Application of the stochastic mesh method in pricing of American-style options

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Abstract
In this paper we present how to perform the pricing of American-style options in practice with use of the stochastic mesh method. We show various types of the estimators. Then we indicate their properties on the examples. We determine the values of parameters which allow to implement a method efficiently. Finally, we present how to use the method on a challenging problem of pricing arithmetic average put option on multiple assets.

1 Introduction
Pricing of high-dimensional American-style options is a very complex financial problem. The most popular methods use Monte Carlo simulation [5–7]. Among them one of the most important is a stochastic mesh method [1–4]. The algorithm relies on simulation a mesh which is next used to create various types of estimators of the option price.

In this paper we give a description how to perform the pricing process and indicate the most important properties of the introduced estimators which could help in practical application of the method. We consider estimators of the upper and lower bounds. On theirs basis it is possible to create confidence intervals as well as the point estimators. Such tools allow to estimate the true option price.

Our numerical examples concern many important aspects of practical option pricing. We examine which parameters values allow to conduct the procedure efficiently. Moreover, we show how to apply the control variates techniques to improve the precision of the estimation. As a result we discuss a challenging problem of pricing arithmetic average put option on five assets. It is not possible to value such an option type analytically. Making use of stochastic mesh method we present how to indicate the option value precisely.
2 Description of the stochastic mesh method

To describe the stochastic mesh method, let \( \{S_t\}_{0,1,...,T} = \{(S^1_t, S^2_t,..., S^n_t)\}_{0,1,...,T} \) be a multi-dimensional stochastic process with discrete time parameter, which represents prices of \( n \) assets. Let us assume that under risk-neutral measure, \( \{S_t\} \) is a Markov process, whose transition probability densities \( f(t, \cdot, \cdot) \) exist and are known.

The price of the American-style option is defined as the maximization of its expected exercise values taken over all stopping times, i.e.:

\[
Q = \max_{\tau = 0, 1, ..., T} E[h(\tau, S_\tau)],
\]

where \( h(\tau, S_\tau) \) can be interpreted as a payoff from the exercise at time \( \tau \). Since \( S_t \) is Markovian, the American-style option value is obtained by dynamic programming principle:

\[
Q(t, x) := \begin{cases} h(t, x) & \text{for } t = T, \\ \max h(t, x), C(t, x) & \text{for } t < T, \end{cases}
\]

where

\[
C(t, x) = E[Q(t + 1, S_{t+1})|S_t = x].
\]

The stochastic mesh method relies on approximation function \( Q(t, \cdot) \) and is described as follows. We generate the mesh which consists of independent random vectors \( X_t(i), \) for \( t = 1, ..., T \) and \( i = 1, ..., b \) from a distribution specified by a certain density function \( g(t, \cdot) \) which is called the mesh density. One of the most reasonable choices of the mesh density is to define it as a uniform mixture of transition densities [4].

In the case of stochastic mesh method the dynamic programming procedure determines the expected value of estimator \( C(t, x) \) by

\[
\hat{C}(t, X_t(i)) = \frac{1}{b} \sum_{j=1}^{b} \hat{Q}^H_b \left( t + 1, X_{t+1}(j) \right) \omega(t, X_t(i), X_{t+1}(j)).
\]

where

\[
\omega(t, x, X_{t+1}(j)) = \frac{f(t, x, X_{t+1}(j))}{g(t + 1, X_{t+1}(j))}.
\]

The introduction of weights \( \omega(t, \cdot, \cdot) \) is necessary as the points in the mesh were sampled from the density \( g(t, \cdot) \). Finally, for \( t = 0 \) we obtain the mesh estimator as

\[
\hat{Q}^H_b = \hat{Q}^H_b(0, S_0).
\]

Once the mesh is constructed, we can define two lower bound estimators. In order to obtain the first one - the path estimator, we generate \( p \) independent trajectory of process \( S_t \) (independent of the mesh points), which are denoted by \( S^k \), for \( k = 1, ..., p \). Each of path \( S^k \) is simulated according to the density function \( f(t, x, \cdot) \). For each of these trajectories we determine first
optimal policy exercise. Because we have only information from the mesh, we calculate the optimal policy exercises determined by the mesh

\[ \hat{\tau}_b^k = \min \{ t : h(t, S_b^k_t) \geq \hat{Q}_b^H(t, S_b^k_t) \}. \] (5)

Finally, the path estimator is equal the average of \( h(\hat{\tau}_k, S_b^k) \), for \( k = 1, \ldots, p \):

\[ \hat{q}_b^L = \frac{1}{p} \sum_{k=1}^{p} h(\hat{\tau}_b^k, S_b^k). \] (6)

The second estimator - half estimator, is constructed with the mesh estimator simultaneously. We partition the set of indices \( I = \{1, \ldots, b\} \) into two subsets - \( A \) and \( A' = I / A \). The half estimator is defined recursively as:

\[ q_b^{LL}(T, x) = h(T, x) \] (7)

and

\[ \hat{q}_b^{LL}(t, x) = \begin{cases} h(t, x), & \text{for } h(t, x) \geq \hat{C}_A(t, x), \\ \hat{C}_A'(t, x), & \text{otherwise}. \end{cases} \] (8)

For every subset \( B \subset I \) we define

\[ \hat{C}_B(t, x) = \frac{1}{|B|} \sum_{l \in B} q_b^{LL}(t + 1, X_{t+1}(l))\omega(t, x, X_{t+1}(l)), \] (9)

where \( |B| \) denotes the number of elements in \( B \).

The properties of introduced estimators are described by the following theorem [1–4]:

**Theorem 1.** (i) The mesh estimator is biased high, i.e.,

\[ E[\hat{Q}_b^H] \geq Q. \] (10)

(ii) The path estimator is biased low, i.e.,

\[ E[\hat{q}_b^L] \leq Q. \] (11)

(iii) The half estimator is biased weak low, i.e.,

\[ E[\hat{q}_b^{LL}(t, x)] \leq E[\hat{Q}_b^H(t, x)], \] (12)

for every \( t \) and \( x \).

Moreover, under some technical assumptions, all defined estimators are asymptotically unbiased.

It allows us to combine the properties of biased high estimator with biased low estimators and define confidence interval and point estimator for the option value. From this time we will use \( Q^H \) and \( Q^L \) to denote high biased and one of the low biased estimators.
We construct the confidence interval as follows. We generate $N$ independent meshes and for each of them we calculate the estimators $\hat{Q}^H_{(i)}$ and $\hat{Q}^L_{(i)}$, for $i = 1, \ldots, N$. Then, we average these individual estimators and get $\overline{Q}^H(N)$ and $\overline{Q}^L(N)$.

The confidence interval for the value $Q$ equals

$$[\overline{Q}^L(N) - \frac{\sigma(\hat{Q})}{\sqrt{N}} \Phi^{-1}(\frac{\alpha}{2}), \overline{Q}^H(N) + \frac{\sigma(\hat{Q})}{\sqrt{N}} \Phi^{-1}(\frac{\alpha}{2})],$$

where $\sigma(\hat{Q})$ is sample standard deviation of $\hat{Q}$ and $\Phi^{-1}(\alpha)$ is the $\alpha$ quantile of the standard normal distribution.

The point estimator is obtained from a linear combination of high biased and one of the low biased estimators as

$$\hat{Q} = \gamma \hat{Q}^H + (1 - \gamma) \hat{Q}^L,$$

where $\gamma$ is a number from an interval $(0, 1)$.

If we constructed the point estimator using path estimator (denoted by $\hat{Q}^P$ - Point1), we should apply the equality (14) only for $t = 0$. In case of half estimator, we should apply the equality for each of mesh point (denoted by $\hat{Q}^{PP}$ - Point2).

The basic version of The Stochastic Mesh Method can turn out inefficient in practice. The algorithm works in a quadratic computational complexity in respect of a number of paths. To improve its efficiency and precision, we can use variance reduction techniques [4]. Very good results can be achieved using control variates. For The Stochastic Mesh Method we can use inner control to reduce a variance in each of mesh point, but also outer control to reduce a variance for final estimators.

### 3 Computational results

In this section we discuss some applications of the stochastic mesh method in the valuation of the American-style options. We assume that the asset prices follow a process:

$$dS^k_t = S^k_t [r dt + \sigma_k dW^k_t],$$

for $k = 1, \ldots, n$, $t \in [0, T]$, where $r$ is the riskless interest rate, $\sigma_k$ is the volatility of $k$-asset and $W^k_t$, for $k = 1, \ldots, n$, are independent standard Brownian motions under risk-neutral measure. In the following examples the parameter $b$ denotes the number of simulated paths.

#### 3.1 Put option on a single asset

The put option is a classical example in which the price of European option differs from the American one. However, the correct option value can be ap-
proximated accurately using CRS lattice model. The payoff function equals:

$$\max\{0, K - S_t\}, \text{ where } K - \text{ a strike price.}$$

The goal of this example is to examine the efficiency and the convergence of various estimators. Furthermore, we compare values given by the particular estimators with the true option price.

The pricing parameters are:

- initial price - \(S_0 = 100\),
- standard deviation - \(\sigma = 40\%\),
- riskless rate - \(r = 10\%\),
- strike price - \(K = 100\),
- duration - \(T = 1\) year,
- number of meshes \(N = 20\).

The value obtained from lattice model equals 11.95. The following figure summarizes the results:

![Figure 1: The values of individual estimators and the true option value.](image)

**Conclusions:**

- All estimators converge to the real option value for every value of \(d\).
- The mesh estimator is the most biased one. This negative property reveals especially for a large values of \(d\). Value \(d = 10\) is sufficient to obtain satisfactory results for all estimators.
- The values of three basis estimators are consistent with the Theorem 1. The value of half estimator can sometimes exceed the correct option price.
Both point estimators are biased high, because both are strongly associated with the mesh estimator which is extremely biased. Second point estimator is less biased, because its value is recursively averaged at every mesh point. To improve the precision of these estimators we could put the $\gamma$ parameter less then 0.5.

The computation times confirm a quadratic computational complexity of the algorithm in a number of paths. Therefore, we should use small meshes and use variance reduction techniques which improve efficiency.

### 3.2 Max-call option

In this example, we examine the impact of the control variates. As the control variates we use the call option on the most expensive asset in a certain time. We test two options, which differs in the prices of assets. We calculate the confidence intervals and point estimators.

The payoff function of max call option of $n$ assets is

$$\max \{0, \max_{i=1,\ldots,n} S_i^t - K\}.$$ 

The pricing parameters are:

- initial prices:
  1. $S_{0}^1 = 90, S_2^0 = 90, S_3^0 = 90$,
  2. $S_{0}^1 = 60, S_2^0 = 80, S_3^0 = 100$,
- standard deviations - $\sigma_k = 20\%$, $k = 1, 2, 3$,
- riskless rate - $r = 2\%$,
- strike price - $K = 100$,
- duration - $T = 1$ year,
- number of meshes - $N = 30$,
- number of exercise dates - $d = 6$.

The values of American and European call options are equal. To determine the value of European option we apply a Monte Carlo method. For the first option it gives the result of 10.57 while for the second one 9.88.

Figures 2, and 3 summarize the results:

Conclusions:

- Confidence intervals are very long and point estimators are inaccurate without using the control variates. The example illustrates that the bias of the estimators reveals especially in high dimensional problems.
<table>
<thead>
<tr>
<th>Technique</th>
<th>b</th>
<th>Confidence interval</th>
<th>Standard error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Path</td>
<td>Half</td>
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<tr>
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<td>64</td>
<td>[8.439 - 14.233]</td>
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<td>1024</td>
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<td>9.834 - 11.801</td>
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<td></td>
<td>256</td>
<td>[10.234 - 10.78]</td>
<td>10.308 - 10.78</td>
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<td></td>
<td>1024</td>
<td>[10.459 - 10.64]</td>
<td>10.471 - 10.64</td>
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</table>

**First option - 10.57 (0.003)**

<table>
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<th>Control variates</th>
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</thead>
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<tr>
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<td>64</td>
<td>256</td>
</tr>
<tr>
<td></td>
<td>b</td>
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<td>[9.737 - 10.042]</td>
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<td>0.022</td>
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<tr>
<td></td>
<td>0.02</td>
<td>0.018</td>
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</table>

**Second option - 9.88 (0.001)**

Figure 2: Confidence intervals for the option values and standards errors of the estimators. The name of confidence interval determines the type of estimator used to calculate the lower confidence limit.

- Application of control variates gives better results than increasing the number of paths. The obtained results are more accurate with less computational time.

- Control variates have the most positive impact on the results of mesh estimator. The highest decrease of the standard error and the bias is observed with this estimator.

- We notice the difference in improving the efficiency between the first and the second option. The standard deviation decreases eight times for the mesh estimator of the second option while for the first option it decreases only three times. This is the result of the differences in prices of the underlying assets of the second option. The price of the first asset is usually the most expensive, so the control is similar to the control of plain call option on a single asset. For the first option it is difficult to control the most expensive asset, because the initial prices are the same. It makes the control inefficient in many points. This example shows that selection of the control variates has significant impact on the results.

- All generated intervals contain the true option value. When we do not use the control variates, the real option value is located closer to the lower confidence limit, while after applying the control variates, it is located in the middle of the interval. This is due to the level of bias of the individual estimators.
Techniques | b | Value of point estimators | Standard error |
--- | --- | --- | --- |
        |  | Point 1 | Point 2 | Point 1 | Point 2 |
First option - 10.57 (0.003) 64 11.286 10.303 0.178 0.356
No control variates 256 11.18 11.072 0.106 0.235
1024 11.063 10.624 0.038 0.056
Control variates 64 10.501 10.245 0.122 0.131
256 10.585 10.323 0.074 0.043
1024 10.589 10.536 0.034 0.026
Second option - 9.88 (0.001) 64 11.028 10.427 0.181 0.423
No control variates 256 10.761 10.178 0.104 0.133
1024 10.235 10.07 0.052 0.086
Control variates 64 9.92 9.859 0.062 0.05
256 9.923 9.811 0.028 0.036
1024 9.866 9.882 0.012 0.013

Figure 3: Point estimators for option values and their standard errors.

- For the high dimensional options, the path estimator determines the lower confidence limit better than the AH estimator. However, after applying the control variates, both estimators produce similar values, which are very close to the correct option price.

- The second point estimator, which is based on the AH estimator, gives better approximation of true option value, when we do not use the control variates. This is due to reduction of the bias for each of the mesh points. On the other hand, this estimator produces higher level of standard error. When we used control variates, both estimators behave similar - both of them approximate the true value very accurately.

### 3.3 Arithmetic average put option

For both, European and American average options, there exist no competitive approach for obtaining an accurate estimate of the option price. Therefore, in this example we indicate the option value using the stochastic mesh method. The payoff function equals:

\[
\max\{0, K - \frac{1}{n} \sum_{k=1}^{n} S^k_t\}.
\]

The control variates is the European geometric average option, which can be easily priced analytically.

The pricing parameters:
• initial prices - \( S_k^b = 100, \ k = 1, \ldots, 5, \)
• standard deviations - \( \sigma_1 = 20\%, \ \sigma_2 = 30\%, \ \sigma_3 = 40\%, \ \sigma_4 = 50\%, \ \sigma_5 = 60\%. \)
• riskless rate - \( r = 5\%, \)
• strike price - \( K = 100, \)
• duration - \( T = 2 \) years,
• number of meshes - \( N = 15, \)
• number of exercise dates - \( d = 6. \)

Figures 4, 5 show the results:

<table>
<thead>
<tr>
<th>b</th>
<th>Confidence interval</th>
<th>Standard error</th>
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<tbody>
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<td></td>
<td>Path</td>
<td>Half</td>
</tr>
<tr>
<td>128</td>
<td>[7.421 - 8.299]</td>
<td>[7.141 - 8.299]</td>
</tr>
<tr>
<td>256</td>
<td>[7.431 - 7.995]</td>
<td>[7.267 - 7.995]</td>
</tr>
<tr>
<td>512</td>
<td>[7.414 - 7.786]</td>
<td>[7.233 - 7.786]</td>
</tr>
<tr>
<td>1024</td>
<td>[7.44 - 7.647]</td>
<td>[7.306 - 7.647]</td>
</tr>
</tbody>
</table>

Figure 4: Confidence intervals for the option values and standards errors of the estimators. The name of confidence interval determines the type of estimator used to calculate the lower confidence limit.

<table>
<thead>
<tr>
<th>b</th>
<th>Value of point estimators</th>
<th>Standard error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Point 1</td>
<td>Point 2</td>
</tr>
<tr>
<td>128</td>
<td>7.765</td>
<td>7.856</td>
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<td>7.812</td>
</tr>
<tr>
<td>512</td>
<td>7.558</td>
<td>7.74</td>
</tr>
<tr>
<td>1024</td>
<td>7.512</td>
<td>7.57</td>
</tr>
</tbody>
</table>

Figure 5: Point estimators and their standard errors.

Valuation:
• We observe the high bias of both point estimators which decline when we increase the value of parameter \( b \). The standard errors for \( b = 1024 \) are still too high and values of estimators are too less stable to conclude that these estimators show the true option value.
• The confidence intervals are also too long, to specify the correct option price. At constant values of the lower confidence limit, we notice gradual
decreasing in the value of the upper confidence limit. The lower confidence limit with the path estimator seems to be especially stable.

- As the confidence interval with the path estimator approximates the true option value better than with the AH estimator and as both point estimators are still decreasing, we can conclude that the fair option value is about 7.5.

References


