A rigorous lower bound for the stability regions of the quadratic map

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Abstract

We establish a lower bound on the measure of the set of stable parameters a for the quadratic map $Q_a(x) = ax(1-x)$. For these parameters, we prove that Q_a either has a single stable periodic orbit or a period-doubling bifurcation. From this result, we also obtain a non-trivial upper bound on the set of stochastic parameters for Q_a.

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1 Introduction

The quadratic map (often referred to as the *logistic map*)

$$Q_a\colon [0,1] \to [0,1] \qquad x \mapsto ax(1-x), \tag{1}$$

is arguably the most studied discrete dynamical system. Beginning with the famous article by Robert May [Ma76], the intricate properties of this dynamical system have gradually been revealed, see e.g. [CE80], [Ep85], [Fe79], [La82], [Ly02]. The latter article succinctly states that almost every parameter $a \in [0,4]$ is either *regular* or *stochastic*. To be more precise, a parameter is called *regular* if Q_a has an attracting cycle. In this case the cycle is unique, and attracts almost all orbits in $[0,1]$. A parameter is called *stochastic* if Q_a has an absolutely continuous invariant measure. In this case, the measure is unique, and almost all orbits in $[0,1]$ are asymptotically equidistributed with respect to it.

For convenience, let us denote the set of regular parameters by $\mathcal{R}$, and the stochastic parameters by $\mathcal{S}$. The result by Lyubich then states that $|\mathcal{R}|+|\mathcal{S}| = 4$, where $|\cdot|$ denotes Lebesgue measure.

Today, it is well-known that $\mathcal{S}$ has positive measure, see e.g. [BC85], [BC91], [Ja81]. Remarkably, until very recently (see [DK08]), there has been no non-trivial estimates on the measure of $\mathcal{S}$. And, as the authors of [DK08] point out, the result $|\mathcal{S} \cap [4-\epsilon, 4]|/\epsilon > 0.98$ with $\epsilon = 10^{-5000}$ is not a good estimate of $|\mathcal{S}|$. Seeing that $\mathcal{R}$ is an open (and dense [Sw92]) set, it trivially follows that it has positive measure. The aim of this work is to try to obtain a good *lower* bound on the measure of $\mathcal{R}$. In light of [Ly02], this automatically translates to a good *upper* bound on the measure of $\mathcal{S}$.

Theorem 1 (Main Theorem) *The set of regular parameters for the quadratic map satisfy the lower bound:* $|\mathcal{R} \cap [2, 4]| \geq 1.6139421085356060 4222$.

We believe that this lower bound is quite close to the true measure. According to non-rigorous numerical experiments ([Si08]) the measure of the regular set, beyond the first period doubling cascade (at $r^\star \approx 3.569945672$) is estimated to be $|\mathcal{R} \cap [r^\star, 4]| \approx 0.0455857$, i.e., $\mathcal{R}$ constitutes roughly 10.6% of the set $[r^\star, 4]$. Our bound captures more than 10.2% of these parameters, which means that our rigorous result is just slightly less than the non-rigorous estimate. Of course, the non-rigorous estimate may be a gross underestimation of the actual measure, but we find this quite unlikely.

The way we obtain our main result is by adaptively partitioning the parameter space $\mathbb{A} = [2, 4]$ into subintervals. For each such subinterval $\mathbb{A}_i$, we attempt to prove that Q_a has a unique, stable periodic orbit, which persists for all $a \in \mathbb{A}_i$. This can be achieved by employing various set-valued versions of fixed point theorems, implemented rigorously using interval analysis.

We also prove the existence of period doubling bifurcations, using a method presented in [WZ08]. Note that, although the set of parameters at which a period doubling occurs has measure zero, this step is not entirely superficial. It allows us to join the two parameter sets that correspond to stable periodic orbits before and after a period doubling. This produces larger connected parameter sets within each period doubling cascade, and thus adds measure to our final bound.

2 Three methods for verifying the existence of a stable orbit.

In our algorithm that computes a lower bound for the measure of $\mathcal{R}$ we use the following three methods for verifying the existence of a stable orbit:

- the Brouwer theorem,
- the method of backward shooting,
- the modified interval Krawczyk operator.

The three methods are increasing in computational complexity, and therefore, when prove the existence of a stable orbit, we first use the Brouwer theorem. If

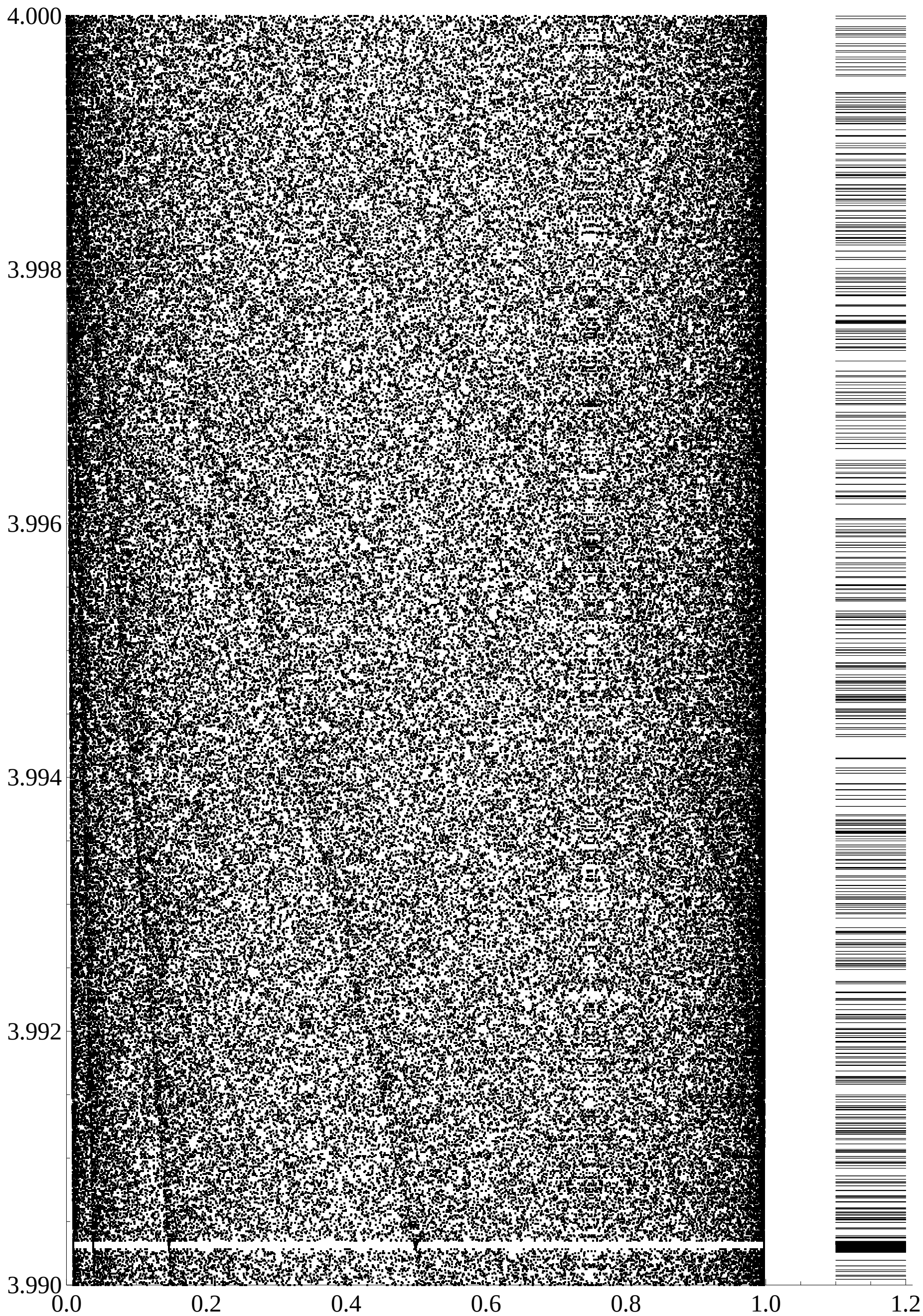

Figure 1: The bifurcation diagram for the quadratic map for parameters $a \in [3.99, 4]$. On the right of the main bifurcation diagram, verified stable windows have been marked as black rectangles.

this method fails, we apply the method of backward shooting. If we still have no success, we switch to the modified interval Krawczyk method, provided the assumptions of this method are satisfied.

2.1 The method based on the Brouwer theorem.

This is the simplest method, and it relies upon the fact that our goal is to prove the existence of a stable periodic orbit. Let $\mathbb{A}$ be a closed interval of parameters for which we expect the existence of a stable periodic orbit for Q_a. Let x_1 be an approximate p-periodic point for the parameter $a = \mathrm{mid}\,(\mathbb{A})$ and let X_1 be a closed interval containing x_1 in its interior. Using interval arithmetic we compute the sequence X_i, $i = 2, \ldots, p$ such that for $a \in \mathbb{A}$

$$Q_a(X_i) \subset X_{i+1}, \quad i = 1, \ldots, p-1$$

Since the orbit is expected to be attracting, it might happen that $X_p \subset X_1$. In that case the Brouwer theorem guarantees that for $a \in \mathbb{A}$ there exists a fixed point $x_1^a \in X_1$ for the map Q_a^p. There remains for us to verify that x_1^a is the p-periodic stable orbit for Q_a.

The verification that p is the principal period of x_1^a can be simply done by checking if the computed enclosure of the trajectory satisfies the condition

$$X_1 \cap X_i = \emptyset, \quad \text{for } i = 2, \ldots, p \text{ such that } i-1 \text{ divides } p \tag{2}$$

In order to verify the stability of the trajectory of x_1^a it is enough to check if

$$|Q'_{\mathbb{A}}(X_1) \cdot Q'_{\mathbb{A}}(X_2) \cdots Q'_{\mathbb{A}}(X_p)| < 1 \tag{3}$$

Summarizing, we had prove the following

Lemma 2 *Assume the Algorithm 1 is called with the arguments $\mathbb{A}, X_1, \ldots, X_p$. If algorithm stops and returns true then for each $a \in \mathbb{A}$ the computed enclosure $(X_1, \ldots, X_p)$ contains unique p-periodic stable orbit for Q_a.*

The above method is very fast, but has some disadvantages compared to the methods described in the following sections. First, given that we always examine an interval of parameters $\mathbb{A}$ rather than a single parameter, the condition $Q_{\mathbb{A}}(X_p) \subset X_1$ might fail.

Second, the initial interval X_1 cannot be too small since we intend to verify the existence of a stable orbit for an interval $\mathbb{A}$. As the "convergence" of X_1 to the periodic orbit is linear (like the contraction of the map), it is very weak for parameters close to bifurcation parameters. Therefore the computed enclosure $(X_1, \ldots, X_p)$ of the periodic orbit may be quite wide. This fact can result in that the stability criterion (3) is not satisfied.

The methods described in the next sections avoid such disadvantages. The computational complexity, however, is larger.

Algorithm 1: verify stability by means of the Brouwer theorem

```
Data: 𝔸 - interval,
      X_1, ..., X_p - intervals
begin
    for i ← 2 to p do
        X_i ← Q_𝔸(X_{i-1});
        if p mod i − 1 = 0 and X_i ∩ X_1 ≠ ∅ then
            return false;
    if Q_𝔸(X_p) ⊄ X_1 then
        return false;
    X_1 ← Q_𝔸(X_p);
    Interval product ← 1;
    for i ← 1 to p do
        product ← Q'_𝔸(X_i) × product;
    if not |product| < 1 then
        return false;
    return true;
end
```

2.2 The method of backward shooting.

One of the most efficient methods for verifying the existence of isolated zeros for a map is the interval Newton method (see e.g. [AH83], [Kr86], [KN84]). Given a smooth function $F : \mathbb{R}^n \to \mathbb{R}^n$, an interval vector[1] X, and a point $\bar{x} \in X$ we define the interval Newton operator by

$$N(F, \bar{x}, X) = \bar{x} - DF^{-1}(X)F(\bar{x}). \tag{4}$$

Theorem 3 *Let X be an interval vector, $\bar{x} \in X$, and $F : \mathbb{R}^n \to \mathbb{R}^n$ be smooth. Assume that $DF(X)$ is invertible as an interval matrix. If the interval Newton operator satisfies*

$$N(F, \bar{x}, X) \subset X$$

then the map F has a unique zero x^* in the set X. Moreover, $x^* \in N(F, \bar{x}, X)$.

This theorem can be adopted to the special situation of proving the existence of periodic orbits for one-dimensional maps, even for very high periods. In this section we recall the method of backward shooting introduced in [Ga02].

The question of the existence of a periodic point for a map $f : \mathbb{R} \to \mathbb{R}$ of principal period p can be transformed to finding a zero of the function

$$F(x_1, \ldots, x_p) = (x_2 - f(x_1), \ldots, x_p - f(x_{p-1}), x_1 - f(x_p)), \tag{5}$$

with the additional assumption that the zero of F has pairwise different coefficients. The following theorem is proved in [Ga02]:

[1] In the sequel we will call a Cartesian product of closed intervals an *interval vector*.

Theorem 4 *Let $X = (X_1, \dots, X_p)$ be an interval vector, $\bar{x} = (\bar{x}_1, \dots, \bar{x}_p) \in X$, and let $f : \mathbb{R} \to \mathbb{R}$ be a smooth map. Define F as in (5). Let us assume that A_k, G_k, H_k, $k = 1, \dots, p$ are intervals such that*

$$\begin{aligned} 0 \notin A_k, &\quad k = 1, \dots, p \\ f'(X_k) \subset A_k, &\quad k = 1, \dots, p \\ f(\bar{x}_k) - \bar{x}_{(k \mod p)+1} \in G_k & \end{aligned} \tag{6}$$

$$\left(1 - A_1^{-1} \cdots A_p^{-1}\right) \sum_{i=1}^{p} A_1^{-1} \cdots A_i^{-1} G_i \subset H_i \tag{7}$$

$$A_k^{-1}\left(H_{(k \mod p)+1} + G_k\right) \subset H_k, \quad k = 2, \dots, p \tag{8}$$

Then $DF(X)^{-1}F(\bar{x}) \subset H = (H_1, \dots, H_p)$, and $N(F, \bar{x}, X) \subset \bar{x} - H$.

The above theorem gives a precise algorithm for proving the existence of periodic points for a map $f : \mathbb{R} \to \mathbb{R}$. If $\bar{x} = (\bar{x}_1, \dots, \bar{x}_p)$ is a good candidate for a periodic orbit, we can choose X to be a small interval vector centered at $\bar{x}$. Then, G_k and H_k can be computed directly by means of the formulas (6–8), provided that none of the A_k contain zero. If $N(F, \bar{x}, X) \subset X$, then by theorem (3) F has a (unique) zero in the set $N(F, \bar{x}, X)$, which corresponds to a periodic point of period p. As a final step, we must also verify that p is the minimal period of the found periodic point, and that this orbit is stable. This can be done by verifying conditions (2–3) in the same way as in Algorithm 1.

Remark 5 *In our algorithm, which verifies the existence of stable periodic orbits for the map Q_a, we use Theorem 3 applied to the map F defined as in* (5) *with $f = Q_{\mathbb{A}}$, where $\mathbb{A}$ is an interval of parameters. The interval Newton operator is computed as described in Theorem 4.*

The method of backward shooting, although very efficient, cannot be used to prove the existence of a superstable periodic orbit, or even an orbit which is very close to the superstable orbit. In this type of situation, we use a special version of the interval Krawczyk operator.

2.3 The interval Krawczyk method for superstable orbits.

A superstable orbit appears when the critical point of Q_a is a periodic point. Let $(x_1, x_2, \dots, x_p)$ be a periodic orbit, and assume that x_1 is the critical point. It is easy to see that, in this situation, we cannot apply the backward shooting method, since $Q'_a(x_1) = 0$. Note, however, that the derivative of $F(x_1, \dots, x_p) = (x_2 - Q_a(x_1), \dots, x_p - Q_a(x_{p-1}), x_1 - Q_a(x_p))$ has the following form:

$$DF(x_1, \dots, x_p) = \begin{bmatrix} -Q'_a(x_1) & 1 & 0 & \dots & 0 \\ 0 & -Q'_a(x_2) & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -Q'_a(x_{p-1}) & 1 \\ 1 & 0 & \cdots & 0 & -Q'_a(x_p) \end{bmatrix}.$$

Since we are assuming that x_1 is a critical point, we have $Q'_a(x_1) = 0$. This means that the solution of the linear equation $DF(x_1, \ldots, x_p) \cdot y = z$, where $y = (y_1, \ldots, y_p)$, $z = (z, \ldots, z_p)$ is given by

$$\begin{cases} y_2 &= z_1 \\ y_3 &= z_2 + Q'_a(x_2)y_2 \\ y_4 &= z_3 + Q'_a(x_3)y_3 \\ \cdots \\ y_p &= z_{p-1} + Q'_a(x_{p-1})y_{p-1} \\ y_1 &= z_p + Q'_a(x_p)y_p \end{cases} \tag{9}$$

Despite this, we cannot use this approach for superstable orbits when solving the linear equation by means of the interval Newton operator. This is because the coefficient $Q'_a(X_1)$ is then not equal to zero - it is an interval *containing* zero. Of course, one might consider using Gaussian elimination for solving the linear system which appears in the interval Newton operator. In this special case, however, the computational complexity can be significantly reduced by using the interval Krawczyk operator instead.

The interval Krawczyk operator is defined by

$$K(F, \bar{x}, X, C) = \bar{x} - C \cdot F(\bar{x}) + (\mathrm{Id} - C \cdot DF(X))(X - \bar{x}) \tag{10}$$

Theorem 6 *Let C be an isomorphism, and let $\bar{x} \in X$, where X is an interval vector. If the interval Krawczyk operator* (10) *satisfies*

$$K(F, \bar{x}, X, C) \subset \operatorname{int} X$$

then the map F has a unique zero in $x^* \in X$. Moreover, $x^* \in K(F, \bar{x}, X, C)$.

Observe that the method only requires that C be an (arbitrary) isomorphism. For the method to work well, however, C should be chosen to be an approximate inverse of $DF(\bar{x})$. The main idea of our approach for proving the existence of superstable periodic orbits (and also nearly superstable ones) is to choose the matrix J as an approximation of $DF(\bar{x})$, and then force the coefficient $J_{11} = 0$. Given this, we choose C to be the inverse of J, which always exists. Furthermore, it can be computed relatively fast – the computational complexity is quadratic with respect to the period p, as we will see below.

Another advantage, in comparison to Gaussian elimination, is that this approach is more memory efficient. In fact, we need not store neither C nor $\mathrm{Id} - C \cdot DF(X)$, which significantly decreases memory usage, especially for large periods – see Algorithm 2. The only storage we need to cater for are the values of $C \cdot F(\bar{x})$ and $(\mathrm{Id} - C \cdot DF(X))(X - \bar{x})$. The value of $C \cdot F(\bar{x})$ can be computed by means of (9). What remains for us to show is the algorithm for the computation of $(\mathrm{Id} - C \cdot DF(X))(X - \bar{x})$.

Lemma 7 *Assume that Algorithm 2 is called with its arguments, and assume that $a_1 = 0$. Then the algorithm always stops and returns an interval vector r*

Algorithm 2: solve linear system

Data: $a_1, \ldots, a_p$ - scalars,
$A_1, \ldots, A_p$ - intervals,
$\bar{x} = (\bar{x}_1, \ldots, \bar{x}_p)$ - a vector,
$X = (X_1, \ldots, X_p)$ - an interval vector

begin
  $r = (r_1, \ldots, r_p)$: an interval vector;
  $r \leftarrow 0$;
  for $i \leftarrow 1$ **to** p **do**
    Interval product $\leftarrow A_i - a_i$;
    $r_{(i \mod p)+1} \leftarrow \big(r_{(i \mod p)+1} + \text{product} \times (X_i - \bar{x}_i)\big)$;
    for $j \leftarrow i+1$ **to** p **do**
      product $\leftarrow$ product $\times a_j$;
      $r_{(j \mod p)+1} \leftarrow \big(r_{(j \mod p)+1} + \text{product} \times (X_j - \bar{x}_j)\big)$;
  return r;
end

which is an upper bound for the interval vector $(Id - C \cdot D)(X - \bar{x})$, where

$$D = \begin{bmatrix} -A_1 & 1 & 0 & \ldots & 0 \\ 0 & -A_2 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -A_{p-1} & 1 \\ 1 & 0 & \cdots & 0 & -A_p \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & -a_2 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -a_{p-1} & 1 \\ 1 & 0 & \cdots & 0 & -a_p \end{bmatrix}^{-1}$$

Proof: It is easy to see that the matrix $M := \mathrm{Id} - C \cdot D$ has the form

$$\begin{bmatrix} a_p \ldots a_2 A_1 & a_p \ldots a_3(A_2 - a_2) & \ldots & a_p(A_{p-1} - a_{p-1}) & A_p - a_p \\ A_1 & 0 & \ldots & 0 & 0 \\ a_2 A_1 & A_2 - a_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{p-2} \ldots a_2 A_1 & a_{p-2} \ldots a_3(A_2 - a_2) & \ldots & 0 & 0 \\ a_{p-1} \ldots a_2 A_1 & a_{p-1} \ldots a_3(A_2 - a_2) & \ldots & A_{p-1} - a_{p-1} & 0 \end{bmatrix} \tag{11}$$

The above form can be obtained by the application of (9) to the columns of D. Hence, the coefficients of the matrix M are given by

$$M_{(j \mod p)+1, i} = \begin{cases} 0 & \text{for } j = 1, \ldots, i-1 \\ A_i - a_i & \text{for } j = i \\ a_j M_{j-1,i} & \text{for } j = i+1, \ldots, p \end{cases} \tag{12}$$

Define $r = (\mathrm{Id} - C \cdot DF(X))(X - \bar{x})$. Then

$$r = \sum_{i=1}^{p} M_{\cdot,i}(X_i - \bar{x}_i), \tag{13}$$

where $M_{\cdot,i}$ denotes the i-th column of the matrix M. The above sum corresponds to the external loop in the algorithm indexed also by i. In the internal loop of the algorithm (indexed by j), the variable 'product' contains precisely the coefficients of the i-th column of the matrix M, as given by the recursive formula (12).

Since all operations used in the algorithm are always well defined in interval arithmetic (extended to NaN cases), the algorithm always returns a valid upper bound for $(\mathrm{Id} - C \cdot D)(X - \bar{x})$. ∎

The above lemma gives us a tool for computing the interval Krawczyk operator for an orbit which is (or is close to) superstable. To be more precise: the interval Krawczyk operator is used to prove the *existence* of a periodic orbit. In addition to this, we need to check if such an orbit has principal period p, and if it is stable. As in the method of backward shooting one needs to verify the conditions (2–3).

3 A method for verifying the existence of period doublings for 1D maps.

The methods described in the earlier sections allow us to prove the existence of stable periodic orbits for parameter ranges well separated from bifurcations. Our aim, however, is to give as good as possible lower bound for the measure of parameters for which stable periodic orbit exists. Therefore, it is important to take into account neighborhoods of parameters for which period doublings occur.

A period doubling occurs when the derivative of a stable periodic orbit decreases through the value -1. When crossing -1, a new stable periodic orbit arises, with a derivative slightly less than 1. The existence of the period doubling can be deduced just from higher derivative information of the map at the bifurcation point. This does not, however, give us any information about the size of the set, or the parameter range, on which we have such dynamics.

In the paper [WZ08], a method for proving the existence of period doubling for higher dimensional maps is proposed. In this method, some computable inequalities for the derivatives allow one to deduce the dynamics of a map in an explicitly given neighborhood of the bifurcation point – both in the phase space and in the parameter space.

Here we recall the main definitions and theorems, adopted to the one-dimensional case. Given a map $f : X \to Y$, we let $\mathrm{dom}\,(f)$ denote the domain of f. For a map $F : X \to X$, we will denote its (restricted) set of fixed points by $\mathrm{Fix}\,(F, U) = \{x \in U \mid F(x) = x\}$.

Definition 1 *Let* $f : \mathbb{R} \supset \mathrm{dom}\,(f) \to \mathbb{R}$ *be* C^1*. Let* $z_0 \in \mathrm{dom}\,(f)$*. We say that* z_0 *is a* hyperbolic fixed point for f *iff* $f(z_0) = z_0$ *and* $|f'(z_0)| \neq 1$*.*

Definition 2 *A subset* $I \subset \mathbb{Z}$ *is called an* interval in $\mathbb{Z}$ *if there exists an closed interval* $[a, b]$*, where* $a, b \in \mathbb{R} \cup \{\pm\infty\}$ *such that* $\mathbb{Z} \cap [a, b] = I$*.*

Definition 3 *Consider a map* $f : \mathbb{R} \supset \text{dom}\,(f) \to \mathbb{R}$*. Let* $x \in \mathbb{R}$*. Any sequence* $\{x_k\}_{k \in I}$*, where* I *is an interval in* $\mathbb{Z}$ *containing zero such that*

$$x_0 = x, \qquad f(x_i) = x_{i+1}, \qquad \textit{for } i, i+1 \in I$$

will be called an orbit through x*. If* $I = \mathbb{Z}_- \cup \{0\}$*, then we will say that* $\{x_k\}_{k \in I}$ *is a full backward orbit through* x*.*

Definition 4 *Consider a continuous map* $f : \mathbb{R} \supset \text{dom}\,(f) \to \mathbb{R}$*.*
Let $Z \subset \mathbb{R}^n$*,* $x_0 \in Z$*,* $Z \subset \text{dom}\,(f)$*. We define*

$$\begin{aligned}
W^s_Z(z_0, f) &= \{z \mid \forall_{n \geq 0} f^n(z) \in Z, \quad \lim_{n\to\infty} f^n(z) = z_0\} \\
W^u_Z(z_0, f) &= \{z \mid \exists\ \{x_n\} \subset Z \textit{ a full backward orbit through } z\textit{, such that} \\
&\qquad \lim_{n\to-\infty} x_n = z_0\} \\
W^s(z_0, f) &= \{z \mid \lim_{n\to\infty} f^n(z) = z_0\} \\
W^u(z_0, f) &= \{z \mid \exists\ \{x_n\} \textit{ a full backward orbit through } z\textit{, such that} \\
&\qquad \lim_{n\to-\infty} x_n = z_0\} \\
Inv^+(Z, f) &= \{z \mid \forall_{n\geq 0} f^n(z) \in Z\} \\
Inv^-(Z, f) &= \{z \mid \exists\ \{x_n\} \subset Z \textit{ a full backward orbit through } z\ \} \\
Inv(Z, f) &= Inv^+(Z, f) \cap Inv^-(Z, f)
\end{aligned}$$

If f *is known from the context, then we will usually drop it and use* $W^s(z_0)$*,* $W^s_Z(z_0)$ *etc instead.*

Definition 5 *[WZ08, Def. 4] Let* $f_a : \mathbb{R} \to \mathbb{R}$*, where a belongs to some interval. We say that* f_a *has a* period doubling bifurcation *at* (a_0, x_0) *iff there exists* $Z = [a_1, a_2] \times X \subset \mathbb{R} \times \mathbb{R}$*, such that the following conditions are satisfied:*

- $(a_0, x_0) \in \text{int}\, Z$*,* $f_{a_0}(x_0) = x_0$
- *there exists a continuous function* $x_{fp} : [a_1, a_2] \to \text{int}\, X$*, such that*

$$Fix(f_a, X) = \{x_{fp}(a)\}$$

- *there exist two continuous curves* $x_i : [a_0, a_2] \to \text{int}\, X$*,* $i = 1, 2$*, such that for* $a \in [a_0, a_2]$*, the following holds:*

$$\begin{aligned}
x_1(a_0) &= x_2(a_0) = x_{fp}(a_0) \\
x_1(a) &\neq x_2(a), \quad a \neq a_0 \\
f_a(x_1(a)) &= x_2(a), \quad f_a(x_2(a)) = x_1(a) \\
Fix(f_a^2, X) &= \{x_1(a), x_2(a), x_{fp}(a)\}
\end{aligned}$$

- *the dynamics:*

for $a \leq a_0$

$$Inv(X, f_a) = \{x_{fp}(a)\}$$

and $x_{fp}(a)$ *is an attracting fixed point for* f_a.

For $a > a_0$ *the maximal invariant set in* X $Inv(X, f_a)$ *is equal to*

$$\overline{W^u_X}(x_{fp}(a), f) \cap \left(\overline{W^s_X}(x_1(a), f_a^2) \cup \overline{W^s_X}(x_2(a), f_a^2)\right)$$

and it is an interval connecting the points $x_1(a)$, $x_2(a)$.

Note that in the above definition, we assume the existence of a stable periodic orbit for each parameter value $a \in [a_1, a_2]$. Even for a_0, which has the neutral eigenvalue -1, the point x_0 is assumed to be attracting.

The following theorem summarizes the numerical method for proving the existence of a period doubling bifurcation for one-dimensional maps.

Theorem 8 *[WZ08, Thm.4, Thm.9] Let* $Z = [a_1, a_2] \times [x_1, x_2]$ *and let* $f_a : \mathbb{R} \to \mathbb{R}$ *be a family of maps, such that* $[x_1, x_2] \subset \mathrm{dom}\,(f_a)$ *for* $a \in [a_1, a_2]$.

A1 *Assume there exists a* C^k*-function* ($k \geq 3$), $x_{fp} : [a_1, a_2] \to (x_1, x_2)$ *such that for* $a \in [a_1, a_2]$

$$Fix(f_a, [x_1, x_2]) = \{x_{fp}(a)\}$$

A2 *Assume that* $G : Z \to \mathbb{R}$, *defined as*

$$G(a, x) = x - f_a^2(x), \quad \text{for } (a, x) \in Z,$$

is a C^k*-function* ($k \geq 3$), *such that*

$$\begin{aligned}
\frac{\partial^3 G}{\partial x^3}(Z) &\subset (0, \infty) \\
\frac{\partial^2 G}{\partial x \partial a}(Z) + \frac{\partial^2 G}{\partial x^2}(Z) x'_{fp}([a_1, a_2]) \cdot [0, 1] &\subset (-\infty, 0), \\
\frac{\partial G}{\partial x}(a_1, [x_1, x_2]) &\subset (0, \infty), \\
G(a_2, x_2) > 0, \qquad G(a_2, x_1) &< 0, \\
\frac{\partial G}{\partial x}(a_2, x_{fp}(a_2)) &< 0.
\end{aligned}$$

A3 *Assume that* f_a *is injective on* $[x_1, x_2]$ *for each* $a \in [a_1, a_2]$, *and*

$$\begin{aligned}
f'_{a_1}(x_{fp}(a_1)) &\in (-1, 1), \\
f'_{a_2}(x_{fp}(a_2)) &< -1, \\
\frac{\partial}{\partial a}\left(f'_a(x_{fp}(a)\right) &< 0, \ \text{for } a \in [a_1, a_2].
\end{aligned}$$

Then there exists a point $(a_0, x_0) \in Z$ such that f has a period doubling bifurcation at (a_0, x_0), with the neighborhood from the definition of the period doubling equal to Z.

Let us observe that all assumptions from the above theorem can be verified using rigorous numerics. The existence of the fixed point curve x_{fp} can be established by means of the interval Newton method. Given an enclosure for $x_{fp}(a)$, we can compute its derivative $x'_{fp}(a)$ by differentiating the equation

$$x_{fp}(a) - f_a(x_{fp}(a)) = 0.$$

The other inequalities are easily handled using interval arithmetic, as long as the set Z is a good candidate for the existence of the period doubling bifurcation.

4 The main algorithm.

In this final section, we will outline the main parts of the algorithm used to bound the measure of stable parameters.

4.1 Finding superstable windows.

It seems that each stable window arises either in the saddle-node bifurcation or in the period doubling bifurcation. Let $x(a_*)$ denote a new born fixed point for $Q^n_{a_*}$ which is an origin of a stable window. In each case $\frac{\partial}{\partial x} Q^n_{a_*}(x(a_*)) = 1$ and this derivative decreases to -1 along the fixed point curve $x(a)$ when the parameter a increases and results in the next period doubling bifurcation. Hence, for some parameter value a_0 we have $\frac{\partial}{\partial x} Q^n_{a_0}(x(a_0)) = 0$ and the orbit is superstable. This means that the critical point for the quadratic map is on the trajectory of $x(a_0)$.

The above considerations give us a method for detecting superstable orbits. It is enough to solve the equation

$$Q^n_a(\tfrac{1}{2}) = \tfrac{1}{2} \tag{14}$$

with respect to a. Since the above equation is polynomial of the degree 2^n it has finite number of solutions, and any method for detecting zeros of this polynomial may be used. In our application we do not need to estimate all solutions of (14) – in fact a lot of them are complex numbers or even when real, they are out of the range $\mathbb{A} = [2, 4]$. In our algorithm we use the Newton method; the details will be presented in the sequel.

Given a good approximation of a superstable periodic point, say $x(a_0)$, we can also estimate very roughly the size of the entire stable window. This can be done by using the Taylor approximation of the curve $\frac{\partial}{\partial x} Q^n_a(x(a))$, where $x(a)$ is a curve of period-n points. In the first approach, we can use the linear approximation. Put

$$s(a) = \frac{\partial}{\partial a}\left(\frac{\partial}{\partial x} Q^n_a(x(a))\right)$$

Clearly $s(a_0)$ gives us a linear approximation of how fast the stability of $x(a_0)$ changes. Hence, given that $\frac{\partial}{\partial x} Q^n_{a_0}(x(a_0)) \approx 0$, we can expect that the stable window is roughly

$$d(a_0) = [a_0 + 1/s(a_0), a_0 - 1/s(a_0)] \tag{15}$$

(note that $s(a_0) < 0$) or at least of this magnitude. It is possible to compute a higher order Taylor expansion of this curve (for example by means of automatic differentiation), and then solve for when it is equal to 1 and -1.

4.2 One iteration of the algorithm.

Assume we have some grid of the parameter domain

$$[2, 4] = \bigcup_{i \in \mathcal{I}} \mathbb{A}_i,$$

where the $\mathbb{A}_i$ are closed intervals (with non-empty interior) such that $\operatorname{int} \mathbb{A}_i \cap \operatorname{int} \mathbb{A}_j = \emptyset$ provided $i \neq j$. To each subinterval $\mathbb{A}_i$ from this grid we attribute one of the following four labels: `unchecked, verified, periodDoubling, small`. The first grid of the interval $\mathbb{A} = [2.4]$ is created by subdividing the interval $\mathbb{A}$ into 100 equal parts; each subinterval in this primary partition is given the label `unchecked`.

Let $N > 1$ be the maximal period of the stable window we are searching for. This is a parameter of our algorithm, and it remains fixed throughout the computations. One iteration of our algorithm consist of the following steps:

1. Search for superstable orbits
2. Fill parameter gaps
3. Search for parameter domains that contain a stable window, but which are not necessary superstable
4. Fill parameter gaps
5. Scan for period doubling

Note that neither our method of searching for superstable orbits, nor the bisection (step 3 below) produces a uniform partition. Without this non-uniform process, the computational times would become enormous. Below we describe how each one of the above steps are realized.

Searching for superstable orbits. For each subinterval $\mathbb{A}_i$, we proceed as follows.

1.1 If the interval $\mathbb{A}_i$ has a width smaller than 10^{-17}, we label it `small`.

1.2 If the interval $\mathbb{A}_i$ is *not* labelled as `unchecked`, we leave it intact, and exit.

1.3 Set $a = \mathrm{mid}(\mathbb{A}_i)$. For each $n = 2, \ldots, N$ the point a is a seed for the standard Newton method which solves equation (14). In this way we obtain a sequence

$$\mathbb{B}_i = \left(\mathbb{B}_i^2, \mathbb{B}_i^3, \ldots, \mathbb{B}_i^N\right)$$

such that for $n = 2, \ldots, N$ either $\mathbb{B}_i^n = \{a_i^n\}$ if $a_i^n \in \mathbb{A}_i$ is a good candidate for the superstable orbit of the period n, or $\mathbb{B}_i^n = \emptyset$, otherwise.

1.4 If $\mathbb{B}_i^n = \emptyset$ for each $n = 2, \ldots, N$ then we leave interval $\mathbb{A}_i$ intact, and exit.

1.5 For each $\mathbb{B}_i^n = \{a_i^n\}$ we try to prove that such a superstable orbit of principal period n exists on the interval of parameters $\mathbb{A}_i^n = a_i^n + [-10^{-17}, 10^{-17}]$. If we are able to prove stability on the interval $\mathbb{A}_i^n$ then we set $\mathbb{B}_i^n = \{\mathbb{A}_i^n\}$, otherwise $\mathbb{B}_i^n = \emptyset$.

1.6 For each $\mathbb{B}_i^n = \{\mathbb{A}_i^n\}$ we proceed as follows. Using the estimation (15) we try to find two numbers $c_i^n, d_i^n \in \mathbb{A}_i$ such that c_i^n is as close as possible to the origin of the window $\mathbb{A}_i^n$ and d_i^n is as close as possible to the end of that window. Next, we try to prove the existence of a stable periodic orbit of period n on the intervals $\mathbb{C}_i^n = c_i^n + [-10^{-17}, 10^{-17}]$ and $\mathbb{D}_i^n = d_i^n + [-10^{-17}, 10^{-17}]$.

1.7 We replace $\mathbb{A}_i$ in the current partition by a set of intervals $\mathbb{A}_i^n$, $\mathbb{C}_i^n$, $\mathbb{D}_i^n$, on which we were able to verify stability and mark them as `verified`. We also insert the remaining parts of $\mathbb{A}_i$, and mark them as `unchecked`.

Filling parameter gaps The above method for finding superstable orbits produces a parameter set with many gaps. At this point of the algorithm, we scan our current partition, and try to locate and fill possible gaps in the computed stable windows. The key point is that we do not need to determine a period of a stable window. We assume that if two subintervals $\mathbb{A}_i$ and $\mathbb{A}_j$ with the same period are separated only by intervals marked as `unchecked`, these might form a gap.

Searching for parameter domains that contain a stable window, but which are not necessary superstable This step might seem superfluous at first, but it is very important for proving the existence of period doublings, as well as for extending the primary window of each detected period doubling cascade. Note that the primary window in a cascade has the largest measure in that cascade, so it is very important to extend this window as much as possible.

3.1 If the interval $\mathbb{A}_i$ has a width smaller than 10^{-17}, we label it `small`.

3.2 If the interval $\mathbb{A}_i$ is *not* labelled as `unchecked`, we leave it intact, and exit.

3.3 Subdivide the parameter $\mathbb{A}_i$ onto two equal parts; say $\mathbb{A}_{i,1}$ and $\mathbb{A}_{i,2}$. Both halves inherit the label of $\mathbb{A}_i$. For each of these parts, we perform the next steps.

level	N	measure	wall time (h:m:s)	per. doublings
1	256	1.60620127942955014935	2 : 05 : 05	14
2	256	1.61118596303518551971	3 : 05 : 08	78
3	256	1.61287812977207803823	1 : 43 : 51	335
4	256	1.61349027421762439933	2 : 39 : 44	1381
5	256	1.61372268390076635341	5 : 18 : 22	4019
6	256	1.61381346643292467144	8 : 47 : 48	9075
7	512	1.61389940246044893686	67 : 26 : 04	20128
8	512	1.61391413966146151119	95 : 46 : 06	41692

Table 1: Computational results in terms of the level of subdivision. Here, the reported wall time corresponds to the current subdivision level only, and is thus not the accumulated time. The listed values of the measure and period doublings, however, correspond to the accumulated amount from all previous levels.

3.4 Set $a = \mathrm{mid}(\mathbb{A}_{i,j})$, and try to determine (non-rigorously) whether the map Q_a admits a stable orbit or not (of any period from the range $2, \dots, N$). If we are not able to find a good candidate for a stable orbit, we mark $\mathbb{A}_{i,j}$ as `unchecked`, and exit.

3.5 If we have a good candidate for stable orbit, we try to prove the existence of this orbit for the parameter domain $a + 10^{-17}[-1, 1]$. If this is not successful, we mark $\mathbb{A}_{i,j}$ as `unchecked` and exit.

3.6 Try to extend the existing stable orbit as much as possible to the left and to the right (this is what makes our partition non-uniform). Then insert to our partition the verified interval of stability, marked `verified`. If the verified domain of stability is not the entire interval $\mathbb{A}_{i,j}$, we also insert the remaining parts and mark them as `unchecked`.

Scanning for period doublings. In this step we traverse the current partition, and try to locate possible period doublings. For each candidate, we try to apply the method described in Section 3. If the method succeeds, we relabel the gap located between the two bounding stable windows as `periodDoubling`.

Collapsing data. In order to decrease the memory usage, we try to collapse windows which are connected, and on which we have the same verified period (it might happen that one window contains stability, the other period doubling).

In Table 1 we present information regarding the computed stability measure, the number of verified period doublings, and computationial effort versus the subdivision level. From this table, it is apparent that proceeding with further subdivisions does not add much to the total stability measure, whereas the computational time increases significantly.

In Table 2 (see the Appendix) the verified lower bound for the total measure of stable windows with a given principal period is presented. Note that the measures do not sum to the total measure obtained in Theorem 1. This is because the sets on which the period doubling is verified is not taken into account in this table.

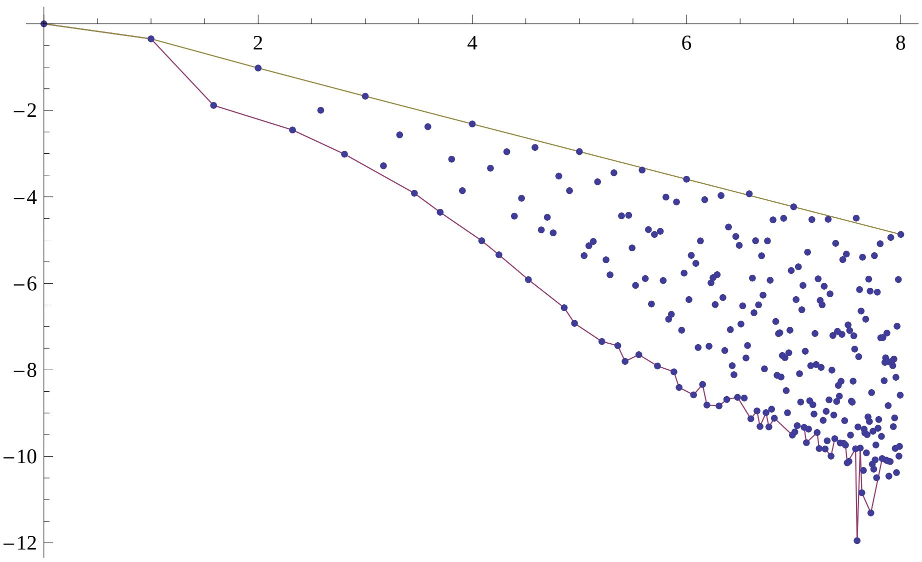

Figure 2: A plot of $\log_2$ of the period versus $\log_{10}$ of the verified measure. Note how the prime periods form the lower line, whereas periods of the form 2^k form the upper line. Also note the lack of some primes after period 222.

In Figure 2, we illustrate an interesting phenomenom; when we plot $\log_2$ of the period versus $\log_{10}$ of the verified measure, all points appear to be confined within a wedge–shaped region. This was also observed in [ST91].

In order to obtain the stability measure reported in Theorem 1, we compute through 8 subdivision levels, with the restriction to the maximal period we are searching for as given in Table 1. We also compute through 18 subdivision levels with the restriction on the maximal period set to 33000, but with the older version of the C++ program, which did not contain the step of searching for superstable orbits. These latter computations took over one year of CPU time. We stress that this large computational effort is a consequence of us verifying the existence of orbits with relatively large periods (up to 33000). For such large periods, the corresponding stable windows do not contribute significantly to the total measure of the stability regions $\mathcal{R}$. Nevertheless, our aim is to maximise the lower bound of $|\mathcal{R}|$; not to be cost-effective. As a final step, we merge the data obtained from both computations.

5 Implementation notes.

The program realizing the computer assisted proof of Theorem 1 has been written by the authors, and the C++ source code is available to download at [W]. The program has been successfully compiled and run on a 32 CPU computer (8 2.2 GHz Athlon Quad Processors) under the Fedora 10 operating system.

The output of the program contains information about the intervals on which we are able to verify stability or period doubling, together with rigorous enclosures for such orbits. The file with this data is about 30GB, and is available from the authors on request.

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Appendix

p	N_w	verified measure
1	1	0.99999999999999999133
2	1	0.449489712730337562154
3	1	0.0130718827972951374093
4	2	0.0952673665554796302024
5	3	0.00352281749622089545605
6	5	0.0100535208500353698182
7	9	0.00097010870416771319627
8	16	0.0211991511493234117709
9	28	0.000524485378625607094258
10	51	0.00270572008244477951906
11	93	0.000121525406628342040605
12	170	0.00417708627231338819503
13	314	4.41715769283789524285e-05
14	579	0.000745325525833007151988
15	1068	0.000138641400453846626653
16	1953	0.00484461107890990462163
17	3507	9.69701606337428946569e-06
18	5972	0.000459273251877917671263
19	9463	4.59268428609099256532e-06
20	13718	0.00110315833152131422337
21	18531	3.58298105005252272987e-05
22	23574	9.29907990940724224482e-05
23	28379	1.216667238331707937e-06
24	32635	0.0013835483469190961673
25	36076	1.71890636980708260556e-05
26	38980	3.37584496156975895353e-05
27	41303	1.46974192968794825642e-05
28	43529	0.000302746238548891365497
29	44699	2.72795058041050419995e-07
30	45166	0.000138997950280466157566
31	45901	1.19517385080121843477e-07
32	47240	0.001111485106359831913879
33	47440	4.3632329300394723931e-06
34	49058	7.37501525907872587662e-06
35	47460	9.3274990966042692353e-06
36	50318	0.000222715347582060421705
37	44075	4.53937424264166944354e-08
38	49191	3.5102554246041819952e-06
39	38467	1.57761156148461534343e-06
40	47982	0.000360835838257135685034
41	31886	3.6311444949173668495e-08
42	45919	3.65113084334731281821e-05
43	24657	1.56848479820545505525e-08
44	42783	3.74661099192274710951e-05
45	18475	6.60043353922822387259e-06
46	39162	8.9946769217335263702e-07
47	11758	2.24632894414451139697e-08
48	37943	0.000415039763801128974707
49	8185	1.29631702846405051643e-06
50	33918	1.75335974632567347803e-05
51	8204	3.33169708217895266245e-07
52	32750	1.35436598711980230036e-05
53	4230	1.23300845397515679291e-08
54	34828	1.59624206681101624139e-05

p	N_w	verified measure
55	3050	1.16600411184837604583e-06
56	31085	9.87863827087817787664e-05
57	6961	1.48295601606722449173e-07
58	29694	1.93045699661180750195e-07
59	1809	9.00775365712466746793e-09
60	36212	7.59640441304722593974e-05
61	1379	3.92577399733161358508e-09
62	28905	8.25950370189967864043e-08
63	7641	1.72763657665161302068e-06
64	30577	0.000256520738423341668893
65	1577	4.22839929780814346993e-07
66	35564	4.43357503740232812561e-06
67	783	2.63448830025575508884e-09
68	30475	2.89387006891677657505e-06
69	8141	3.28413728109024083412e-08
70	26952	9.56719570414628697752e-06
71	463	4.59780049363173182186e-09
72	38243	8.63115591216488762388e-05
73	360	1.53214927247138921196e-09
74	21973	3.5159473734790863686e-08
75	9815	1.02819050732663383463e-06
76	28051	1.35693201555380990597e-06
77	1066	3.22918023934731096558e-07
78	26060	1.58600911115609234658e-06
79	252	1.47106215988949962092e-09
80	29387	0.000107394074560007712904
81	8453	4.68557416541306309687e-07
82	12798	2.80936450511391072604e-08
83	144	2.07398621193677357155e-09
84	33850	2.0120764728698758636e-05
85	3503	8.52777142382535324749e-08
86	8596	1.24863019281690190709e-08
87	8197	7.72277030042581302816e-09
88	22564	1.21041500725691496265e-05
89	65	2.31937124736160404925e-09
90	20920	7.57842147977236257894e-06
91	1740	1.14816160706569803107e-07
92	19664	3.00570422225900080182e-07
93	7722	2.23444688511424682797e-09
94	4051	1.87703480516911852538e-08
95	4793	3.66334682228526589057e-08
96	30683	0.000117688693258389187254
97	31	7.37530394178884107781e-10
98	4869	1.32069927886481475188e-06
99	7915	2.0867437799509253038e-07
100	23096	9.63996436359167499508e-06
101	26	1.12145802757049328768e-09
102	13166	3.17131057843292665788e-07
103	27	4.90537400629764697157e-10
104	19024	4.3369611247329360737e-06
105	13346	5.33186985297298340902e-07
106	1440	1.05734072372142690055e-08
107	21	1.022770400272918212e-09
108	29324	9.55522620181385728966e-06
109	16	4.77940104376753382986e-10

p	N_w	verified measure
110	8431	1.1812725373037610832e-06
111	3925	1.23393548395571428511e-09
112	21841	2.93298849702130267486e-05
113	18	7.62759379095701017803e-10
114	12290	1.31535680928220391483e-07
115	5508	7.49251626692114258699e-09
116	15499	6.900509117693971739e-08
117	4848	7.20411398869025276781e-08
118	676	6.82326176896369540614e-09
119	3606	2.137405886049742465e-08
120	37923	3.18396039204213530255e-05
121	1324	1.917376117968085103e-08
122	521	3.30492569617681086758e-09
123	1603	1.01977183858750763967e-09
124	14699	2.47198570699474345913e-08
125	5500	8.23257322989795481405e-08
126	17957	1.97838960298976282437e-06
127	7	3.10795777568734221674e-10
128	19373	5.89883105963581472891e-05
129	1078	3.66784072948836414874e-10
130	9080	4.20631583707099451774e-07
131	9	5.11994356633405089951e-10
132	26838	2.42218211973419493555e-06
133	4221	8.16669937274763779733e-09
134	258	1.79819775280969096354e-09
135	9226	2.44965392485725977356e-07
136	18501	8.9812239105089772262e-07
137	4	4.6859563581844176916e-10
138	10647	2.68571779735420762547e-08
139	6	2.0663487045567696665e-10
140	24598	5.25549481856829765747e-06
141	504	4.2746522120606922357e-10
142	133	1.93015680543563716842e-09
143	3404	1.25144653095182745162e-08
144	33746	2.97889222889103841285e-05
145	4713	1.56418397266773478993e-09
146	120	9.49043677846150846023e-10
147	4786	6.91553730240192232603e-08
148	8294	1.33181957673571965683e-08
149	5	3.57004470907459903195e-10
150	19365	1.2754626365378204933e-06
151	5	1.5113572155930898866e-10
152	16225	4.0395330228970398323e-07
153	3899	1.14160096519218268174e-08
154	8891	3.18795315769641526837e-07
155	3856	6.78931672600663338457e-10
156	19989	8.543413353039909508e-07
157	3	1.4798593015070971024e-10
158	65	1.09752049917873073515e-09
159	194	2.27780292457596678091e-10
160	24807	3.02876161675914943455e-05
161	4378	2.01331796655275507835e-09
162	13760	5.73849525635191162753e-07
163	5	1.01435233476609987235e-10
164	3552	9.87795133557990701068e-09
165	6657	6.2416601692705242499e-08
166	40	8.98260122967481366985e-10
167	2	2.55999052902258772413e-10
168	33068	8.44847806316835481244e-06

p	N_w	verified measure
169	1767	1.8610426908306082483e-09
170	9306	7.75032402385630175834e-08
171	3643	4.3573504804328422324e-09
172	2444	2.47015120705997015538e-09
173	3	2.03688357196599167942e-10
174	8948	5.46092922444618711353e-09
175	5652	6.64362200962411580951e-08
176	16945	3.54646454374493887152e-06
177	77	1.99879849650808072425e-10
178	20	6.69900202367526564906e-10
179	1	1.81419202577304261581e-10
180	29960	4.75820449395659636482e-06
181	2	7.09528416922844407111e-11
182	8760	1.09307333107837648112e-07
183	62	7.63824476758545589661e-11
184	11409	8.12132364155868785582e-08
185	1061	3.09529817358808467809e-10
186	7577	1.8821592145078225311e-09
187	3251	1.79110255641747040389e-09
188	1175	5.49032037476700207623e-09
189	6446	6.14531474908697195048e-08
190	8362	3.01641478581051586394e-08
191	1	1.49110541839206356407e-10
192	26859	3.21324108153351294115e-05
193	1	1.11756394069462139385e-12
194	17	4.79882695430047800933e-10
195	4386	2.02441377710783909638e-08
196	7459	7.18425075670221791047e-07
197	2	1.55131276314418586892e-10
198	14414	2.30098677430094419871e-07
199	1	1.4394758171901167465e-11
200	19143	4.02909050146917774304e-06
201	32	4.71774375208205976051e-11
202	10	4.22747300490508570725e-10
203	2095	3.47202185621911518787e-10
204	14003	1.49018551693236994682e-07
205	419	1.20529853459006019989e-10
206	13	3.17498380451169959215e-10
207	2925	8.15956036678139318141e-10
208	15354	1.24971310225805744276e-06
209	2794	6.37096150141180928017e-10
210	20665	6.58688601611881127673e-07
211	1	4.90333836901390984764e-12
212	409	2.98467600135833566188e-09
213	10	6.59314716537628964232e-11
214	8	3.79725573250791481605e-10
215	258	5.05727809890083035427e-11
216	27283	4.36158776567318988426e-06
217	1406	8.30777391325199232597e-11
218	9	1.83606336948485826532e-10
219	10	3.20000926798402751672e-11
220	13469	6.27076812462429647455e-07
221	2301	4.45699145125433293657e-10
222	2514	7.09567133131902560983e-10
224	19383	8.23850415667803231967e-06
225	4746	5.51746486888472775778e-08
226	6	2.88586158442294404836e-10
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228	11343	5.57340890138587757852e-08

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233	1	8.18837829572732767147e-11
234	9536	7.08296236553822289306e-08
235	130	7.86033204670799623415e-11
236	202	1.495247332851043498e-09
237	11	3.48109145675845610413e-11
238	6522	1.51726968623637492595e-08
239	1	7.58494384921898356922e-11
240	36400	1.15294318147581356659e-05
242	3716	1.58947918067493115801e-08
243	1925	1.24415053402888997613e-08
244	142	4.84619781370571045542e-10
245	759	1.76788815934615650693e-08
246	1058	7.74593838940088041944e-10
247	1627	1.53152881652782180133e-10
248	6675	6.75415567116637027834e-09
249	2	4.22043962278184636716e-11
250	6493	1.02538442511802482815e-07
252	22283	1.22486141447206153821e-06
253	1536	1.01155065010058442532e-10
254	4	1.69353072364264445326e-10
255	2813	2.59583316799226471172e-09
256	12327	1.35366643798659827547e-05
258	653	2.72059625478979216773e-10
259	286	3.41191703796123180226e-11
260	11907	2.05938494813847142173e-07
261	1065	1.52759486917591891242e-10
262	2	1.96686378989285870489e-10
263	1	2.87478591279310968787e-13
264	19566	9.80300638316359759794e-07
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267	7	3.34174302812906276472e-11
268	79	3.71120459233612032257e-10
270	14535	3.11016055777168370922e-07
272	11198	2.32390117338790447299e-07
273	2238	3.90077191359493313794e-09
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275	1292	6.55838387373861020357e-09
276	7747	1.08072344977196838078e-08
279	575	3.52271530088682505166e-11
280	19073	2.16561362701090179272e-06
282	319	4.05198325847261342503e-10
284	28	1.06867019075175539999e-09
285	2331	9.00028809943873020494e-10
286	5752	8.28558049557545639452e-09
287	102	2.92420397791925967823e-11
288	26801	9.50318098398247358694e-06
289	363	1.55973038304391131703e-11
290	3788	9.69104700133949226171e-10
291	3	3.28789100979831605542e-12
292	19	2.43357794853238385002e-10
294	5848	8.05145010808038658068e-08
295	13	2.6396698845840282921e-11
296	1985	2.54021873783046026052e-09
297	1760	5.44000582861775550114e-09
298	1	2.69944656905440893979e-15

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303	2	2.10503576375531409326e-11
304	9015	9.60568618405335805654e-08
305	8	9.99460215110603567013e-12
306	5690	8.32956720459888944053e-09
308	9612	1.51741400689639829857e-07
309	3	1.17033555156048629797e-11
310	2435	2.49474347863848366913e-10
312	13904	3.28591576098831439356e-07
315	3662	2.41236501616684528848e-08
316	11	2.29022591689831411799e-10
318	105	1.7372986686400859746e-10
319	230	1.34097444520059339013e-11
320	17324	8.19797986926812556618e-06
322	3502	1.15320845841399011178e-09
323	354	5.71498620996480077849e-12
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325	803	1.776314742648818501e-09
326	1	3.60323750003832543598e-15
327	1	6.57320638891567554651e-12
328	731	2.33152009677939364707e-09
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330	8406	6.67121231238791689844e-08
332	9	2.94044038250118600075e-10
333	85	1.10730691109395262828e-11
334	1	8.3319837574508115452e-12
336	24223	3.01852421677570488534e-06
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344	485	5.45217979271150898768e-10
345	1316	1.80082394742736573434e-10
348	3807	2.14708923123446693282e-09
350	4365	7.91153934552590865081e-08
351	1285	1.31548023732733221358e-09
352	11422	9.59780129438292037292e-07
354	50	7.38209393787159218192e-11
356	4	1.32117570356138358534e-10
357	1205	4.66795671127256950328e-10
358	1	5.73755105925943809098e-12
360	25576	2.21276322956518584073e-06
361	76	4.66203031040945958807e-13
362	1	4.45357726391870656357e-13
363	588	4.98110518760627729584e-10
364	7262	4.14048574298025995333e-08
366	31	5.66953723551977262218e-11
368	5307	1.75552925009425653258e-08
369	35	1.05845241425706060312e-11
370	410	8.24403017640057411697e-11
371	7	5.89999598656076873482e-12
372	2404	6.07508208452442466374e-10
374	2927	7.35033434291765264135e-10
375	1386	5.08463279856943839441e-09
376	213	1.00565732499315052539e-09
377	50	2.32391384781538423709e-12

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385	874	2.78167789171702983797e-09
387	13	7.74055732705170029817e-13
388	2	4.92084751852611690381e-11
390	5743	1.75744183717515406462e-08
391	54	3.52437480241607481446e-13
392	5173	2.80804042494548627973e-07
393	1	9.68506958659576255855e-14
395	1	1.06436365797379917097e-12
396	10415	1.20159816404622615815e-07
399	924	1.46892991910799186606e-10
400	10965	1.43922879372094560113e-06
402	13	2.70759909718615299212e-11
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404	2	1.01355401285405100076e-10
405	1680	6.31650627787826157711e-09
406	741	1.62885576554527844451e-10
407	14	2.27054920004521321175e-13
408	6891	4.34370420506879212352e-08
410	136	4.43144243088972045186e-11
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413	5	6.09234017401316663154e-13
414	2234	4.20015274296692275868e-10
416	8534	3.2471492999986382999e-07
418	1820	2.0392443940073223807e-10
420	14602	4.15608721490018373501e-07
423	10	1.2194433830770012861e-13
424	71	6.98134743808490054739e-10
425	469	2.02649066364385399552e-10
426	7	4.30395315741127770792e-11
427	2	1.42516857690178078144e-13
428	4	7.4364516714636730299e-11
429	790	2.11923907242339559787e-10
430	92	5.5887169891910559727e-12
432	16020	1.67044264700547812064e-06
434	392	2.54916339773036959571e-11
435	277	2.19568083333065988683e-11
436	2	3.30837861446664494025e-11
437	27	2.03133950632539139747e-14
438	6	1.60390082291905766709e-12
440	8554	2.32805795916286861758e-07
441	946	2.51513861257088300327e-09
442	1432	1.09045624309758726334e-10
444	369	1.30991824375328858032e-10
448	10182	2.19921083578853565621e-06
450	5701	6.81370985547175705976e-08
451	2	4.90102750050347424349e-15
452	1	8.88078673100256565931e-14
455	738	5.83592011304742697142e-10
456	4513	1.33458296211128035758e-08
459	566	1.10629608989290528775e-10
460	2575	1.51762641516228757865e-09
462	3875	1.2279167517633901574e-08
464	1816	3.72782777758630390341e-09
465	113	8.90887710824905809659e-12
468	6600	2.67747084640220650575e-08
469	2	1.40968248359130132741e-12

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473	4	1.9420250997603805132e-13
474	5	1.21866505602141739217e-11
475	341	6.02061211215504377137e-11
476	3213	3.33967695945874831764e-09
477	3	2.88749774958463323316e-12
480	19347	3.74792087095383516515e-06
481	2	4.04101665324452241634e-14
483	389	2.11538746588968429307e-11
484	2279	3.97778451128281973492e-09
486	1935	1.3116444083421593958e-08
488	15	9.65298826115235097944e-11
490	1222	1.81205483808059841788e-08
492	110	3.93086973561607155858e-11
493	3	1.36257758548419261047e-13
494	719	2.41512497934498471253e-11
495	1153	1.65950646822433933725e-09
496	1072	6.91385064000363258607e-10
497	1	1.44151184044982727528e-14
498	3	1.04744498664877117022e-11
500	2465	6.57837836575986961707e-08
502	1	2.55261686354230077356e-11
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506	507	3.0719837697795116892e-11
507	256	1.71990861223131341617e-11
510	2487	1.67623497829456902242e-09
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513	16	1.22095844719283252999e-12
516	6	2.0872930120430233103e-13
518	4	2.26243069417764175455e-13
520	545	3.16232846211267348835e-08
522	10	6.40748305108518323436e-14
525	62	1.17687558548289217253e-09
528	957	2.29569912349979199373e-07
532	101	1.17417377687098478667e-10
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540	1038	1.46165916865880277098e-07
544	276	2.51742548694998402592e-08
546	137	3.68559668895152015011e-10
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552	94	3.61501270758346482381e-10
556	1	2.07384884005207270619e-11
558	9	6.55671914331912297769e-13
560	1256	6.70327007334979921294e-07
561	8	4.1048262879039842943e-13
564	4	1.53744278592882199774e-11
567	48	2.43151527558327362755e-10
570	54	4.93243759638190004679e-11
572	119	1.86208339329405059281e-10
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575	4	7.47015296842512555031e-15
576	1800	2.66651592608756898839e-06
578	4	1.76045159352988811463e-13
580	15	9.10767121442557225919e-13
585	27	2.28643547237755706547e-11
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592	6	5.69116865179603026448e-11
594	143	6.44021189459206833128e-10

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598	9	1.38628691859210562143e-13
600	1005	3.36770259626783738383e-07
605	7	2.27237456282280980702e-12
608	159	6.82432254521352088528e-09
609	1	1.29260751408066809631e-14
612	121	2.25683559730918692487e-10
616	448	1.7950820768199149402e-08
620	5	2.35666521020139185794e-14
621	4	7.93031005447142334219e-14
624	629	4.3791715737375239148e-08
625	3	1.25901772080744267512e-10
627	4	4.09498823739085082707e-14
630	383	1.03442019975738541682e-08
637	12	3.36762865857376247902e-12
638	2	8.84145187618479155844e-15
640	1120	2.0527302517655242331e-06
644	37	2.22930066334087184643e-11
646	2	1.36652841820072978862e-14
648	738	1.25460073577450537119e-07
650	54	1.80044884600175092437e-10
656	3	6.54908206658422198387e-11
660	453	1.00597978354606543139e-08
665	10	1.31711698839187540955e-13
666	3	1.13548059843537885172e-13
672	1444	8.56510101975016535647e-07
675	42	2.12535093691013088346e-10
676	44	1.88338149329642101648e-11
680	172	1.19071373247403577977e-09
682	1	4.32813507256213370056e-16
684	72	3.71837931539137356296e-11
686	23	2.50064468047470067624e-10
688	3	2.50916741811185728395e-11
690	12	4.04565487013841540431e-13
693	25	2.18898482239743286115e-11
696	17	3.80391638327148839949e-12
700	324	3.01428516311704614017e-08
702	57	3.80185910964919715482e-11
704	527	1.83185128695530566323e-07
710	1	2.73915525678125604259e-12
714	31	7.39434381780146310348e-12
715	7	6.05321999122554466055e-13
720	1887	7.05579485690485674887e-07
722	1	1.48752538065011208346e-16
726	32	1.6575213857700221709e-11
728	250	2.61358546632152466405e-09
729	8	1.60572273793374575845e-11
730	2	4.18040168453925886638e-14
735	27	9.88518403417792956844e-11
736	40	1.02616516846343319713e-09
740	3	1.15252416985067718258e-11
744	12	7.9270361975913861113e-12
748	36	6.3494183868972875473e-12
750	75	2.84647557824771468837e-09
752	1	4.09828421199520676055e-17
754	1	2.05998412772245842461e-17
756	479	1.61220390199837831924e-08
759	1	1.07769695945059140074e-16
760	70	1.4081922000003244122e-10

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780	202	8.15061277151252205897e-10
782	1	4.43902727076395109407e-14
784	369	6.88380924790833043092e-08
792	515	1.28835511686159476241e-08
798	14	1.47326833892236219725e-12
800	650	4.18762281319089157972e-07
805	3	3.10450450069499339634e-15
810	151	2.09467494131226894538e-09
812	5	1.03710954754698825298e-11
816	157	9.61918258371094458958e-10
819	4	1.27023391804925722681e-13
820	1	8.83212773750141622031e-15
825	14	1.19587638403029217216e-11
828	11	5.29068759289041956606e-13
832	279	4.13586658794349443236e-08
833	2	1.32018094373131944508e-13
836	25	1.19635009364349453875e-12
840	983	1.18551047246801682933e-07
850	10	4.35139207652368575907e-12
855	5	1.78383132917736553225e-13
858	24	2.49250424987079721717e-12
864	1157	4.82980160197580069004e-07
868	1	1.0579644799113552267e-14
870	3	1.38076008959764351403e-12
875	6	2.59332527240835331028e-11
880	450	3.0673142643227249704e-08
882	90	4.41232968348068133047e-10
884	11	1.79039075232090283407e-13
891	4	1.47963043727050713727e-12
896	607	4.97512492082970159135e-07
900	378	1.98595036511681893865e-08
910	31	1.40852555313708549534e-11
912	88	3.15394833080157033223e-10
918	15	4.22161654592412283193e-13
920	17	2.57394344553574239143e-12
924	160	6.90243518808655864838e-10
928	10	5.95693646320005765205e-11
930	1	1.89366751446318204444e-15
931	1	7.28518807779909849387e-15
935	3	2.14416158439423298887e-14
936	220	1.1238388919563441215e-09
938	1	2.00197931149448393739e-14
940	1	9.88789779221566078604e-13
945	25	3.0464403352400970526e-11
950	7	4.54089023327330920665e-13
952	51	3.49582496197292647011e-11
960	1516	9.97820564639434123833e-07
966	7	4.73555656493873655677e-14
968	50	1.04705775950605928237e-10
972	135	2.02491484083827666396e-09
975	5	1.3937867787000568498e-12
980	127	3.23273840877914830916e-09
984	1	4.8479952691193872738e-12
988	5	2.29213349689505463402e-14
990	76	1.1067172222059551423e-10
992	2	1.17142602042152876862e-12

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1001	3	7.38547677875400765402e-14
1008	927	1.19193268609913902156e-07
1012	3	3.47395723299115388727e-14
1014	7	2.48196211846685166336e-13
1020	45	2.16502341228441697041e-11
1024	327	6.68559892306888961522e-07
1040	16	7.44776625354742760621e-11
1050	7	6.47966142547362622395e-11
1056	76	7.7612096313541684145e-09
1072	1	7.05499872931392157938e-12
1080	137	1.26050035227295853524e-08
1088	5	4.27856838283377505228e-10
1092	1	2.01661604082303824725e-17
1100	5	4.45526298145648702587e-12
1104	1	2.01661604082303824725e-17
1120	156	1.26051267759818078074e-07
1125	1	2.01661604082303824725e-17
1134	1	5.26686875536308551915e-12
1140	1	2.01661604082303824725e-17
1144	1	2.01661604082303824725e-17
1152	253	6.10567644449304045007e-07
1170	1	2.01661604082303824725e-17
1176	24	5.56366844088607148677e-10
1184	1	2.01661604082303824725e-17
1188	2	4.03323208164607649451e-17
1200	157	5.58073092254366548426e-08
1215	1	2.01661604082303824725e-17
1216	3	1.59302432356511758371e-12
1224	4	8.06646416329215298902e-17
1232	24	4.69212817471448273565e-11
1248	25	2.34445507712316425497e-10
1260	22	5.99486903261198245119e-11
1280	153	4.29530000367519398941e-07
1296	87	9.42396631803225914847e-09
1300	1	2.01661604082303824725e-17
1320	18	2.09381784913725832453e-11
1344	181	1.70660756297841623153e-07
1350	1	1.64668621620289723495e-11
1352	1	2.01661604082303824725e-17
1360	2	4.03323208164607649451e-17
1368	2	4.03323208164607649451e-17
1372	2	4.03323208164607649451e-17
1386	1	2.01661604082303824725e-17
1392	1	3.93118697317174081718e-14
1400	25	1.32888534074351860603e-09
1408	41	1.35467981270554621576e-08
1428	1	2.01661604082303824725e-17
1440	292	1.19849110882547932322e-07
1456	3	6.04984812246911474176e-17
1458	2	4.03323208164607649451e-17
1500	11	7.61010807322215487858e-11
1512	19	1.0492414099019797824e-10
1520	2	4.03323208164607649451e-17
1536	170	4.36508358502354920638e-07
1540	4	8.06646416329215298902e-17
1560	4	8.06646416329215298902e-17
1568	29	4.11252392557015511443e-09
1584	13	7.48487556656574071212e-12

p	N_w	verified measure
1600	93	8.37857453239547467216e-08
1620	12	2.47292923985509283114e-11
1632	4	1.75302552422179624969e-11
1664	10	3.41086777528795681569e-10
1680	120	5.40150254342217084336e-09
1728	208	9.33806746942957704993e-08
1744	1	2.3505438047355386999e-12
1760	21	7.20544088804453419783e-11
1764	1	2.01661604082303824725e-17
1792	65	9.26499040202876517069e-08
1800	50	7.92415121584136294963e-10
1824	4	8.06646416329215298902e-17
1848	4	8.06646416329215298902e-17
1872	6	1.20996962449382294835e-16
1904	1	2.01661604082303824725e-17
1920	262	2.163420860449693367e-07
1944	9	1.81495443674073442253e-16
1960	4	1.40516183758099266754e-11
1980	1	2.01661604082303824725e-17
2000	15	1.29177164234425823075e-09
2016	123	4.52573798142424932323e-09
2040	3	6.04984812246911474176e-17
2048	43	1.35853521534765042666e-07
2080	5	1.00830802041151912363e-16
2100	8	1.6132928326584305978e-16
2112	22	2.01244349526744081835e-10
2156	1	2.01661604082303824725e-17
2160	92	1.56601677936060723617e-09
2176	1	2.01661604082303824725e-17
2184	1	2.01661604082303824725e-17
2200	4	8.06646416329215298902e-17
2240	89	1.89759388814249307931e-08
2268	2	4.03323208164607649451e-17
2304	187	1.46981382817372441263e-07
2340	1	2.01661604082303824725e-17
2352	12	2.62160085306994972143e-16
2376	2	4.03323208164607649451e-17
2400	121	7.68120562294472009057e-09
2464	2	4.03323208164607649451e-17
2496	5	1.00830802041151912363e-16
2520	19	3.83157047756377266978e-16
2560	99	9.52879874516806418816e-08
2592	56	4.48746794497884016195e-10
2640	5	1.00830802041151912363e-16
2688	119	2.46360268030005463702e-08
2700	6	1.20996962449382294835e-16
2720	1	2.01661604082303824725e-17
2800	18	9.85830763810396915048e-11
2816	14	3.9972266664169142647e-10
2880	224	1.57514175023500124784e-08
2912	2	4.03323208164607649451e-17
2940	1	2.01661604082303824725e-17
3000	7	1.41163122857612677308e-16
3024	18	3.62990887348146884506e-16
3040	1	2.01661604082303824725e-17
3072	103	9.33353577474860945928e-08
3120	1	2.01661604082303824725e-17
3136	13	9.7852865133668531783e-11
3168	10	2.01661604082303824725e-16

p	N_w	verified measure
3200	66	1.13766802092244573297e-08
3240	6	1.20996962449382294835e-16
3328	5	1.00830802041151912363e-16
3360	61	8.18627051996784205201e-11
3456	133	1.13324545385727615265e-08
3520	8	1.6132928326584305978e-16
3528	1	2.01661604082303824725e-17
3584	41	1.46504951935213373337e-08
3600	24	2.22905021263902769491e-11
3640	1	2.01661604082303824725e-17
3696	1	2.01661604082303824725e-17
3780	2	4.03323208164607649451e-17
3840	165	3.35846411221672053182e-08
3888	5	1.00830802041151912363e-16
3920	2	4.03323208164607649451e-17
3960	1	2.01661604082303824725e-17
4000	13	1.18540902788177859861e-10
4032	60	1.91539026826725855557e-10
4096	27	2.24732561321123319731e-08
4160	1	2.01661604082303824725e-17
4200	4	8.06646416329215298902e-17
4224	15	4.40836603332606102867e-16
4320	41	6.33671505025079628837e-11
4416	1	2.01661604082303824725e-17
4480	47	4.44286632657331370666e-10
4500	1	2.01661604082303824725e-17
4608	109	2.75429096201380840814e-08
4704	4	8.06646416329215298902e-17
4752	1	2.01661604082303824725e-17
4800	59	4.65015083080458357934e-10
4928	2	4.03323208164607649451e-17
4992	4	8.06646416329215298902e-17
5000	1	2.01661604082303824725e-17
5040	8	1.6132928326584305978e-16
5120	46	1.69437647755695587245e-08
5184	27	3.28448097712541464688e-12
5280	2	4.03323208164607649451e-17
5376	71	1.51430259672825917594e-09
5400	2	4.03323208164607649451e-17
5600	6	1.20996962449382294835e-16
5632	2	4.03323208164607649451e-17
5760	111	1.09496048512902621752e-09
6000	5	1.00830802041151912363e-16
6048	4	8.06646416329215298902e-17
6144	58	1.70946220685863625732e-08
6272	3	6.04984812246911474176e-17
6300	1	2.01661604082303824725e-17
6336	2	4.03323208164607649451e-17
6400	38	6.63067415210272237402e-10
6480	4	8.06646416329215298902e-17
6720	17	3.42824726939916502033e-16
6912	80	8.68082543131407091686e-10
7040	1	2.01661604082303824725e-17
7168	23	9.41381698288390467155e-10
7200	12	2.62160085306994972143e-16
7680	78	3.80943674846943858281e-09
7776	1	2.01661604082303824725e-17
7840	2	4.03323208164607649451e-17
8000	7	1.41163122857612677308e-16
8064	28	4.11358252148941172521e-12
8192	8	4.29929311379292911077e-09
8400	2	4.03323208164607649451e-17
8448	5	1.00830802041151912363e-16
8640	8	1.6132928326584305978e-16
8960	14	2.82326245715225354616e-16
9216	63	2.5629170070928797287e-09
9408	2	4.03323208164607649451e-17
9600	23	4.30581091614978817006e-12
10080	1	2.01661604082303824725e-17
10240	30	2.16034582193810886785e-09
10368	5	1.00830802041151912363e-16
10752	25	5.8277964702299622779e-11
11200	1	2.01661604082303824725e-17
11264	1	2.01661604082303824725e-17
11520	50	3.45301736422859439912e-11
12288	25	2.52287638046258533286e-09
12544	1	2.01661604082303824725e-17
12800	7	1.6132928326584305978e-16
13440	5	1.00830802041151912363e-16
13824	30	3.00239722406808740018e-11
14336	7	3.0324076799939647664e-11
15360	30	1.3809309164919403301e-10
16000	1	2.01661604082303824725e-17
16128	2	4.03323208164607649451e-17
16384	2	4.58952424241959588969e-10
17280	1	2.01661604082303824725e-17
17920	7	1.41163122857612677308e-16
18432	19	7.0838524282579307112e-11
19200	1	2.01661604082303824725e-17
20480	6	1.48670596666899967886e-10
21504	4	8.06646416329215298902e-17
23040	1	2.01661604082303824725e-17
24576	7	1.96098825112206442967e-10
25600	1	2.01661604082303824725e-17
27648	3	6.04984812246911474176e-17
28672	2	4.03323208164607649451e-17

Table 2: The lower bound of the total stability measure of windows with respect to the principal period p. Also the number of detected windows N_w is given. This means that we found N_w *distinct* orbits of principal period p.