Computer-assisted proofs in dynamics
Part III: differential inclusions

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Outline of Part III:

1. Differential inclusions (problems that lead to)
2. Validated integration of differential inclusions
   - logarithmic norms
   - component-wise estimates
3. CAPD library: differential inclusions' solvers
4. Examples of applications:
   - integration of piecewise-smooth systems
   - globally attracting fixed point for the Burgers PDE
   - existence of periodic orbits for Kuramoto-Sivashinsky PDE

References to third part

P. Zgliczyński. Rigorous numerics for dissipative Partial Differential Equations II.Periodic orbit for the Kuramoto-Sivashinsky PDE - a computer-assisted proof , Foundations of Computational Mathematics, 4 (2004), 157–185

T. Kapela, P. Zgliczyński, A Lohner-type algorithm for control systems and ordinary differential inclusions, Discrete and Continuous Dynamical Systems B, 11 (2009), 365-385.

P. Zgliczyński, Rigorous Numerics for Dissipative PDEs III. An effective algorithm for rigorous integration of dissipative PDEs, Topological Methods in Nonlinear Analysis, 36 (2010) 197C262

J. Cyranka, Efficient and generic algorithm for rigorous integration forward in time of dPDEs: Part I, Journal of Scientific Computing, Vol. 59, 1 (2014), 28-52

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ODE (non-autonomous):

$$x'(t) = f(t, x(t))$$

$f : \mathbb{R} \times \mathcal{H} \to \mathcal{H}$ – vector field

Differential inclusion:

$$x'(t) \in f(t, x(t), u(t))$$

$f : \mathbb{R} \times \mathcal{H} \times U \to \mathcal{P}(\mathcal{H})$ – multivalued

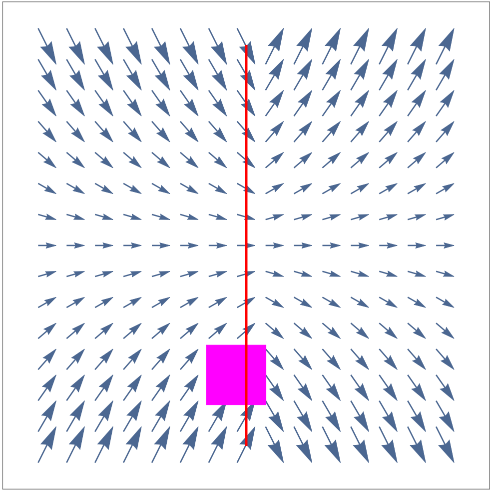
Problems that lead to differential inclusions

Piecewise-smooth systems

$$\dot{x}(t) = f(t, x(t))$$

$[X]$ – the set we want to propagate

Space-dependent inclusion



Control systems

$$\dot{x}(t) = f(x(t), u(t))$$

- $f : \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n$ is a $\mathcal{C}^1$ in x
- $U \subset \mathbb{R}^m$ is a set of admissible control values
- $u(t) \in U$ for all t

Time-dependent inclusion

Problems that lead to differential inclusions

Definition (Reachable Set)

Point y is reachable from point x in time T if there exists control u such that $\varphi(T, x, u) = y$.

Reachable set from point x is the set of all point reachable from x in some time T.

Goal:

Provide algorithm which computes rigorous approximation for reachable set. Upper and inner aproximation needed.

Problems that lead to differential inclusions

Definition (Reachable Set)

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Goal:

Provide algorithm which computes rigorous approximation for reachable set. Upper and inner aproximation needed.

Stiff ODEs:

Example

$$x' = f(x, y), \qquad y' = -Ly + g(x, y)$$

where $g \approx 0$ and $L \gg 1$.

Solve instead

$$x'(t) = f(x(t), y(t)), \qquad y(t) \in [Y]$$

and control $[Y]$ by analytic estimates

Problems that lead to differential inclusions

Dissipative PDE $\rightsquigarrow$ **Infinite dimensional ODE**

Represent a solution as a Fourier series

$$u(t,x) = \sum_{k\in-\infty}^{\infty} a_k(t)e^{ikx}$$

Substituting $u(t,x)$ to PDE we get a system of ODE's

$$\dot{a}_k(t) = F(a_0, a_1, a_{-1}, a_2, a_{-2}, \ldots), \quad k \in \mathbb{Z}$$

Decompose variables as

$$\begin{aligned} x(t) &= (a_0(t), a_1(t), a_{-1}(t), \ldots, a_N(t), a_{-N}(t)) \\ y(t) &= (a_{N+1}(t), ...) \end{aligned}$$

If there are apriori bounds on $y(t)$ then we end up with a differential inclusion.

Warning:

Perturbation may be time dependent!
Cannot use an ODE solver with interval parameter.

Differential inclusion: Perturbed oscillator

$$x' \in y + [-\epsilon, \epsilon], \qquad y' \in -x + [-\epsilon, \epsilon]$$

Fixed parameter

For fixed $\delta \in [-\epsilon, \epsilon]^2$

$$\begin{aligned} x' &= y + \delta_1 \\ y' &= -x + \delta_2 \end{aligned}$$

All solutions remain BOUNDED!

This is a Hamiltonian system

$$H(x, y) = \frac{1}{2}\left((x - \delta_2)^2 + (y + \delta_1)^2\right)$$

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Perturbation may be time dependent!
Cannot use an ODE solver with interval parameter.

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Differential inclusion: Perturbed oscillator

$$x' \in y + [-\epsilon, \epsilon], \qquad y' \in -x + [-\epsilon, \epsilon]$$

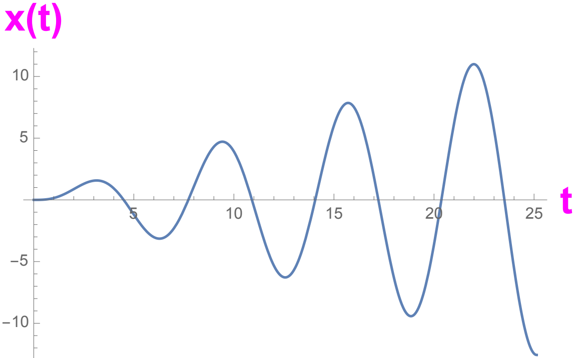
Resonant forcing

$$x'' = -x + \epsilon \sin t$$

All solutions are UNBOUNDED!

$$x(t) = \left(x(0) - \frac{\epsilon t}{2}\right)\cos(t) + \frac{1}{2}(2x'(0) + \epsilon)\sin(t)$$

Solution for
$x(0) = x'(0) = 0$
$\epsilon = 1$



Algorithm

Standing assumptions:

$$\begin{aligned} x'(t) &= f(x(t), y(t)) \\ &\in f(x(t), [W_y]) \end{aligned}$$

where

- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a $\mathcal{C}^1$ function
- $y : \mathbb{R} \to \mathbb{R}^m$ is measurable and bounded on any compact interval
- we can compute $y([t_0, t_0 + h]) \in [W_y]$

Time dependence:

The method works for non-autonomous vector fields.

Notation:

$[y_0]$ - set of unknown functions $\mathbb{R} \to \mathbb{R}^m$

$\varphi(t, x_0, y_0(t))$ - a solution to

$$x' \in f(x, [y_0(t)]), \quad x(0) = x_0$$

$\overline{\varphi}(t, x_0, y_c)$ - a solution to

$$x' = f(x, y_c), \quad x(0) = x_0, \quad y_c = \text{const}$$

One step of the algorithm

INPUT:

- t_k, h_k - current time and a time step
- $[x_k] \subset \mathbb{R}^n$ such that $\varphi(t_k, [x_0], [y_0(t_k)]) \subset [x_k]$.

OUTPUT:

- $t_{k+1} = t_k + h_k$ – new time
- $[x_{k+1}] \subset \mathbb{R}^{n_1}$ such that $\varphi(t_{k+1}, [x_0], [y_0(t_{k+1})]) \subset [x_{k+1}]$.

One step of the algorithm – main parts

1. **Generation of a priori bounds for φ**
   Find a convex and compact set $[W_2] \subset \mathbb{R}^n$, such that
   $$\varphi([0,h_k],[x_k],[y_0([t_k,t_k+h])]) \subset [W_2].$$
2. **Computation of $\overline{\varphi}$**
   Fix $y_c \in y_0([t_k,t_k+h])]$ and use any ODE solver to compute
   $$\begin{aligned} \overline{\varphi}([0,h_k],[x_k],y_c) &\subset [W_1] \quad \text{– convex, compact} \\ \overline{\varphi}(h_k,[x_k],y_c) &\subset [\overline{x}_{k+1}] \end{aligned}$$
3. **Add influence of perturbation**
   Compute $[\Delta] \subset \mathbb{R}^n$, such that
   $$\begin{aligned} \varphi(t_{k+1},[x_0],[y_0(t_{k+1})]) &\subset \overline{\varphi}(h_k,[x_k],y_c)+[\Delta] \\ &\subset [\overline{x}_{k+1}]+[\Delta] \\ &=: [x_{k+1}] \end{aligned}$$

Generation of a priori bounds

A priori bound $[W_y]$ for unknown function:

$$[y_0([t_k, t_k + h])] \subset [W_y]$$

Comment:

This is problem dependent.

- in piecewise-smooth systems this is known explicitly
- in the context of dissipative PDEs the whole story is more complicated, because $[W_y]$ is x-dependent – details later

In what follows we assume $[W_y]$ is computed.

Generation of a priori bounds

A priori bound $[W_2]$ for differential inclusion:

$$\varphi([0, h_k], [x_k], [W_y]) \subset [W_2].$$

Warning:

Perturbation $y(t)$ may not be continuous!
Cannot differentiate and use High Order Enclosure

First Order Enclosure:

$$[x_k] + [0, h_k] * [f([W_2], [W_y])]_I \subset int[W_2]$$

$$\Downarrow$$

$$\varphi([0, h_k], [x_k], [W_y]) \subset [W_2]$$

Generation of a priori bounds

A priori bound $[W_2]$ for differential inclusion:

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$$\Downarrow$$

$$\varphi([0, h_k], [x_k], [W_y]) \subset [W_2]$$

Strategy for computing influence of inclusion:

$$
\begin{aligned}
x_1'(t) &= f(x_1, y_c) \\
x_2'(t) &= f(x_2, y(t)) \\
&\in f(x_2, [W_y])
\end{aligned}
$$

where

- $y_c \in [Y]$ – constant, usually centre of $[W_y]$
- $y(t) \in [W_y]$ – unknown function

Measure the difference $|x_1(t) - x_2(t)| \subset [\Delta]$

Two methods for computing $[\Delta]$:

- logarithmic norms
- component-wise estimates

Propagation of errors in ODEs:

$$x' = f(x)$$

L - Lipschitz constant

$$|f(x) - f(y)| \le L|x - y|$$

Then

$$|x(t) - y(t)| \le e^{Lt}|x - y|, \qquad t \ge 0$$

This is very bad estimate

Example

$$x' = -10x$$

Predicted growth e^{10t}!

Logarithmic norms

Definition

Logarithmic norm of a square matrix A:

$$\mu(A) = \limsup_{h \to 0^+} \frac{\|\mathrm{Id} + Ah\| - 1}{h},$$

where $\| \cdot \|$ is a given matrix norm.

Fact

Logarithmic norm is not a norm.

It can be negative!

Logarithmic norms

Easy to compute:

1. for max norm $\|x\|_1$

$$\mu(\mathbf{A}) = \max_{\mathbf{j}}(\mathbf{a_{jj}} + \sum_{\mathbf{i} \neq \mathbf{j}} |\mathbf{a_{ij}}|)$$

2. for Euclidean norm $\|x\|_2$

$$\mu(\mathbf{A}) = \text{largest eigenvalue of } (\mathbf{A} + \mathbf{A^T})/\mathbf{2}$$

3. for sum norm $\|x\|_\infty$

$$\mu(\mathbf{A}) = \max_{\mathbf{i}}(\mathbf{a_{ii}} + \sum_{\mathbf{j} \neq \mathbf{i}} |\mathbf{a_{ij}}|)$$

Theorem (Hairer, Nørsett, Wanner (1987), Thm. I.10.6)

$x(t)$ – solution to

$$x'(t) = f(t, x(t)), \quad x \in \mathbb{R}^n.$$

$\nu(t) : \mathbb{R} \to \mathbb{R}^n$ – piecewise smooth.

If

$$\mu\left(\frac{\partial f}{\partial x}(t, \eta)\right) \leq \kappa(t) \quad \text{for } \eta \in [x(t), \nu(t)]$$

$$|\nu'(t) - f(t, \nu(t))| \leq \delta(t).$$

Then for $t \geq t_0$ we have

$$|x(t) - \nu(t)| \leq e^{L(t)} \left(|x(t_0) - \nu(t_0)| + \int_{t_0}^{t} e^{-L(s)} \delta(s) ds\right),$$

with $L(t) = \int_{t_0}^{t} \kappa(\tau) d\tau$.

Corollary (fundamental estimate):

- Z – convex set
- $x_2([0,T]) \subset Z$ – a smooth function
- $x_1([0,T]) \subset Z$ – a solution to $x'(t) = f(t, x(t))$
- $\mu\left(Df(Z)\right) \leq \kappa$
- $\|x_2'(t) - f(t, x_1(t))\| \leq \delta$

If $\kappa \neq 0$ then

$$|x_2(t) - x_1(t)| \leq e^{\kappa t}|x_2(0) - x_1(0)| + \delta\frac{e^{\kappa t} - 1}{\kappa}$$

If $\kappa = 0$ then

$$|x_2(t) - x_1(t)| \leq |x_2(0) - x_1(0)| + \delta t$$

Example

$$x' = -10x$$

Predicted growth e^{-10t}

Lemma (Component-wise estimate)

Assume that

- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *is* $\mathcal{C}^1$
- $y : [t_0, t_0 + h] \to \mathbb{R}^m$ *– bounded and measurable*
- $y([t_0, t_0 + h]) \subset [W_y]$ *– convex, compact*
- $y_0 \in [W_y]$
- $x_1, x_2 : [t_0, t_0 + h] \to \mathbb{R}^n$ *are weak solutions to*

$$\begin{aligned} x_1' &= f(x_1, y_0), \quad x_1(t_0) = x_0, \\ x_2' &= f(x_2, y(t)), \quad x_2(t_0) = x_0. \end{aligned}$$

- $[W_1] \subset [W_2] \subset \mathbb{R}^n$ *are convex and compact*
- $x_1(t) \in [W_1], \quad x_2(t) \in [W_2]$ *for* $t \in [t_0, t_0 + h]$.

Lemma (continuation)

Then for $t \in [t_0, t_0 + h]$ and $i = 1, \ldots, n$ there holds

$$|x_{1,i}(t) - x_{2,i}(t)| \leq \left(\int_{t_0}^{t} e^{J(t-s)} C \, ds \right)_i ,$$

where

$$\begin{aligned}
[\delta] &= \{ f(x, y_c) - f(x, y) \mid x \in [W_1], y \in [W_y] \}, \\
C_i &\geq \sup |[\delta_i]| , \quad i = 1, \ldots, n \\
J_{ij} &\geq \begin{cases} \sup \frac{\partial f_i}{\partial x_j}([W_2], [W_y]) & \text{if } i = j, \\ \sup \left| \frac{\partial f_i}{\partial x_j}([W_2], [W_y]) \right| & \text{if } i \neq j. \end{cases}
\end{aligned}$$

Influence of inclusion – logarithmic norms

INPUT:

- $[W_y] \supset [y_0(t_k, t_k + h])$ – enclosure for uknown function
- $[W_1]$ – enclosure for unperturbed system

$$\overline{\varphi}([0,h],[x_k],y_c) \subset [W_1]$$

- $[W_2] \supset [W_1]$ enclosure for differential inclusion

$$\varphi([0,h],[x_k],[W_y]) \subset [W_2]$$

Computation of $[\Delta]$:

Fix any norm $\|\cdot\|$, preferably $\|x\|_\infty = \max_i |x_i|$

1. $[\delta] = [\{f(x,y_c) - f(x,y) \mid x \in [W_1], y \in [W_y]\}]_I$.
2. $C = \|[\delta]\|$
3. $l = \text{right}\left(\mu(\frac{\partial f}{\partial x}([W_2],y_c))\right)$
4. If $l \neq 0$, then $D = \frac{C(e^{lh}-1)}{l}$.
   If $l = 0$, then $D = Ch$
5. $[\Delta] = [-D, D]^n$

Influence of inclusion – component-wise estimates

INPUT:

- $[W_y] \supset [y_0(t_k, t_k + h])$ – enclosure for uknown function
- $[W_1]$ – enclosure for unperturbed system

$$\overline{\varphi}([0,h],[x_k],y_c) \subset [W_1]$$

- $[W_2] \supset [W_1]$ enclosure for differential inclusion

$$\varphi([0,h],[x_k],[W_y]) \subset [W_2]$$

Computation of $[\Delta]$:

1. We set

$$\begin{aligned}
[\delta] &= [\{f(x,y_c) - f(x,y) \mid x \in [W_1], y \in [W_y]\}]_I \\
C_i &= \mathrm{right}(|[\delta_i]|), \quad i = 1,\ldots,n \\
J_{ij} &= \begin{cases} \mathrm{right}\left(\frac{\partial f_i}{\partial x_i}([W_2],[W_y])\right) & \text{if } i = j, \\ \mathrm{right}\left(\left|\frac{\partial f_i}{\partial x_j}([W_2],[W_y])\right|\right). & \text{if } i \neq j. \end{cases}
\end{aligned}$$

2. $D = \int_0^h e^{J(h-s)} C \, ds$
3. $[\Delta_i] = [-D_i, D_i]$, for $i = 1,\ldots,n$

Exponent of a matrix – independent story

Approach 1 (better): solve linear differential equation

Approach 2 (faster): sum Taylor series

Fact:

$$\int_0^t e^{A(t-s)} C \, ds = t \left(\sum_{n=0}^{\infty} \frac{(At)^n}{(n+1)!} \right) \cdot C$$

$$A_m := \frac{(At)^m}{(m+1)!}.$$

For the remainder term we will use the following estimate

$$\|A_{N+k}\| \leq \|A_N\| \cdot \left\| \frac{At}{N+2} \right\|^k$$

Hence if $\left\|\frac{At}{N+2}\right\| < 1$, then

$$\begin{aligned}
\left\|\sum_{m>N} A_m\right\| &\leq \|A_N\| \cdot \left\|\frac{At}{N+2}\right\| \cdot \left(1 - \left\|\frac{At}{N+2}\right\|\right)^{-1} \\
&= \|A_N\| \cdot \frac{\|At\|}{N+2-\|At\|} =: r
\end{aligned}$$

And finally,

$$\sum_{m=0}^{\infty} A_m = \sum_{m=0}^{N} A_m + [-r, r]^n \tag{1}$$

Wrapping effect

Representation of a set, for example

$$[X] = x_0 + C[r_0] + B[r]$$

Unperturbed systems solved by:

$$\overline{X}(h) \subset \Phi(x_0) + (D\Phi([X])C)[r_0] + (D\Phi([X])C)[r] + [R]$$

Differential inclusion solved by:

$$X(h) \subset \Phi(x_0) + (D\Phi([X])C)[r_0] + (D\Phi([X])C)[r] + [R] + [\Delta]$$

Use the same strategies as for ODEs to propagate products (provided $[\Delta]$ is relatively small)

Differential inclusions in CAPD

- **IMultiMap** - class that represents vector field written in the form $f(x) + [y]$
- **InclRect2Set** - representation of a set in the form of doubleton
- **CWDiffInclSolver** - solver for differential inclusions that uses component-wise estimates to compute $[\Delta]$
- **LNDiffInclSolver** - solver for differential inclusions that uses logarithmic norm to compute $[\Delta]$

```cpp
#include <iostream>
#include "capd/capdlib.h"
using namespace capd;
int main(){
  // f is an unperturbed vector field
  IMap f("var:x,y;fun:y,(1-x^2)*y-x;");
  // we define a perturbation e(t) \in [-eps,eps]
  IMap perturb("par:e;var:x,y;fun:e,e;");
  perturb.setParameter("e", interval(-1e-4, 1e-4));
  // We set right hand side of differential inclusion to f + perturb
  IMultiMap rhs(f, perturb);
  //  We set up two differential inclusions (order 20)
  // (they differ in the way they handle perturbations)
  CWDiffInclSolver cwSolver(rhs, 20, IMaxNorm());
  LNDiffInclSolver lnSolver(rhs, 20, IEuclLNorm());
  // constant time step, just for this example (not recommended)
  cwSolver.setStep(1./128); lnSolver.setStep(1./128);
  // Representation of initial condition for diff. incl.
  InclRect2Set lnSet({2.,0.}), cwSet({2.0, 0.0});
  // We perform some numnber of steps with constant time step
  for(int i = 0; i < 128; ++i) {
    lnSet.move(lnSolver);
    cwSet.move(cwSolver);
  }
  std::cout.precision(10);
  std::cout << "LN method:\n" << IVector(lnSet) << std::endl;
  std::cout << "CW method:\n" << IVector(cwSet) << std::endl;
}
```

```
/* Output:
LN method:
{[1.507948164, 1.50834031],[-0.7803484048, -0.7800877445]}
CW method:
{[1.508005535, 1.508282938],[-0.7803100148, -0.7801261345]}
*/
```

```
#include <iostream>
#include "capd/capdlib.h"
#include "capd/poincare/TimeMap.hpp"
using namespace capd;
using namespace std;
typedef poincare::TimeMap<CWDiffInclSolver> CWTimeMap;
int main(){
  // f is an unperturbed vector field
  IMap f("var:x,y;fun:y,(1-x^2)*y-x;");
  // we define a perturbation e(t) \in [-eps,eps]
  IMap perturb("par:e;var:x,y;fun:e,e;");
  perturb.setParameter("e", interval(-1e-4, 1e-4));
  // We set right hand side of differential inclusion to f + perturb
  IMultiMap rhs(f, perturb);
  // component-wise based solver
  CWDiffInclSolver cwSolver(rhs, 20, IMaxNorm());
  // class for long-time integration with this solver
  CWTimeMap tm(cwSolver);
  // Representation of initial condition for diff. incl.
  InclRect2Set set({2.,3.});
  cout.precision(13);
  cout << "phi(1,(2,3))=\n" << tm(1.,set);
}
/* Output:
phi(1,(2,3))=
{[2.300371385204, 2.300624075276],[-0.4798629375598, -0.4797786804589]}
*/
```

Integration of dissipative PDEs

Kuramoto-Sivashinsky PDE:

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x, \qquad \nu > 0$$

where $(t, x) \in [0, \infty) \times \mathbb{R}$

Odd and periodic boundary conditions:

$$\begin{aligned} u(t,0) &= u(t,2\pi) \\ u(t,-x) &= -u(t,x) \end{aligned}$$

Expand solutions as Fourier series:

$$u(t,x) = \sum_{k=-\infty}^{\infty} b_k(t) e^{ikx}$$

Using PDE and boundary conditions:

$$\dot{a}_k = k^2(1-\nu k^2)a_k - k\sum_{n=1}^{k-1} a_n a_{k-n} + 2k \sum_{n=1}^{\infty} a_n a_{n+k}$$

where $b_k = ia_k$ and $k = 1, 2, 3, \ldots$.

Infinite dimensional system of ODEs.

ODE:

$$\dot{a}_k = k^2(1-\nu k^2)a_k - k\sum_{n=1}^{k-1} a_n a_{k-n} + 2k\sum_{n=1}^{\infty} a_n a_{n+k}$$

Linear part (from Laplacian):

$$\dot{a}_k = k^2(1-\nu k^2)a_k$$

- k^{th} mode is unstable for $k < \frac{1}{\sqrt{\nu}}$
- k^{th} mode is stable for $k > \frac{1}{\sqrt{\nu}}$
- the modes with $k >> \frac{1}{\sqrt{\nu}}$ should be irrelevant for the dynamics

KS PDE – some known analytical results

- **Foias, Temam**:
  the existence of global attractor, the functions from attractor are analytic
  (Fourier series converge at geometric rate)

- **Foias, Nicolaenko, Sell, Temam, Rossa, Jolly**:
  the existence of finite dimensional inertial manifold
  (not of much use in rigorous numerics)

No analytical results dynamics more complicated than fixed points bifurcating from zero solution

Some computer-assisted proofs for KS PDE

There are several computer-assisted proofs concerning dynamics of the KS PDE.

- branches of steady states
- attracting periodic orbits
- hyperbolic periodic orbits
- connecting orbits between steady states
- chaos

Goal:
give some details of computer-assisted proof of

Theorem (Zgliczyński)

There are periodic solutions (both stable and unstable) for various parameter values $\nu \approx 0.1215, 0.1212, 0.125, 0.032, 0.02991$

Methodology:

- **Poincaré map for finite dimensional projection:**

$$\Pi_m := \{(a_1, \ldots, a_m) : a_1 = a_3\}, \qquad P_m \colon \Pi_m \to \Pi_m$$

- **periodic points for finite dimensional projection:**
  show that there is $M > 0$ such that for all $m > M$ there is a fixed point x_m for P_m

- **convergence:**
  using some compactness argument show that x_m has a convergent subsequence to a fixed point for full infinite dimensional Poincaré map.

General idea of integration of dissipative PDEs

Impose the following structure of PDE:

$$u_t = Lu + N(u, Du, \dots, D^r u),$$

where

- $u \in \mathbb{R}, \quad x \in \mathbb{T} = \mathbb{R}/2\pi$
- L – linear operator
- N – polynomial
- $D^s u$ denotes s-th order derivative of u
- L is diagonal in the Fourier basis $\{e^{ikx}\}_{k\in\mathbb{Z}}$

$$Le^{ikx} = \lambda_k e^{ikx}$$

and the eigenvalues λ_k satisfy

$$\begin{aligned} \lambda_k &= -v(|k|)|k|^p \\ 0 &< v_0 \le v(|k|) \le v_1, \qquad \text{for } |k| > K_- \\ p &> r \end{aligned}$$

The last assumption is crucial:

for large k linear part dominates nonlinear near $a_k = 0$.

Corresponding ODE in the Fourier basis:

$$u(t,x) = \sum_k u_k(t) e^{ikx}$$

$$\frac{du_k}{dt} = \lambda_k u_k + N_k(u), \qquad \text{for all } k \in \mathbb{Z}$$

Split $u = (p, q)$:

- $p \in X$ - finite dimensional part which contain observed relevant dynamics
- $q \in T \subset X^{\perp}$ - infinite dimensional compact tail on which the dynamics is strongly contracting

Dynamics in X– differential inclusion:

$$\frac{dp}{dt} \in P(Lp + N(p + T)), \qquad p \in X$$

where P is a projection onto X.

Dynamics in T– infinite set of inequalities:

$$\lambda_k u_k + N_k^- < \frac{du_k}{dt} < \lambda_k u_k + N_k^+$$

where $N_k^\pm$ are computable constants.

Consistency:

T is varying in time. We need some consistency conditions in order to integrate differential inclusion.

Notation: $\mathcal{H}$ – Hilbert space,
$e_1, e_2, \dots$ – an orthogonal basis in $\mathcal{H}$
$\mathcal{X}_m$ - subspace spanned by $e_1, \dots, e_m$
P_m, Q_m – projections onto $\mathcal{X}_m$ and $\mathcal{X}_m^\perp$

$$\begin{aligned} p_m = P_m a &:= (a_1, a_2, \dots, a_m) \\ q_m = Q_m a &:= (a_{m+1}, a_{m+2}, \dots) \end{aligned}$$

Vector field:

$$\dot{a} = F(a) = L(a) + N(a)$$

Problem:

F is not continuous, with dense domain in H.

Standing (admissibility) assumption:

$F_k \circ P_n$ is a C^1-function for $n, k \in \mathbb{N}$

The method of self-consistent bounds (special case)

Fix $0 < m \leq M$ (integers)

Definition

(W, T, m, M) is a self-consistent a-priori bounds for F if:

- $W \subset \mathcal{X}_m$ is a compact set and
- $T = \prod_{k>m} T_k$, where $T_k = [a_k^-, a_k^+]$ (T=tail)

Moreover, the following three conditions are satisfied.

[C1] For $k > M$ there holds $0 \in T_k$.

[C2] Let $\widehat{a}_k := \max |a_k^{\pm}|$ for $k > m$. Then $\sum_{k>m} \widehat{a}_k^2 < \infty$. In particular

$$W \oplus T \subset \mathcal{H}$$

[C3] The function $u \to F(u)$ is continuous on $W \oplus T \subset \mathcal{H}$. Moreover, $\sum_{k \in I_{>m}} \widehat{f}_k^2 < \infty$, where

$$\widehat{f}_k = \max \{|F_k(u)| : u \in W \oplus T\} .$$

The method of self-consistent bounds (special case)

Definition

(W, T, m, M) is topologically self-consistend bound for F if additionally

[C4]

$$
\begin{aligned}
a_k = a_k^+ \quad &\Rightarrow \quad \dot{a}_k < 0 \\
a_k = a_k^- \quad &\Rightarrow \quad \dot{a}_k > 0
\end{aligned}
$$

C1, C2, C3 – convergence
C4 – isolation and a priori bounds

Finite representation of $W \oplus T$

W – finite dimensional object.
(doubleton, tripleton, etc.)

Polynomial decay of tail:

$$|a_k^{\pm}| = C/k^s$$

$C \geq 0$ and $s \geq 2$

Geometric decay of tail:

$$|a_k^{\pm}| = Cq^k$$

$C \geq 0$ and $0 < q < 1$

Why is it possible to obtain rough enclosure?

Recall the form

$$u_t = Lu + N(u, Du, \ldots, D^r u)$$

Lemma (bound for nonlinear part)

If $|a_k| \leq C/k^s$, $|a_0| \leq C$ and $s > r$ then there exists $D = D(C, s)$

$$|N_k| \leq \frac{D}{k^{s-r}}, \qquad |N_0| \leq D$$

Lemma (Isolation)

Assume $L(a)_k = -k^p a_k$, $p > r$. If $|a_k| \leq \frac{C}{k^s}$, $|a_{k_0}| = \frac{C}{k_0^s}$, then

$$\frac{d|a_{k_0}|}{dt} \leq -|k_0|^p |a_{k_0}| + |N_{k_0}(a)| \leq$$
$$-C|k_0|^{p-s} + D|k_0|^{r-s}$$

Rough enclosure - data

INPUT:

- $a = (a_k)_{k>0} = ([X], C, s)$, i.e. $|a_k| \leq C/k^s$
- $h > 0$ – time step

OUTPUT:

- $W = (W_k)_{k>0} = ([Y], D, s_0)$ such that

$$a([0,h]) \subset W$$

Rough enclosure - algorithm

Set $W := a$

repeat (possible infinite loop):

1. enlarge slightly the constant C in W
   (there are some heuristics)
2. compute bound for nonlinear part $[N_k^-, N_k^+] := N_k(W)$
   (finite dimensional part + analytic estimates)
3. set finite part $[Y]$ = enclosure for differential inclusion
   (finite dimensional Galerkin projection + the above estimate on nonlinear part)

until $\frac{d|a_k|}{dt}(W) < 0$ for all $k > m$

Result:

If the above stops, then we obtain

- tail T which is forward invariant over the time step
- $[Y]$ – enclosure for differential inclusion

Apparent problem:

Decay power s_0 in obtained enclosure is smaller than s in the initial condition.

Tail evolution

For $k > m$ we have

$$\lambda_k a_k + N_k^- < \frac{da_k}{dt} < \lambda_k a_k + N_k^+,$$

Set

$$b_k^\pm = \frac{N_k^\pm}{-\lambda_k}$$

Decay of tail coefficients:

$$T(h)_k^\pm = \left(T(0)_k^\pm - b_k^\pm\right) e^{\lambda_k h} + b_k^\pm$$

Note that

$$|b_k^\pm| \leq D/k^{s-r+p}$$

where

- p – decay of eigenvalues λ_k
- r – order of derivative in nonlinear term

Smoothing effect:

If $p > r$ then we can even improve decay power.

Periodic orbits in the KS-equation

Important property of the algorithm:

PDE integrator computes simultaneously solutions to all n-dimensional Galerkin projections with $n > m$.

Attracting periodic orbit:

P – Poincaré map
$B = W \oplus T$ – set on section
If $P(B) \subset B$ then

- for all $n > m$ finite dimensional flow induced by Galerkin projection has a periodic orbit x_n (Brouwer theorem)
- B – is a compact set in infinite dimensional space
- x – condensation point of x_n

Periodic point for KS-equation $\nu = 0.127$

Theorem (Zgliczyński, Symmetric attracting orbit)

Let $u_0(x) = \sum_{k=1}^{10} -2a_k \sin(kx)$, where a_k are given in table below. There exists a function $u^*(t,x)$, the classical solution of KS for $\nu = 0.127$, such that

$$\|u_0 - u^*(0,\cdot)\|_{L_2} < 8.1 \cdot 10^{-4},$$
$$\|u_0 - u^*(0,\cdot)\|_{C^0} < 6.5 \cdot 10^{-4}$$

such that u^* is periodic with respect to t.

Coordinates of u_0:

$a_1 = 2.012088e-01$	$a_2 = 1.289978$
$a_3 = 2.012152e-01$	$a_4 = -3.778654e-01$
$a_5 = -4.231056e-02$	$a_6 = 4.316137e-02$
$a_7 = 6.940373e-03$	$a_8 = -4.156441e-03$
$a_9 = -7.945097e-04$	$a_{10} = 3.315994e-04$

Proof uses Brouwer Thm. and rigorous integration of KS-PDE

Periodic point for KS-equation $\nu = 0.1215$

Theorem (Zgliczyński, symmetric unstable orbit)

Let $u_0(x) = \sum_{k=1}^{11} -2a_k \sin(kx)$, where a_k are given in table below. There exists a function $u^*(t,x)$, the classical solution of KS for $\nu = 0.1215$, such that

$$\|u_0 - u^*(0,\cdot)\|_{L_2} < 1.27 \cdot 10^{-3},$$
$$\|u_0 - u^*(0,\cdot)\|_{C^0} < 8.26 \cdot 10^{-4}$$

such that u^* is periodic with respect to t.

Coordinates of u_0:

$a_1 = 2.450027e-01$	$a_2 = 1.041500e+00$
$a_3 = 2.449985e-01$	$a_4 = -2.760754e-01$
$a_5 = -4.371320e-02$	$a_6 = 2.531380e-02$
$a_7 = 6.345919e-03$	$a_8 = -1.996779e-03$
$a_9 = -6.177148e-04$	$a_{10} = 1.184863e-04$
$a_{11} = 5.269771e-05$	

Viscous Burgers equation case study

Example (Burgers equation)

$$u_t(t,x) + u(t,x)\cdot u_x(t,x) - \nu u_{xx}(t,x) = \mathbf{F}(\mathbf{t},\mathbf{x})$$

where $t \in [t_0, \infty),\ x \in \mathbb{R}$ and

$$\begin{aligned} u(t,x) &= u(t, x+2\pi), \quad t \in [t_0,\infty),\ x \in \mathbb{R} \\ F(t,x) &= F(t, x+2\pi), \quad t \in \mathbb{R},\ x \in \mathbb{R} \\ u(t_0,x) &= u_0(x), \quad t_0 \in \mathbb{R},\ x \in \mathbb{R} \end{aligned}$$

where $\nu > 0$.

Goal:
show that for a non-trivial forcing F there is **globally** attracting fixed point in some class of initial conditions.

Some properties of the equation

- Equation in Fourier basis

$$\frac{da_k}{dt} = -i\frac{k}{2}\sum_{k_1\in\mathbb{Z}} a_{k_1}\cdot a_{k-k_1} + \lambda_k a_k + f_k(t), \quad t\in[t_0,\infty),\ k\in\mathbb{Z},$$

- Global existence and uniqueness for real solutions.

$$a_k = \overline{a_{-k}}, \quad f_k(t) = \overline{f_k(t)} \text{ for } t\in\mathbb{R}.$$

- Energy absorbing l^2 ball

$$\frac{d\,E(\{a_k\})}{d\,t} < 0, \text{ as long as } E(\{a_k\}) > \frac{\sup_{t\in\mathbb{R}} E(\{f_k(t)\})}{\nu^2}$$

Theorem (Cyranka)

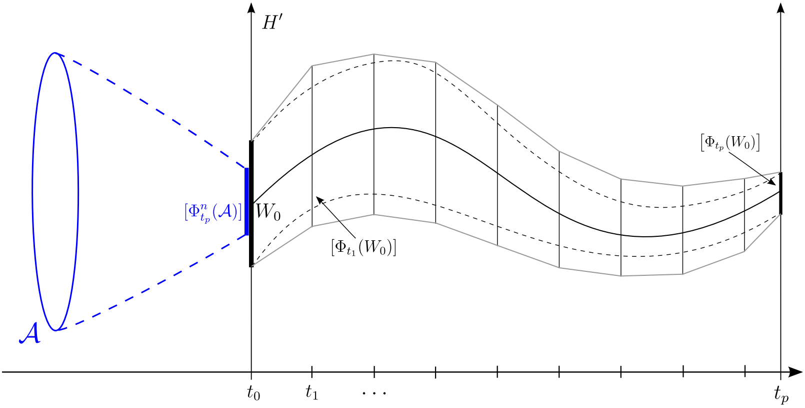
For $\nu = 2$ and $f \in S_2$, where

$$S_2 = \{x \mapsto p(x) + q(x) + r(x)\}$$

$$\begin{aligned}
p(x) &= -0.6\sin(x) + 0.7\cos(2x) + 0.7\sin(2x) - 0.8\cos(3x) - 0.8\sin(3x) \\
q(x) &= \sin(t)\left[-0.6\cos(x) + 0.7\cos(2x) + 0.7\sin(2x) - 0.8\cos(3x) - 0.8\sin(3x)\right] \\
r(x) &= \sum_{k=1}^{3} \beta_k(t)\sin(kx) + \gamma_k(t)\cos(kx),\ \beta_k(t), \gamma_k(t) \in \left[-5\cdot 10^{-5}, 5\cdot 10^{-5}\right]\ \forall\, t,
\end{aligned}$$

there exists a classical solution defined on $\mathbb{R}$ which attracts exponentially any initial data u_0 satisfying $u_0 \in C^4$ and $\int_0^{2\pi} u_0(x)\,dx = \pi$.

Methodology



t_p - period of dominant part of nonautonomous part of forcing.

Calculate the Lipschitz constant of Φ_{t_p} on W_0 using the interval enclosure

$$[W] := \bigcup_{i=0}^{n} [t_i, t_{i+1}] \times [\varphi(t_i, [0, t_{i+1} - t_i], [x_i])] \,.$$

Lipschitz constant of Φ_{t_p} is bounded by

$$L = Ce^{l}, \qquad l = \sum_{i=0}^{n} l_i \cdot (t_{i+1} - t_i) P_{i \mapsto i+1},$$

where l_i are Logarithmic norms calculated locally on each part of $[W]$.

If $l < 0$ then the existence of a locally attracting orbit within W is claimed.