K > 0$, then, because of the analicity of the right hand side of (1), the same applies for $K + \epsilon$ for sufficiently small ϵ.

Fix such an ϵ. In the sequel W will stand for $[-K, K] \times [-K, K]$ with $K = 3.0 + \epsilon$ and we will consider only the modified flow $\bar{\varphi}$ and the associated modified Poincaré map. Therefore, to simplify the notation, we will drop the bars and we will write simply φ and F for the modified flow and Poincaré map.

Remark 3.1. *The choice of ϵ additionally guarantees that every trajectory crossing the boundary of $[-K, K] \times [-K, K] \times \mathbb{R}$ in the outward direction, will never come back.*

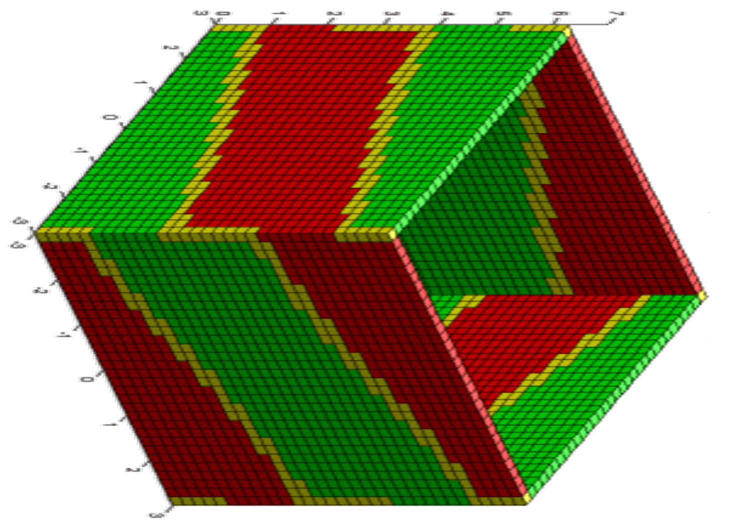

FIGURE 2. A three dimensional view of the four sides of $[-K, K] \times [-K, K] \times [0, T]$ for $K = 3.0$.

4. ISOLATED INVARIANT SETS AND THE FIXED POINT INDEX.

We now briefly recall some concepts from topological dynamics, which we need in this paper. We treat the Poincar'e map as a generator of a discrete dynamical system. In general, let X denote a fixed, locally compact, metric space and let $f : X \to X$ be a continuous map considered as the generator of a discrete dynamical system.

We say that a compact set $S \subset X$ is *invariant* with respect to f if $f(S) = S$. The *invariant part* of $N \subset X$ with respect to f is defined as the union of all invariant subsets of N. It will be denoted by $\mathrm{Inv}(N, f)$. The set S is called *isolated invariant*, if it admits a compact neighborhood N such that $S = \mathrm{Inv}(N, f)$. The neighborhood N is then called an *isolating neighborhood* of S.

Let $N \subset X$ be an isolating neighborhood for f and let $S = \mathrm{Inv}(N, f)$. A *weak index pair* for f and S in N is a pair of compact sets $P = (P_1, P_2)$, where $P_2 \subset P_1 \subset N$ (comp. [7]), satisfying the following three properties:

(i) $f(P_2) \cap P_1 \subset P_2$,

(ii) $\operatorname{bd}_f P_1 := P_1 \cap \operatorname{cl}(f(P_1) \setminus P_1) \subset P_2$,
(iii) $S \subset \operatorname{int}(P_1 \setminus P_2)$.

Since in this paper we will only consider weak index pairs, in the sequel we will refer to them simply as index pairs.

One can prove (see [7]) that if $P = (P_1, P_2)$ is an index pair, then f induces the following map of pairs

$$f_P : (P_1, P_2) \to (P_2 \cup f(P_1), P_2 \cup (f(P_1) \setminus P_1)).$$

Moreover, the inclusion

$$\iota_P : (P_1, P_2) \to (P_2 \cup f(P_1), P_2 \cup (f(P_1) \setminus P_1)).$$

induces an isomorphism of Alexander-Spanier homologies. Therefore we have a well defined endomorphism

$$\chi(P, f) := \chi_P := f_P^*(\iota_P^*)^{-1} : H^*(P_1, P_2) \to H^*(P_1, P_2),$$

called the *index map* of P.

The index map carries information about the fixed points of f in S. Let

$$\operatorname{Fix}(f) := \{ x \in X \mid f(x) = x \}$$

denote the set of fixed points of f. A subset $K \subset \operatorname{Fix}(f)$ is called *isolated* if there are no other fixed points in some neighborhood of K. If X is an ENR, then for every isolated set K of fixed points of f its *fixed point index* $\operatorname{ind}(K, f)$ is defined (see [2]). The fundamental feature of the fixed point index is that

$$\operatorname{ind}(K, f) \neq 0 \;\Rightarrow\; K \neq \emptyset. \tag{2}$$

Wneh S is an isolated invariant set, then $S \cap \operatorname{Fix}(f)$ is an isolated set of fixed points and

$$\operatorname{ind}(S, f) := \operatorname{ind}(S \cap \operatorname{Fix}(f), f) = \Lambda(\chi_P) \tag{3}$$

for any index pair P for S (see [4]). Here $\Lambda(\chi_P)$ denotes the Lefschetz number of χ_P given by

$$\Lambda(\chi_P) := \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}(\chi_P)_i.$$

From computational point of view it is sometimes more convenient to use a special form of isolating neighborhood, called isolating block. A compact subset $N \subset X$ is called an *isolating block* if

$$f^{-1}(N) \cap N \cap f(N) \subset \operatorname{int} N.$$

Let N be an isolating block for f. We say that (N, N^e) is a *block pair* for f and N if

(i) $f(N^e) \cap N = \emptyset$,

(ii) $f(N) \cap \operatorname{bd} N \subset N^e$,
(iii) $N^e \subset \operatorname{bd} N$.

It is straightforward to verify that every block pair for an isolating block is an index pair for it. Note that N^e is not uniquely determined. Nevertheless, whenever it is clear from the context, we will write briefly χ_N for the index map of (N, N^e).

5. Topological sections.

A potential method for a computer assisted proof of a variant of Theorem 2.1 for $\eta = 1$ could be based on the geometric crtiterion for chaos presented in [8], discrete version of the criterion used in [11]. In particular, such an approach would require finding χ_M and χ_N for certain two isolating blocks for the Poincaré map F.

The standard rigorous numerical technique to compute the index maps χ_N and χ_M requires constructing first a good multivalued representation of the Poincaré map F. However, for reasons explained earlier, in the case of equation (1) this is hopeless. A substantial improvement may be obtained by introducing intermediate sections. Let $\{t_i\}_{i\in\mathbb{Z}_n} \subset [0,T]$ be an increasing sequence of numbers such that $t_0 = 0$. For $i \in \mathbb{Z}_n$ define

$$X_i := W \times \{t_i\}.$$

The sets X_i constitute the so called *intermediate sections*. There are well defined intermediate Poincaré maps

$$F_i : X_i \to X_{i+1}$$

for $i \in \mathbb{Z}_n$ and

$$F = F_{n-1}F_{n-2}\cdots F_0.$$

Finiding multivalued enclosures for the intermediate Poincaré maps F_i and composing the enclosures gives rise to an enclosure of the total Poincaré maps F, which usually gives a much better approximation of F than the enclosure computed directly. Unfortunately in the discussed case even this approach does not provide a satisfactory enclosure.

However, since the information we are after is topological (more precisely we need only the map induced in cohomology by F), an alternative is to try to associate a kind of index map with every F_i and compose them.

It will be convenient to consider first a general setting. Assume that for $i \in \mathbb{Z}_n$ we have a collection of pairwise disjoint compact metric spaces $\{X_i\}$ and continuous maps $f_i : X_i \to X_{i+1}$. In the sequel we will substitute the indtermediate sections for X_i and the partial Poincaré maps for f_i. Put

$$\begin{aligned} X &:= \bigcup_{i=0}^{n-1} X_i \\ f &:= \bigcup_{i=0}^{n-1} f_i : X \to X. \end{aligned}$$

For $A \subset X$ we will use the notation $A_i := A \cap X_i$. Note that $\{A_i\}_{i=0}^{n-1}$ constitutes a decomposition of A. We will consider X as a topological set in which $U \subset X$ is open if and only if U_i is open for every $i \in \mathbb{Z}_n$. Obviously f is continuous in this topology, so we may consider it as a generator of a semidynamical system on X. Note that

$$f(X_i) \subset X_{i+1} \text{ for } i \in \mathbb{Z}_n. \tag{4}$$

Let K be an isolating block for f and assume $K^e \subset \operatorname{bd} K$ is such that (K, K^e) is a block pair for F. It follows from (4) that

$$\chi_K(H^*(K_i, K_i^e)) \subset H^*(K_{i+1}, K_{i+1}^e).$$

Put

$$\chi_{K_i} := \chi(K, f)|_{H^*(K_i, K_i^e)}.$$

Since

$$H^*(K, K^e) = \bigoplus_{i=0}^{n-1} H^*(K_i, K_i^e)$$

we get

$$\chi(K, f) = \begin{bmatrix} 0 & 0 & \cdots & \chi_{K_{n-1}} \\ \chi_{K_0} & 0 & \cdots & 0 \\ 0 & \chi_{K_1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & & \chi_{K_{n-2}} & 0 \end{bmatrix}. \tag{5}$$

Therefore $\chi(K, f)^n$ is a diagonal matrix with every entry on the diagonal equal to

$$\chi_{K_{n-1}} \chi_{K_{n-2}} \cdots \chi_{K_0}.$$

Put $S := \operatorname{Inv}(K, f)$. Then S is an isolated invariant set for f^n. As in [5] one can prove that $\chi(S, f^n) = \chi(S, f)^n$. Now observe that the sets X_i are invariant with respect to f^n. Thus each S_i is invariant with respect to f^n and formula (5) implies that

$$\chi(S_0, f^n|_{X_0}) \ = \ \chi_{K_{n-1}} \chi_{K_{n-2}} \cdots \chi_{K_0}.$$

By (3)

$$i(S_0, f^n|_{X_0}) = \Lambda(\chi_{K_{n-1}}\chi_{K_{n-2}}\cdots\chi_{K_0}), \tag{6}$$

Formula (6) may be considered as lifting the method of intermediate sections to the topological level, because to find the fixed point index we only need enclosures of the partial Poincaré maps, which are available even in the case of very strong expansion, if only the intermediate sections are chosen sufficiently close one to the other.

6. Chaotic dynamics.

Now we go back to equation (1). Given $p, n \in \mathbb{N}$ put

$$\mathcal{T}_n^p := \{\, t = (t_i)_{i\in\mathbb{Z}_n} \mid 0 = t_0 < t_1 < t_2 < \ldots < t_{n-1} < pT \,\}.$$

For $t \in \mathcal{T}_n^p$ we put $|t| := n$. Now take $t \in \mathcal{T}_n^p$ and $s \in \mathcal{T}_m^q$. We define the concatenation $st \in \mathcal{T}_{n+m}^{p+q}$ by

$$(st)_i := \begin{cases} t_i & \text{if } i < n, \\ pT + s_{i-n} & \text{otherwise.} \end{cases}$$

Let $t \in \mathcal{T}_n^p$. Put $X_i^t := W_0 \times \{t_i\}$ and let $f_i^t : X_i^t \to X_{i+1}^t$ denote the partial Poincaré map. Let $X^t := \bigcup_{i=0}^{n-1} X_i^t$ and $f^t := \bigcup_{i=0}^{n-1} f_i^t$. Note that $(f^t)^n = F^p$. Given $A \subset X^t$ and $B \subset X^s$ we define the concatenation $BA \subset X^{st}$ by

$$(BA)_i := \begin{cases} A_i & \text{if } i < n, \\ B_{i-n} & \text{otherwise.} \end{cases}$$

Let $t \in \mathcal{T}_n^1$ and $s \in \mathcal{T}_m^1$. Assume $N \subset X^t$ is an isolating block for f^t and $M \subset X^s$ is an isolating block for f^s. Moreover, put $S_N := \operatorname{Inv}(N, f^t)_0$, $S_M := \operatorname{Inv}(M, f^s)_0$ and assume that

$$M_0 = N_0 \tag{7}$$

$$S_M \subset S_N \tag{8}$$

$$\chi(S_N, F) = -\operatorname{id}_{\mathbb{Q}},\ \chi(S_M, F) = \operatorname{id}_{\mathbb{Q}}. \tag{9}$$

Note that S_N and S_M are isolated invariant sets respectively for $(f^t)^n = F$ and $(f^s)^m = F$, so equations in (9) make sense. Put

$$S := S_N, S_0 := S_M, S_1 := S \setminus S_0.$$

Similarly as in the proof of Theorem 2 in [11] or Lemma 6 in [8] one can show that S_1 is compact. Therefore we have a decomposition of S into the union of two compact sets

$$S = S_0 \cup S_1.$$

Put $\Sigma_2 := \{\, c : \mathbb{Z} \to \{0,1\} \,\}$ and define $\rho : S \to \Sigma_2$ by $\rho(x)_i = j$ if and only if $F^i(x) \in S_j$. To show that ρ constitutes a semiconjugacy onto the shift dynamics on Σ_2 it is emough to show that ρ is surjective. For this end it is enough to show that for every periodic $c \in \Sigma_2$ there is a periodic point $x \in S$ such that $\rho(x) = c$.

For this end fix the principal period $p \in \mathbb{N}$ of c and consider $\tau : I_p \to \{t, s\}$. We will identify τ with the concatenation

$$\tau_{p-1}\tau_{p-2}\cdots\tau_0.$$

For $j = 0, 1, 2, \ldots p-1$ let

$$K^j := \begin{cases} M & \text{if } \tau(j) = s, \\ N & \text{if } \tau(j) = t. \end{cases}$$

Let $K := K^{p-1}K^{p-2}\cdots K^0$ denote the respective concatenation of isolating blocks M and N. Condition (7) guarantees that K is an isolating block for f^τ. In particular $K_0 = N_0 = M_0$. Let $S_K := \operatorname{Inv}(K, f^\tau)_0$. Then S_K is an isloated invariant set for $F^p = (f^\tau)^{|\tau|}$ and by (6)

$$\operatorname{ind}(S_K, F^p) = \Lambda(\chi_{K_{|\tau|-1}}\chi_{K_{n-2}}\cdots\chi_{K_0}).$$

By grouping we get

$$\operatorname{ind}(S_K, F^p) = \Lambda(\chi(K^{p-1}, F)\chi(K^{p-2}, F)\cdots\chi(K^0, F)).$$

Therefore

$$\operatorname{ind}(S_K, F^p) = (-1)^k,$$

where k denotes the number of appearances of N in K.

Let

$$\Sigma^p := \{\, c : I_p \to \{0,1\} \,\}$$

and for $\tau : I_p \to \{t, s\}$ let

$$\Sigma^\tau := \{\, c \in \Sigma^p \mid \tau(i) = s \;\Rightarrow\; c(i) = 0 \,\}.$$

Put

$$\operatorname{Fix}^\tau := \operatorname{Fix}(F^p) \cap S_K$$

and for $c \in \Sigma^\tau$ put

$$\operatorname{Fix}^c := \{\, x \in \operatorname{Fix}^\tau \mid F^i(x) \in c(i) \,\}.$$

Using the decomposition

$$\operatorname{Fix}^\tau = \bigcup_{c \in \Sigma^\tau} \operatorname{Fix}^c$$

and the additivity of the fixed point index one can prove by the induction argument analogous to Lemma 1 in [11] or Lemma 12 in [8] that

$$\operatorname{ind}(\mathrm{Fix}^c, F^p) = (-2)^{\mathrm{card}\{\, i \in I_p \mid c(i) = 1 \,\}}. \tag{10}$$

Since

$$\Sigma^p = \bigcup_{\tau} \Sigma^\tau$$

we see from (10) that for any $c \in \Sigma^p$ the fixed point index $\operatorname{ind}(\mathrm{Fix}^c, F^p) \neq 0$, which by (2) implies that F has a periodic point $x \in S$. Moreover, since $x \in \mathrm{Fix}^c$, we have $F^i(x) \in S_{c(i)}$. Therefore, using the standard argument, we come to the conclusion that $\rho : S \to \Sigma_2$ is a semiconjugacy.

7. Rigorous numerics.

Now, in order to prove chaotic dynamics for the equation (1) with $\eta = 1$ it remains to show that there are choices of $t \in \mathcal{T}^1_n$ and $s \in \mathcal{T}^1_m$ and isolating blocks $N \subset X^t$, $M \subset X^s$ such that conditions (7)-(9) are satisfied. To do so we apply rigorous numerics.

Take $n = 80$ and for $i \in I_{80}$ put $t_i := i\pi/40$, $N_i = [-3, 3] \times [-3, 3]$. Take $m = 20$ and for $i \in I_{20}$ as s_i and M_i take the values in Table 1.

Observe that (7) is immediate and (8) follows from the fact that every trajectory starting from S_M remains in $W \times \mathbb{R}$. Therefore by Remark 3.1 it also stays in $S_N \times \mathbb{R}$, because otherwise it would not return to $S_N \times \mathbb{R}$ and in consequence to $S_M \times \mathbb{R}$.

Finally (9) may be verified by rigorously enclosing the partial Poincaré maps, finding the respective index maps and composing them. The details are based on the machinery of isolating blocks for multivalued representations developed in [7].

We completed such a verification succesfully on a machine with a 3.6 GHz Pentium IV in about 20 minutes. Therefore we have proved the following theorem.

Theorem 7.1. *For $\eta = 1$ the Poincaré map of the equation*

$$z' = (1 + e^{i\eta t}|z|^2)\bar{z}.$$

admits a chaotic invariant set, which is semiconjugate to symbolic dynamics on two symbols.

Conlcuding let us say that the presented method of topological sections may be extremly helpful in solving problems, where other approaches fail because of rapid growth of error estimates. It may be applied not only to Poincaré maps in time periodic non-autonomous

i	s_i	M_i
0	0	$[-3,3]\times[-1.5,1.5]$
1	0.0373779	$[-0.598513,0.598513]\times[-2.37269,2.37269]$
2	0.143098	$[-0.471825,0.471825]\times[-1.82283,1.82283]$
3	0.33732	$[-0.36717,0.36717]\times[-1.35041,1.35041]$
4	0.636343	$[-0.281645,0.281645]\times[-0.955435,0.955435]$
5	1.05424	$[-0.213114,0.213114]\times[-0.637908,0.637908]$
6	1.60358	$[-0.161148,0.161148]\times[-0.397826,0.397826]$
7	2.29583	$[-0.130109,0.130109]\times[-0.23519,0.23519]$
8	3.14159	$[-0.15,0.15]\times[-0.15,0.15]$
9	3.98736	$[-0.23519,0.23519]\times[-0.185304,0.185304]$
10	4.67961	$[-0.397826,0.397826]\times[-0.267075,0.267075]$
11	5.22895	$[-0.637908,0.637908]\times[-0.38326,0.38326]$
12	5.64684	$[-0.955435,0.955435]\times[-0.531551,0.531551]$
13	5.79635	$[-1.15292,1.15292]\times[-0.62253,0.62253]$
14	5.94587	$[-1.35041,1.35041]\times[-0.713509,0.713509]$
15	6.04298	$[-1.58662,1.58662]\times[-0.823082,0.823082]$
16	6.14009	$[-1.82283,1.82283]\times[-0.932654,0.932654]$
17	6.19295	$[-2.09776,2.09776]\times[-1.06305,1.06305]$
18	6.24581	$[-2.37269,2.37269]\times[-1.19344,1.19344]$
19	6.2645	$[-2.68635,2.68635]\times[-1.34672,1.34672]$

TABLE 1. Values defining the isolating block M.

differential equations, but also to Poincaré maps in autonomous equations and t translations.

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