Heteroclinic Connections between Periodic Orbits in Planar Restricted Circular Three Body Problem - A Computer Assisted Proof

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Abstract

The restricted circular three-body problem is considered for the following parameter values $C = 3.03$, $\mu = 0.0009537$ - the values for *Oterma* comet in the Sun-Jupiter system. We present a computer assisted proof of an existence of homo- and heteroclinic cycle between two Lyapunov orbits and an existence of symbolic dynamics on four symbols built on this cycle.

1 Introduction and statement of results

In paper [11] methods of dynamical system theory were used (see also [13]) to explain rapid transitions from heliocentric orbits outside the orbit of Jupiter to heliocentric orbits inside the orbit of Jupiter and vice versa for Jupiter comets *Oterma* and *Gehrels 3*. To model this problem authors in [11] used planar circular restricted three-body problem and established that for a parameters corresponding to Sun-Jupiter-Oterma system rapid transitions of Oterma are explained by transversal intersections of stable and unstable manifolds of two periodic orbits around libration points L_1 and L_2. In fact the existence of symbolic dynamics on three symbols was claimed.

The goal of this paper is develop and test tools which allow with computer assistance to prove the results claimed in [11].

Before we state our main results we give a short description of the planar restricted circular three-body problem. We follow the paper [11] and use the notation introduced there. Let S and J be two bodies called Sun and Jupiter, of masses $m_s = 1 - \mu$ and $m_j = \mu$, $\mu \in (0, 1)$, respectively. They rotate in the

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plane in circles counter clockwise about their common center and with angular velocity normalized as one. Choose a rotating coordinate system (synodical coordinates) so that origin is at the center of mass and the Sun and Jupiter are fixed on the x-axis at $(-\mu, 0)$ and $(1-\mu, 0)$ respectively. In this coordinate frame the equations of motion of a massless particle called the comet or the spacecraft under the gravitational action of Sun and Jupiter are (see [11] and references given there)

$$\ddot{x} - 2\dot{y} = \Omega_x(x, y), \qquad \ddot{y} + 2\dot{x} = \Omega_y(x, y), \tag{1.1}$$

where

$$\Omega(x, y) = \frac{x^2 + y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1-\mu)}{2}$$
$$r_1 = \sqrt{(x+\mu)^2 + y^2}, \qquad r_2 = \sqrt{(x-1+\mu)^2 + y^2}$$

Equations (1.1) are called the equations of the planar circular restricted three-body problem (PCR3BP). They have a first integral called the *Jacobi integral*, which is given by

$$C(x, y, \dot{x}, \dot{y}) = -(\dot{x}^2 + \dot{y}^2) + 2\Omega(x, y). \tag{1.2}$$

We consider PCR3BP on the hypersurface

$$\mathcal{M}(\mu, C) = \{(x, y, \dot{x}, \dot{y}) \mid C(x, y, \dot{x}, \dot{y}) = C\}, \tag{1.3}$$

and we restrict our attention to the following parameter values $C = 3.03$, $\mu = 0.0009537$ - the parameter values for *Oterma* comet in the Sun-Jupiter system (see [11]).

The projection of $\mathcal{M}(\mu, C)$ onto position space is called a Hill's region and gives the region in the (x, y)-plane, where the comet is free to move. The Hill's region for the parameter considered in this paper is shown on Figure 1 in white, the forbidden region is shaded. The Hill's region consists of three regions: an interior (Sun) region, an exterior region and Jupiter region.

In [11] a very good numerical evidence was given for the following facts for the Sun-Jupiter-Oterma system:

0. an existence of Lyapunov orbits L_1^* and L_2^* around libration points L_1 and L_2, respectively. Both orbits are hyperbolic and are located in the Jupiter region.

1. There exists a transversal heteroclinic orbit connecting L_1^* and L_2^*. There exists a transversal heteroclinic orbit connecting L_2^* and L_1^*. Both orbits are in the Jupiter region. These orbits were discovered for the first time in [11].

2. there exists a transversal homoclinic orbit to L_1^* in the interior (Sun) region,

3. there exists a transversal homoclinic orbit to L_2^* in the exterior region.

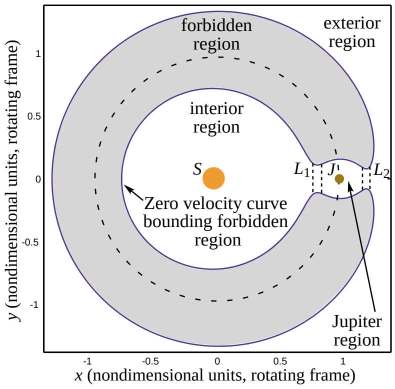

Figure 1: Hills region for PCR3BP with $C = 3.03$, $\mu = 0.0009537$ from [11].

By transversal hetero- and homoclinic orbit, we mean that appropriate unstable and stable manifolds intersect transversally. For example in assertion **1**: the stable manifold of L_1^* intersect transversally the unstable manifold of L_2^*.

It is now standard in dynamical system theory (see [11] and references given there) to derive from assertions [0-3] an existence of symbolic dynamics on four symbols S, L_1^*, L_2^*, X which the following allowed transition

$$S \to S, L_1^*, \quad L_1^* \to L_1^*, S, L_2^* \quad L_2^* \to L_1^*, L_2^*, X, \quad X \to X, L_2^*.$$

In [11](section 1.4) an existence of symbolic dynamics on three symbols, only, was claimed. Instead of two symbols L_1^* and L_2^* one symbol J for Jupiter region was used.

From the point of view of rapid transition of Oterma from interior region to an exterior region and vice versa, an existence of heteroclinic orbits between L_1^* and L_2^* claimed in assertion 1 was of a special importance, as they are an indication of an existence of a dynamical channel joining an interior with an exterior region.

The following two theorems summarize main results of our paper

Theorem 1.1 *For PCR3BP with* $C = 3.03$, $\mu = 0.0009537$ *there exist two periodic solutions in the Jupiter region,* L_1^* *and* L_2^* *, called Lyapunov orbits, and there exists heteroclinic connections between them, in both directions. Moreover for both orbits* L_1^* *and* L_2^* *there exists a homoclinic orbit in interior and exterior region, respectively.*

The next theorem says that the homo- and heteroclinic connections whose existence is established in Theorem 1.1 are topologically transversal i.e. give rise to the symbolic dynamics, just as in the case of an existence of transversal intersections of stable and unstable manifolds.

Theorem 1.2 *For PCR3BP with* $C = 3.03$, $\mu = 0.0009537$ *there exist a symbolic dynamics on four symbols* $\{S, X, L_1, L_2\}$ *corresponding to Sun and exterior regions and vicinity of* L_1 *and* L_2*, respectively.*

A precise statement of this theorem with all necessary details about the symbolic dynamics is given as Theorem 7.1 in Section 7.

Hence we have proved assertion **0**, but we did'nt proved assertions **1-3**, as we did'nt checked that stable and stable manifolds of Lyapunov orbits intersect transversally. Instead we had proved that there is enough topological transversality present to build a symbolic dynamics on it. The use of topological tools was essential for the success of this work, as the rigorous computation of stable and unstable manifolds appears to be much more difficult than the computations reported in this paper.

Contents of our paper may be described as follows. In Section 2 we continue our brief description of PCR3BP, we define suitable Poincaré maps and state their symmetry properties. In Section 3 we present the main topological tool used in this paper - the notion of covering relation. In Section 4 we describe how to link a local hyperbolic behavior with covering relations to obtain homo- and heteroclinic orbits. In Sections 5 and 6 we report the results of our rigorous computations for PCR3BP and we prove Theorem 1.1. In Section 7 we show how to use symmetries of PCR3BP together with covering relations to complete the proof of Theorem 1.2. Section 8 contains the details of the numerical part of proof, we mainly discuss the question of an efficient approach to a verification of covering relations. We also include all initial data, so that a willing reader with his own code can verify our claims. Our C++-source code is available on-line (see[18]). At this side the reader will also find files describing basic classes used in our programm and we give the names of functions, which perform proofs of numerical lemmas. In Section 9 we discuss some natural extensions of our results.

What is new in this paper besides giving a proof for some results from [11]? First of all it shows how to successfully link numerically cheap C^0-methods (the covering relations) with much more numerically expensive C^1-methods (a local hyperbolicity). This was previously done for maps (see [6]), only. The main obstacle in applications to ODEs was the lack of an efficient C^1 ODE solver. Such a solver - called a C^1-Lohner algorithm - was recently proposed by the second author in [23].

Other novelties in this paper are some theoretical improvements in the theory of covering relations. We use an abstract definition of covering relation from [7] and we show how to use a symmetry which involves a time reflection (for Poincaré map this correspond to taking an inverse map) together with covering relations. Both these improvements (in numerical algorithms and in theory of covering relations) allow to reduce the computation time considerably (probably by two orders of magnitude). The total computation time on a PC using a 1.1Ghz Celeron processor is less than 40 minutes.

2 Properties of PCR3BP: Poincaré maps and symmetries

In this section we continue our brief description of PCR3BP we started in the Introduction and we introduce various notations which will be used throughout the paper.

The PCR3BP has three unstable collinear equilibrium points on the Sun-Jupiter line, called L_1, L_2 and L_3 (see Fig. 1.4 in [11]), whose eigenvalues include one real and one imaginary pair. The value of C (Jacobi constant) at the point L_i will be denoted by C_i. An linearization at L_i for $i = 1, 2$ for the parameter range considered here, shows that these points are of center-saddle type (see [11]). By theorem of Moser [16] it follows that for $C < C_i$ and $|C - C_i|$ small enough, there exist hyperbolic periodic orbits, L_i^*, around L_i, called Lyapunov orbits. Observe that for a fixed value of $C < C_i$ an existence of Lyapunov orbit L_i^* is not settled by the Moser theorem and has to be proved.

As was mentioned in the Introduction we restrict our attention to the following parameter values $C = 3.03$, $\mu = 0.0009537$ - the parameter values for *Oterma* comet in the Sun-Jupiter system (see [11]). Since we work with fixed parameter values we usually drop the dependence of various objects defined throughout the paper on μ and C, so for example $\mathcal{M} = \mathcal{M}(\mu, C)$. For our parameter values we have $C_2 > C > C_3$ (this means that we are considering Case 3 from section 3.1 in [11]).

We consider Poincaré sections: $\Theta = \{(x, y, \dot{x}, \dot{y}) \in \mathcal{M} | y = 0\}$, $\Theta_+ = \Theta \cap \{\dot{y} > 0\}$, $\Theta_- = \Theta \cap \{\dot{y} < 0\}$.

On $\Theta_\pm$ we can express $\dot{y}$ in terms of x and $\dot{x}$ as follows

$$\dot{y} = \pm\sqrt{2\Omega(x, 0) - \dot{x}^2 - C} \tag{2.4}$$

Hence the sections $\Theta_\pm$ can be parameterized by two coordinates $(x, \dot{x})$ and we will use this identification throughout the paper. More formally, we have the transformation $T_\pm : \mathbb{R}^2 \to \Theta_\pm$ given by the following formula

$$T_\pm(x, \dot{x}) = (x, 0, \dot{x}, \pm\sqrt{2\Omega(x, 0) - \dot{x}^2 - C}\ ) \tag{2.5}$$

The domain of $T_\pm$ is given by an inequality $2\Omega(x, 0) - \dot{x}^2 - C \geq 0$.

Let $\pi_{\dot{x}} : \Theta_\pm \longrightarrow \mathbb{R}$ and $\pi_x : \Theta_\pm \longrightarrow \mathbb{R}$ denote the projection onto $\dot{x}$ and x coordinate, respectively. We have $\pi_{\dot{x}}(x_0, \dot{x}_0) = \dot{x}_0$ and $\pi_x(x_0, \dot{x}_0) = x_0$.

We will say that $(x, \dot{x}) \in \Theta_\pm$ meaning that $(x, \dot{x})$ represents two-dimensional coordinates of a point on $\Theta_\pm$. Analogously we give a meaning to the statement $M \subset \Theta_\pm$ for a set $M \subset \mathbb{R}^2$.

We define the following Poincaré maps between sections

$$\begin{aligned} P_+ &: \Theta_+ \to \Theta_+ \\ P_- &: \Theta_- \to \Theta_- \\ P_{\frac{1}{2},+} &: \Theta_+ \to \Theta_- \\ P_{\frac{1}{2},-} &: \Theta_- \to \Theta_+. \end{aligned}$$

As a rule the sign $+$ or $-$ tells that the domain of the maps $P_\pm$ or $P_{\frac{1}{2},\pm}$ is contained in $\Theta_\pm$ (the same sign). Observe that

$$P_+(x) = P_{\frac{1}{2},-} \circ P_{\frac{1}{2},+}(x), \qquad P_-(x) = P_{\frac{1}{2},+} \circ P_{\frac{1}{2},-}(x)$$

whenever $P_+(x)$ and $P_-(x)$ are defined. These identities express the following simple fact: to return to Θ_+ we need to cross Θ with negative $\dot{y}$ (this is $P_{\frac{1}{2},+}$ first and then we return to Θ with $\dot{y} > 0$ (this is $P_{\frac{1}{2},-}$).

Sometimes we will drop signs in $P_\pm$ and $P_{\frac{1}{2},\pm}$, hence $P(z) = P_+(z)$ if $z \in \Theta_+$ and $P(z) = P_-(z)$ if $z \in \Theta_-$, a similar convention will be applied to $P_{\frac{1}{2}}$.

2.1 Symmetry properties of PCR3BP

Notice that PCR3BP has the following symmetry

$$R(x, y, \dot{x}, \dot{y}, t) = (x, -y, -\dot{x}, \dot{y}, -t), \tag{2.6}$$

which expresses the following fact, if $(x(t), y(t))$ is a trajectory for PCR3BP, then $(x(-t), -y(-t))$ is also a trajectory for PCR3BP. From this it follows immediately that

$$\begin{aligned} &\text{if} \quad P_\pm(x_0, \dot{x}_0) = (x_1, \dot{x}_1), \qquad \text{then} \quad P_\pm(x_1, -\dot{x}_1) = (x_0, -\dot{x}_0) \\ &\text{if} \quad P_{\frac{1}{2},\pm}(x_0, \dot{x}_0) = (x_1, \dot{x_1}), \qquad \text{then} \quad P_{\frac{1}{2},\mp}(x_1, -\dot{x}_1) = (x_0, -\dot{x}_0) \end{aligned} \tag{2.7}$$

We will denote also by R the map $R : \Theta_\pm \to \Theta_\pm$ $R(x, \dot{x}) = (x, -\dot{x})$ for $(x, \dot{x}) \in \Theta_\pm$. Now eq. (2.7) can be written as

$$\begin{aligned} &\text{if} \quad P_\pm(x_0) = x_1, \qquad \text{then} \quad P_\pm(R(x_1)) = R(x_0) \\ &\text{if} \quad P_{\frac{1}{2},\pm}(x_0) = x_1, \qquad \text{then} \quad P_{\frac{1}{2},\mp}(R(x_1)) = R(x_0) \end{aligned} \tag{2.8}$$

3 Topological tools

In this section we present main topological tools used in this paper. The crucial notion is that of *a covering relation*. This notion in various forms was introduced in papers [20, 21, 22, 24]. Here we follow the most recent and most general version introduced in [7] and the reader is referred there for proofs.

3.1 h-sets

Notation: For a given norm in $\mathbb{R}^n$ by $B_n(c, r)$ we will denote an open ball of radius r centered at $c \in \mathbb{R}^n$. When the dimension n is obvious from the context we will drop the subscript n. Let $S^n(c, r) = \partial B_{n+1}(c, r)$, by the symbol S^n we will denote $S^n(0, 1)$. We set $\mathbb{R}^0 = \{0\}$, $B_0(0, r) = \{0\}$, $\partial B_0(0, r) = \emptyset$.

For a given set Z, by $\operatorname{int} Z$, $\overline{Z}$, ∂Z we denote the interior, the closure and the boundary of Z, respectively. For the map $h : [0, 1] \times Z \to \mathbb{R}^n$ we set $h_t = h(t, \cdot)$. By Id we denote an identity map. For a map f, by $\operatorname{dom}(f)$ we will denote the domain of f. Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ a continuous map we will say that $X \subset \operatorname{dom}(f^{-1})$ if the map $f^{-1} : X \to \mathbb{R}^n$ is well defined and continuous.

Definition 3.1 *A h-set, N, is the object consisting of the following data*

- $|N|$ *- a compact subset of* $\mathbb{R}^n$*, a support of* N
- $u(N), s(N) \in \{0, 1, 2, \dots\}$*, such that* $u(N) + s(N) = n$
- *a homeomorphism* $c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$*, such that*
$$c_N(|N|) = \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1).$$

We set

$$\begin{aligned} N_c &= \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1), \\ N_c^- &= \partial\overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1) \\ N_c^+ &= \overline{B_{u(N)}}(0,1) \times \partial\overline{B_{s(N)}}(0,1) \\ N^- &= c_N^{-1}(N_c^-), \quad N^+ = c_N^{-1}(N_c^+) \end{aligned}$$

Hence a h-set, N, is a product of two closed balls in some coordinate system. The numbers, $u(N)$ and $s(N)$, stand for the dimensions of nominally unstable and stable directions, respectively. The subscript c refers to the new coordinates given by homeomorphism c_N. We will call N^- (N_c^-) an exit set of N and N^+ (N_c^+) an entry set of N. Observe that if $u(N) = 0$, then $N^- = \emptyset$ and if $s(N) = 0$, then $N^+ = \emptyset$.

Definition 3.2 *Let N be a h-set. We define a h-set N^T as follows*

- $|N^T| = |N|$
- $u(N^T) = s(N)$, $s(N^T) = u(N)$
- *We define a homeomorphism* $c_{N^T} : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N^T)} \times \mathbb{R}^{s(N^T)}$*, by*
$$c_{N^T}(x) = j(c_N(x)),$$
where $j : \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)} \to \mathbb{R}^{s(N)} \times \mathbb{R}^{u(N)}$ *is given by* $j(p,q) = (q,p)$.

∎

Observe that $N^{T,+} = N^-$ and $N^{T,-} = N^+$. This operation is useful in the context of inverse maps, as it was first pointed out in [1].

3.2 Covering relations

For $n > 0$ and a continuous map $f : S^n \to S^n$ by $d(f)$ we denote the degree of f [4]. For $n = 0$ we define the degree, $d(f)$, as follows. Observe first that $S^0 = \{-1, 1\}$. We set

$$d(f) = \begin{cases} 1, & \text{if } f(1) = 1 \text{ and } f(-1) = -1, \\ -1, & \text{if } f(1) = -1 \text{ and } f(-1) = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

Definition 3.3 *Assume $n > 0$. Let $f : \overline{B_n}(0,1) \to \mathbb{R}^n$, such that*

$$0 \notin f(\partial B(0,1))). \tag{3.10}$$

We define a map $s_f : S^{n-1} \to S^{n-1}$ by

$$s_f(x) = \frac{f(x)}{\|f(x)\|}. \tag{3.11}$$

Definition 3.4 *Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f : |N| \to \mathbb{R}^n$ be continuous. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s$. Let w be a nonzero integer. We say that*

$$N \stackrel{f,w}{\Longrightarrow} M$$

(N f-covers M with degree w) iff the following conditions are satisfied

1. *there exists a continuous homotopy $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$, such that the following conditions hold*

$$\begin{aligned} h_0 &= f_c, & (3.12)\\ h([0,1], N_c^-) \cap M_c &= \emptyset & (3.13)\\ h([0,1], N_c) \cap M_c^+ &= \emptyset & (3.14) \end{aligned}$$

2.1 *If $u > 0$, then there exists a map $A : \mathbb{R}^u \to \mathbb{R}^u$, such that*

$$\begin{aligned} h_1(p,q) &= (A(p), 0), \quad \text{where } p \in \mathbb{R}^u \text{ and } q \in \mathbb{R}^s & (3.15)\\ A(\partial B_u(0,1)) &\subset \mathbb{R}^u \setminus \overline{B_u(0,1)} & (3.16) \end{aligned}$$

Moreover, we require that

$$d(s_A) = w,$$

2.2 *If $u = 0$, then*

$$\begin{aligned} h_1(x) &= 0 \qquad \text{for } x \in N_c & (3.17)\\ w &= 1. & (3.18) \end{aligned}$$

Intuitively, $N \stackrel{f}{\Longrightarrow} M$ if f stretches N in the 'nominally unstable' direction, so that its projection onto 'unstable' direction in M covers in topologically nontrivial manner projection of M. In the 'nominally stable' direction N is contracted by f. As a result N is mapped across M in the unstable direction, without touching M^+. An example of covering relation on the plane with one unstable direction is shown on Figure 3.

Definition 3.5 *Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. Assume that $g^{-1} : |M| \to \mathbb{R}^n$ is well defined and continuous. We say that $N \stackrel{g,w}{\Longleftarrow} M$ (N g-backcovers M with degree w) iff $M^T \stackrel{g^{-1},w}{\Longrightarrow} N^T$.*

The following theorem proved in [7] is one of main tools used in this paper. Various versions of this theorem (without backcovering) using slightly weaker notions of covering relations or even without an explicitly defined notion of covering relation were given in [20, 21, 22, 24]. In the planar case this theorem with backcovering was stated also in [1].

Theorem 3.6 *Assume N_i, $i = 0, \ldots, k$, $N_k = N_0$ are h-sets and for each $i = 1, \ldots, k$ we have either*

$$N_{i-1} \overset{f_i, w_i}{\Longrightarrow} N_i \tag{3.19}$$

or $|N_i| \subset dom(f_i^{-1})$ and

$$N_{i-1} \overset{f_i, w_i}{\Longleftarrow} N_i. \tag{3.20}$$

Then there exists a point $x \in \operatorname{int}|N_0|$, such that

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \in \operatorname{int}|N_i|, \quad i = 1, \ldots, k \tag{3.21}$$

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x \tag{3.22}$$

Obviously we cannot make any claim about the uniqueness of x in Theorem 3.6.

3.3 Covering relation on the plane with one nominally expanding direction ($u = 1$)

In this section we discuss the case, when $u = s = 1$, hence we have only one nominally expanding and one nominally contracting direction. The basic idea here is: the set N^- consists from two disjoint components and all possible values of the degree w in covering relation are ± 1. This allows to give sufficient conditions for an existence of covering relations, which are relatively easy to verify.

Definition 3.7 *Let N be a h-set, such that $u(N) = s(N) = 1$. We set*

$$\begin{aligned} N_c^{le} &= \{-1\} \times [-1, 1] \\ N_c^{re} &= \{1\} \times [-1, 1] \\ S(N)_c^l &= (-\infty, -1) \times \mathbb{R} \\ S(N)_c^r &= (1, \infty) \times \mathbb{R}. \end{aligned}$$

We define

$$N^{le} = c_N^{-1}(N_c^{le}), \quad N^{re} = c_N^{-1}(N_c^{re}),$$
$$S(N)^l = c_N^{-1}(S(N)^l), \quad S(N)^r = c_N^{-1}(S(N)^r).$$

We will call N^{le}, N^{re}, $S(N)^l$ and $S(N)^r$ the left edge, the right edge, the left side and right side of N, respectively.

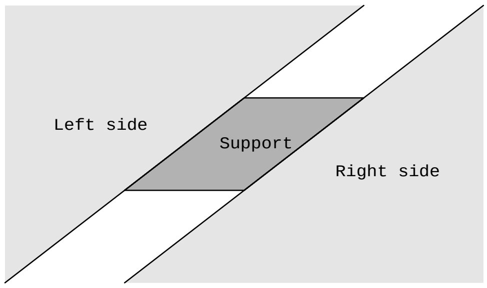

Figure 2: An example of h-set on the plane.

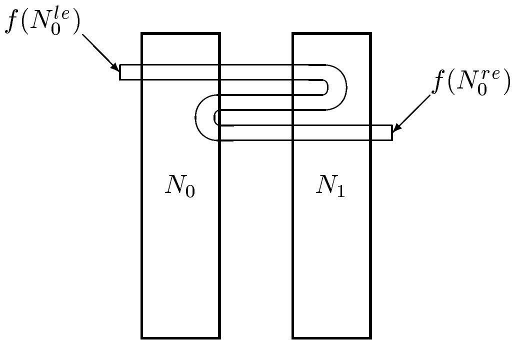

Figure 3: An example of an $f-$covering relation: $N_0 \Longrightarrow N_0, N_1$

It is easy to see that $N^- = N^{le} \cup N^{re}$.

The triple $(|N|, \overline{S(N)^l}, \overline{S(N)^r})$ is a t-set from [2]. As in [2] we will use the following notation for $S(N)^{r,l}$.

$$N^l = \overline{S(N)^l}, \qquad N^r = \overline{S(N)^r}$$

Remark 3.8 *For all h-sets used in this paper the support is a parallelogram. A usual picture of a h-set is given in Figure 2.*

A typical picture illustrating covering relation on the plane with one 'unstable' direction is given on Figure 3.

The following theorem was proved in [7] for any $n > 1$ and $u(N) = 1$. Here we rewrite it for the planar case in a slightly different notation (we use N^l and N^r for $S(N)^l$ and $S(M)^r$, respectively).

Theorem 3.9 *Let $n = 2$ and let N, M be two h-sets in $\mathbb{R}^n$, such that $u(N) = u(M) = 1$ and $s(N) = s(M) = 1$. Let $f : |N| \to \mathbb{R}^n$ be continuous.*

Assume that there exists $q_0 \in \overline{B}_s(0,1)$, such that following conditions are satisfied

$$
\begin{aligned}
f(c_N([-1,1] \times \{q_0\})) &\subset \operatorname{int}(M^l \cup |M| \cup M^r) \quad (3.23)\\
f(|N|) \cap M^+ &= \emptyset, \quad (3.24)
\end{aligned}
$$

and one of the following two conditions holds

$$
\begin{aligned}
f(N^{le}) \subset M^l \quad &\text{and} \quad f(N^{re}) \subset M^r \quad (3.25)\\
f(N^{le}) \subset M^r \quad &\text{and} \quad f(N^{re}) \subset M^l. \quad (3.26)
\end{aligned}
$$

Then there exists $w = \pm 1$, such that

$$N \overset{f,w}{\Longrightarrow} M$$

3.4 Representation of the h-sets

In this paper we use very simple h-sets, namely the support is a parallelogram.

A h-set is defined by specifying the triple $N = t(c, u, s)$, where $c, u, s \in \mathbb{R}^2$, such that u, s are linearly independent. We set

$$
\begin{aligned}
|N| &= \{x \in \mathbb{R}^2 \mid \exists_{t_1,t_2 \in [-1,1]} \quad x = c + t_1 u + t_2 s\}\\
&= c + [-1,1] \cdot u + [-1,1] \cdot s\\
c_N(t_1, t_2) &= c + t_1 u + t_2 s
\end{aligned}
$$

In this representation c is a center point of the parallelogram N, u represents an oriented half-length in the 'unstable' direction and s is an oriented half-length in the 'stable' direction. See Fig. 2 for an example of h-set in this representation.

We have

$$
\begin{aligned}
N^{le} &= c - u + [-1,1] \cdot s\\
N^{re} &= c + u + [-1,1] \cdot s\\
N^l &= c + (-\infty, -1]u + (-\infty, \infty)s\\
N^r &= c + [1, \infty)u + (-\infty, \infty)s.
\end{aligned}
$$

We introduce notions of top and bottom edges of N, N^{te} and N^{be} by

$$
\begin{aligned}
N^{be} &= c + [-1,1] \cdot u - s\\
N^{te} &= c + [-1,1] \cdot u + s
\end{aligned}
$$

Let us recall that the symmetry $R : \mathbb{R}^2 \to \mathbb{R}^2$ introduced in section 2.1 was given by

$$R(x_1, x_2) = (x_1, -x_2)$$

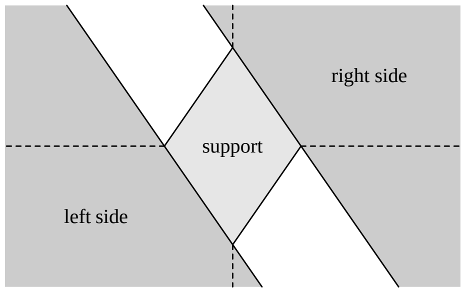

Figure 4: An example of an R-symmetric h-set

Definition 3.10 *A h-set, N, will be called* an R-symmetric h-set *if $N = t(c,u,s)$ for some $c,u,s \in \mathbb{R}^2$, such that*

$$\begin{aligned} R(c) &= c \\ R(u) = s \quad &or \quad R(u) = -s \end{aligned}$$

Figure 4 shows an example of a R-symmetric h-set. Symmetry properties of such h-set are apparent.

3.5 Action of R on h-sets

The symmetry of $P_{1/2,\pm}$ and $P_\pm$ expressed in (2.7) relates the maps and theirs inverse, hence beside mapping the support of N by R it will switch also the nominally stable and unstable directions. This motivates the following definition of the action of the symmetry R on h-sets

Definition 3.11 *Let N be a h-set. We define a h-set $R(N)$ as follows*

- $|R(N)| = R(|N|)$
- $u(R(N)) = s(N)$ *and* $s(R(N)) = u(N)$
- *the homeomorphism* $c_{R(N)} : \mathbb{R}^n \to \mathbb{R}^{u(R(N))} \times \mathbb{R}^{u(S(N))}$ *is given by*

$$c_{R(N)} = c_{N^T} \circ R^{-1}$$

∎

Observe that according to above definition we have

$$\begin{aligned} R(N)^- = R(N^+) &= R(N^{T,-}) \\ R(N)^+ = R(N^-) &= R(N^{T,+}) \\ R(t(c,u,s)) &= t(R(c),R(s),R(u)) \end{aligned} \tag{3.27}$$

As an immediate consequence of equation (3.27) we obtain

Lemma 3.12 *Let $N = t(c,u,s)$ be an R-symmetric h-set. Then $R(N) = N$.*

We have the following easy lemma

Lemma 3.13 *Let $f_i : dom(f_i) \subset \mathbb{R}^2 \to \mathbb{R}^2$ for $i = 1,2$ continuous and invertible on some open sets. Assume that*

$$\textit{if} \quad f_1(x) = x_1, \qquad \textit{then} \qquad f_2(Rx_1) = R(x). \tag{3.28}$$

Let N_1, N_2 be h-sets such that

$$N_1 \stackrel{f_1}{\Longrightarrow} N_2.$$

Then

$$R(N_2) \stackrel{f_2}{\Longleftarrow} R(N_1)$$

Proof: From Def. 3.11 and the assumed symmetry it follows immediately that

$$R(N_1)^T \stackrel{f_2^{-1}}{\Longrightarrow} R(N_2^T). \tag{3.29}$$

∎

From above Lemma and (2.8) we obtain

Corollary 3.14 *If $N_1 \stackrel{P_\pm}{\Longrightarrow} N_2$, then $R(N_2) \stackrel{P_\pm}{\Longleftarrow} R(N_1)$.*
If $N_1 \stackrel{P_{\frac{1}{2},+}}{\Longrightarrow} N_2$, then $R(N_2) \stackrel{P_{\frac{1}{2},-}}{\Longleftarrow} R(N_1)$.
If $N_1 \stackrel{P_{\frac{1}{2},-}}{\Longrightarrow} N_2$, then $R(N_2) \stackrel{P_{\frac{1}{2},+}}{\Longleftarrow} R(N_1)$.

4 C^1 tools

The goal of this section is to describe the tools which allow in the presence of hyperbolic fixed points for a map to prove an existence of homo- and heteroclinic trajectories.

In this section we recall the results from [6] with some additions (see also [19] where the method was outlined for the first time). In the symbol of covering relation we will drop the degree part, hence we will use $N \stackrel{f}{\Longrightarrow} M$ instead of $N \stackrel{f,w}{\Longrightarrow} M$ for some nonzero w.

4.1 General theorems

Let $P : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map. For any set X we define an interval matrix $[DP(X)] \subset \mathbb{R}^{n\times n}$ to be an interval enclosure of $DP(X)$ given by

$$M \in [DP(X)] \quad \text{iff} \quad \inf_{x\in X} DP(x)_{ij} \leq M_{ij} \leq \sup_{x\in X} DP(x)_{ij} \quad i,j = 1,2,\dots,n$$

Lemma 4.1 *Let N be a convex set. Assume $x_0, x_1 \in N$. Then*

$$P(x_1) - P(x_0) \in [DP(N)] \cdot (x_1 - x_0). \tag{4.30}$$

Moreover, there exists a matrix $M \in [DP(N)]$ such that

$$P(x_1) - P(x_0) = M \cdot (x_1 - x_0) \tag{4.31}$$

Proof:

$$P(x_1) - P(x_0) = \int_0^1 \frac{dP}{dt}(x_0 + t(x_1 - x_0))dt =$$
$$\int_0^1 \frac{\partial P}{\partial x}(x_0 + t(x_1 - x_0))dt \cdot (x_1 - x_0)$$

To finish the proof observe that

$$M = \int_0^1 \frac{\partial P}{\partial x}(x_0 + t(x_1 - x_0))dt \in [DP(N)]$$

∎

Let $Id : \mathbb{R}^n \to \mathbb{R}^n$ denote the identity map.

Theorem 4.2 *Let N be a convex set. Assume that*

$$0 \notin det([DP(N)] - Id) = \{t = det(M - Id) \mid M \in [DP(N)]\},$$

then N contains at most one fixed point of P.

Proof: Assume that $x_0, x_1 \in N$ are fixed points for P. Then from lemma 4.1 it follows that

$$x_1 - x_0 = P(x_1) - P(x_0) = M \cdot (x_1 - x_0) \tag{4.32}$$

for some matrix $M \in [DP(N)]$. Hence $x_1 - x_0$ is in the kernel of $M - Id$. From our assumption it follows that $\det(M - Id) \neq 0$, hence $x_1 = x_0$. ∎

Consider a two–dimensional function $f(x) = (f_1(x), f_2(x))^T$, where $x = (x_1, x_2)^T$. We assume that $f(0) = 0$, i.e. 0 is a fixed point of f. For a convex set U, such that $0 \in U$ we define intervals $\boldsymbol{\lambda}_1(U)$, $\boldsymbol{\varepsilon}_1(U)$, $\boldsymbol{\varepsilon}_2(U)$ and $\boldsymbol{\lambda}_2(U)$ by

$$Df(U) = \begin{pmatrix} \boldsymbol{\lambda}_1(U) & \boldsymbol{\varepsilon}_1(U) \\ \boldsymbol{\varepsilon}_2(U) & \boldsymbol{\lambda}_2(U) \end{pmatrix}. \tag{4.33}$$

Since $f(0) = 0$ then from Lemma 4.1 it follows that

$$f_1(x) \in \boldsymbol{\lambda}_1(U)x_1 + \boldsymbol{\varepsilon}_1(U)x_2$$
$$f_2(x) \in \boldsymbol{\varepsilon}_2(U)x_1 + \boldsymbol{\lambda}_2(U)x_2$$

Let

$$\varepsilon'_1(U) = \sup\{|\varepsilon| : \varepsilon \in \boldsymbol{\varepsilon}_1(U)\}, \qquad \varepsilon'_2(U) = \sup\{|\varepsilon| : \varepsilon \in \boldsymbol{\varepsilon}_2(U)\},$$
$$\lambda'_1(U) = \inf\{|\lambda_1| : \lambda_1 \in \boldsymbol{\lambda}_1(U)\}, \qquad \lambda'_2(U) = \sup\{|\lambda_2| : \lambda_2 \in \boldsymbol{\lambda}_2(U)\}.$$

Let us define the rectangle N_{α_1,α_2} by

$$N_{\alpha_1,\alpha_2} = [-\alpha_1, \alpha_1] \times [-\alpha_2, \alpha_2].$$

Definition 4.3 *[6, Def. 1] Let x_* be a fixed point for the map f. We say that f is* hyperbolic *on $N \ni x_*$, if there exists a local coordinate system on N, such that in this coordinate system*

$$x_* = 0 \tag{4.34}$$
$$\varepsilon_1'(N)\varepsilon_2'(N) < (1-\lambda_2'(N))(\lambda_1'(N)-1). \tag{4.35}$$
$$N = N_{\alpha_1,\alpha_2}, \tag{4.36}$$

where $\alpha_1 > 0$, $\alpha_2 > 0$ are such that the following conditions are satisfied

$$\frac{\varepsilon_1'(N)}{\lambda_1'(N)-1} < \frac{\alpha_1}{\alpha_2} < \frac{1-\lambda_2'(N)}{\epsilon_2'(N)}. \tag{4.37}$$

It is easy to see that that for the map f to be hyperbolic on N it is necessary that $\lambda_1' > 1$,$\lambda_2' < 1$ and the linearization of f at x_* is hyperbolic with one stable and unstable direction.

Theorem 4.4 *[6, Thm. 3] Assume that f is hyperbolic on N. Then*

1. *if $f^k(x) \in N$ for $k \geq 0$, then $\lim_{k\to\infty} f^k(x) = x_*$,*

2. *if $y_k \in N$ and $f(y_{k-1}) = y_k$ for $k \leq 0$, then $\lim_{k\to-\infty} y_k = x_*$.*

The next theorem shows how we can combine $\mathcal{C}^0$- and $\mathcal{C}^1$-tools to prove the existence of asymptotic orbits with prescribed itinerary.

Theorem 4.5 *[6, Thm. 4] Assume that g is hyperbolic on N_m and f hyperbolic on N_0. Let $x_g \in N_m$ be a fixed point for g and $x_f \in N_0$ be a fixed point for f.*

1. *If*

$$N_0 \overset{f_0}{\Longrightarrow} N_1 \overset{f_1}{\Longrightarrow} N_2 \overset{f_2}{\Longrightarrow} \dots \overset{f_{m-1}}{\Longrightarrow} N_m \overset{g}{\Longrightarrow} N_m \tag{4.38}$$

then there exists $x_0 \in N_0$ such that

$$\begin{aligned} f_{i-1}\circ f_{i-2}\circ\dots\circ f_0(x_0) &\in N_i \qquad \text{for} \quad i=1,\dots,m,\\ g^k\circ f_{m-1}\circ\dots\circ f_0(x_0) &\in N_m \qquad \text{for} \quad k>0,\\ \lim_{k\to\infty} g^k\circ f_{m-1}\circ\dots\circ f_0(x_0) &= x_g. \end{aligned}$$

2. *If*

$$N_0 \overset{f}{\Longrightarrow} N_0 \overset{f_0}{\Longrightarrow} N_1 \overset{f_1}{\Longrightarrow} N_2 \overset{f_2}{\Longrightarrow} \dots \overset{f_{m-1}}{\Longrightarrow} N_m \tag{4.39}$$

then there exists a sequence $(x_k)_{k=-\infty}^0$, $f(x_k) = x_{k+1}$ for $k < 0$ such that

$$\begin{aligned} x_k &\in N_0 \qquad \text{for} \quad k\leq 0,\\ f_{i-1}\circ f_{i-2}\circ\dots\circ f_0(x_0) &\in N_i \qquad \text{for} \quad i=1,\dots,m,\\ \lim_{k\to-\infty} x_k &= x_f. \end{aligned}$$

3. *If*

$$N_0 \overset{f}{\Longrightarrow} N_0 \overset{f_0}{\Longrightarrow} N_1 \overset{f_1}{\Longrightarrow} N_2 \overset{f_2}{\Longrightarrow} \dots \overset{f_{m-1}}{\Longrightarrow} N_m \overset{g}{\Longrightarrow} N_m \tag{4.40}$$

then there exists a sequence $(x_k)_{k=-\infty}^{0}$, $f(x_k) = x_{k+1}$ *for* $k < 0$ *such that*

$$\begin{aligned} x_k \in N_0, \quad k \leq 0, & \\ f_{i-1} \circ f_{i-2} \circ \cdots \circ f_0(x_0) \in N_i \qquad & \textit{for} \quad i = 1, \dots, m, \\ g^n \circ f_{m-1} \circ \cdots \circ f_0(x_0) \in N_m \qquad & \textit{for} \quad n > 0, \\ \lim_{k \to -\infty} x_k = x_f, & \\ \lim_{k \to \infty} g^k \circ f_{m-1} \circ \cdots \circ f_0(x_0) = x_g. & \end{aligned}$$

The above theorem can be used without any modifications for proving the existence of trajectories converging to periodic orbits. In this case we consider higher iterates of maps f and g in (4.38), (4.39) and (4.40).

4.2 How to prove an existence of an heteroclinic orbit, fuzzy sets.

To prove an existence of an heteroclinic orbit we want to use the third assertion in Theorem 4.5 for $g = f$, but in order to make the exposition easier to follow we use two different maps f and g. Observe that to apply this theorem directly one needs to know an exact location of two fixed points $x_f \in N_0$ and $x_g \in N_m$, because the sets N_0 and N_m are centered on x_f and x_g respectively. But exact coordinates of x_f and x_g are usually unknown. We overcome this obstacle in three steps as follows

1. Finding very good estimates for x_f and x_g. In this paper we use an argument based on symmetry to obtain tight bounds for x_f and x_g. In [6] a rigorous interval Newton algorithm was used. Let us denote by M_f and M_g obtained estimates for x_f and x_g, respectively.

We choose one fixed point $x_f \in M_f$ and $x_g \in M_g$ for further considerations.

2. C^1-computations, hyperbolicity We choose a set U_f, $M_f \subset U_f$, on which we compute rigorously $[Df(U_f)]$. Then we have to choose a coordinate system, in which the matrix $[Df(U_f)]$ will be as close as possible to the diagonal one. In this paper we have chosen numerically obtained stable and unstable eigenvectors. Let us denote these eigenvectors by u and s, where u corresponds to unstable direction and s is pointing in the stable direction. Assume that this process gives us a coordinate frame in which

$$\varepsilon_1'(U_f)\varepsilon_2'(U_f) < (1 - \lambda_2'(U_f))(\lambda_1'(U_f) - 1). \tag{4.41}$$

From (4.41) it follows easily that there exists $\alpha_1 > 0$, $\alpha_2 > 0$ such that

$$\frac{\varepsilon_1'(U_f)}{\lambda_1'(U_f) - 1} < \frac{\alpha_1}{\alpha_2} < \frac{1 - \lambda_2'(U_f)}{\epsilon_2'(U_f)}, . \tag{4.42}$$

Observe that above inequality specifies only the ratio α_1/α_2, hence we can find a pair (α_1, α_2) such that condition (4.42) and the following condition holds

$$M_f + \alpha_1 \cdot [-1,1] \cdot u + \alpha_2 \cdot [-1,1] \cdot s \subset U_f \tag{4.43}$$

We now define a h-set N_0 by

$$N_0 = t(x_f, \alpha_1 u, \alpha_2 s). \tag{4.44}$$

Obviously f is hyperbolic on N_0. Observe that the hyperbolicity implies uniqueness of x_f in N_0.

We do similar construction for g to obtain $N_m = t(x_g, \beta_1 \bar{u}, \beta_2 \bar{s})$.

3. Covering relations for fuzzy h-sets. We have to verify the following covering relations

$$N_0 \overset{f}{\Longrightarrow} N_0 \overset{f}{\Longrightarrow} N_1 \tag{4.45}$$

$$N_{m-1} \overset{f_{m-1}}{\Longrightarrow} N_m \overset{g}{\Longrightarrow} N_m. \tag{4.46}$$

As was mentioned above we don't know the h-sets N_0, N_m explicitly, but we know that

$$N_0 \in \widetilde{N}_0 = \{t(c, \alpha_1 u, \alpha_2 s) \mid c \in M_f\}$$
$$N_m \in \widetilde{N}_m = \{t(c, \beta_1 \bar{u}, \beta_2 \bar{s}) \mid c \in M_g\}.$$

Above equations define *a fuzzy h-set*, as a collection of h-sets. We can now extend the definition of covering relations to fuzzy h-sets as follows.

Definition 4.6 *Let f be a continuous map on the plane. Assume $\widetilde{N}_1$, $\widetilde{N}_2$ are fuzzy h-sets (collections of h-set) and R is a h-set.*

- *we say that $\widetilde{N}_1 \overset{f}{\Longrightarrow} R$ iff $M \overset{f}{\Longrightarrow} R$ for all $M \in \widetilde{N}_1$.*
- *we say that $R \overset{f}{\Longrightarrow} \widetilde{N}_1$ iff $R \overset{f}{\Longrightarrow} M$ for all $M \in \widetilde{N}_1$.*
- *we say that $\widetilde{N}_1 \overset{f}{\Longrightarrow} \widetilde{N}_2$ iff $M_1 \overset{f}{\Longrightarrow} M_2$ for all $M_1 \in \widetilde{N}_1$ and $M_2 \in \widetilde{N}_2$.*

With the above definition is obvious that to prove the covering relations in equations (4.45) and (4.46) it is enough to show that

$$\widetilde{N}_0 \overset{f}{\Longrightarrow} \widetilde{N}_0 \overset{f}{\Longrightarrow} N_1 \tag{4.47}$$

$$N_{m-1} \overset{f_{m-1}}{\Longrightarrow} \widetilde{N}_m \overset{g}{\Longrightarrow} \widetilde{N}_m. \tag{4.48}$$

In practice (in rigorous numerical computations) it is convenient to think about a fuzzy h-set $\widetilde{N}$ as an parallelogram with thickened edges, hence all tools developed to verify covering relations for h-sets can be easily extended to fuzzy h-sets.

5 The Lyapunov orbits.

In this section we present a computer assisted proof of an existence and hyperbolicity of the Lyapunov orbits around libration points. Hence we realize here step 1 and 2 from section 4.2 on our way to the proof of an existence of heteroclinic connection.

As in previous section in the symbol of covering relation we will drop the degree part, hence we will use $N \overset{f}{\Longrightarrow} M$ instead of $N \overset{f,w}{\Longrightarrow} M$ for some nonzero w.

5.1 Existence of Lyapunov orbits.

Theorem 5.1 *Let* $x_1 = 0.92080349132074001\,96$, $x_2 = 1.081929486841799903$.

- *There exists a fixed point* $L_1^* = (x_1^*, 0) \in \Theta_+$ *for* P_+, *such that*

$$|x_1^* - x_1| < \eta_1 = 6 \cdot 10^{-14} \tag{5.49}$$

- *There exists a fixed point* $L_2^* = (x_2^*, 0) \in \Theta_-$ *for* P_- *such that*

$$|x_2^* - x_2| < \eta_2 = 10^{-13} \tag{5.50}$$

Proof: We consider two intervals $I_1 := [x_1 - \eta_1, x_1 + \eta_1] \times \{0\} \subset \Theta_+$, $I_2 := [x_2 - \eta_2, x_2 + \eta_2] \times \{0\} \subset \Theta_-$. The location of x_i is schematically shown on Figure 5.

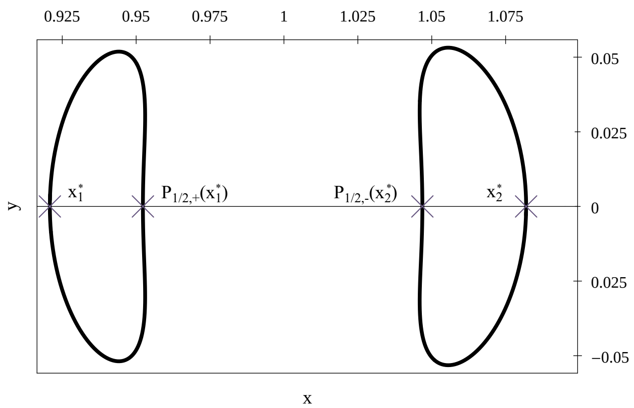

Figure 5: The Lyapunov orbits and the location of x_i^*.

Let us recall, that by $\pi_{\dot{x}} : \Theta \longrightarrow \mathbb{R}$ we denote the projection onto $\dot{x}$ coordinate. With a computer assistance we proved the following

Lemma 5.2 *The maps $P_{\frac{1}{2},+} : I_1 \longrightarrow \Theta_-$ and $P_{\frac{1}{2},-} : I_2 \longrightarrow \Theta_+$ are well defined and continuous. Moreover we have the following properties*

$$\pi_{\dot{x}}(P_{\frac{1}{2},+}(x_1 - \eta_1, 0)) < 0, \qquad \pi_{\dot{x}}(P_{\frac{1}{2},+}(x_1 + \eta_1, 0)) > 0 \tag{5.51}$$

$$\pi_{\dot{x}}(P_{\frac{1}{2},-}(x_2 - \eta_2, 0)) < 0, \qquad \pi_{\dot{x}}(P_{\frac{1}{2},-}(x_2 + \eta_2, 0)) > 0 \tag{5.52}$$

Figures 6 and 7 display rigorous enclosures for $P_{\frac{1}{2},+}(x_1 \pm \eta_1, 0)$ and $P_{\frac{1}{2},-}(x_2 \pm \eta_2, 0)$, respectively.

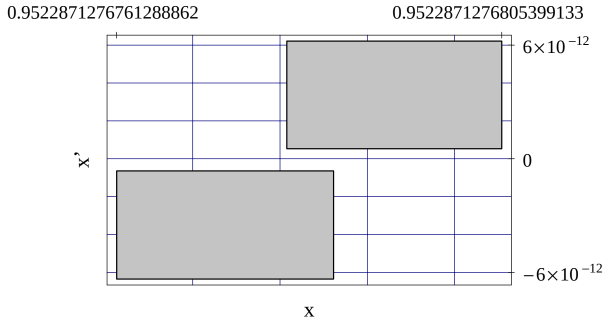

Figure 6: Rigorous enclosure of $P_{\frac{1}{2},+}(x_1 - \eta_1, 0)$ (a box in lower left corner) and $P_{\frac{1}{2},+}(x_1 + \eta_1, 0)$ (a box in upper right corner)

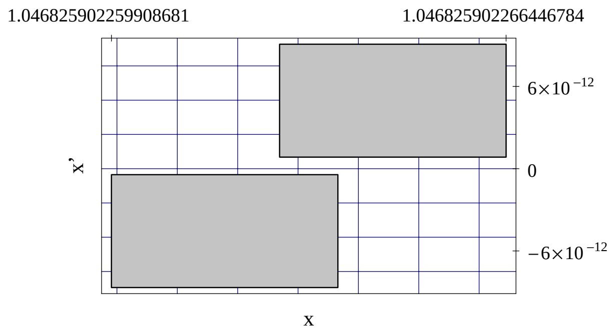

Figure 7: Rigorous enclosure of $P_{\frac{1}{2},-}(x_2 - \eta_2, 0)$ (a box in lower left corner) and $P_{\frac{1}{2},-}(x_2 + \eta_2, 0)$ (a box in upper right corner)

Now we are ready to finish the proof of Theorem 5.1. From Lemma 5.2 and the Darboux property it follows that there exist points $x_1^* \in \mathrm{int}(I_1)$ and

$x_2^* \in \mathrm{int}(I_2)$, such that

$$P_{\frac{1}{2},+}(x_1^*,0) = (x_1^0,0) \tag{5.53}$$

$$P_{\frac{1}{2},-}(x_2^*,0) = (x_2^0,0). \tag{5.54}$$

An application of symmetry properties of $P_{1/2,\pm}$ (see eq. (2.7)) gives

$$P_+(x_1^*,0) = (x_1^*,0) \tag{5.55}$$

$$P_-(x_2^*,0) = (x_2^*,0). \tag{5.56}$$

∎

5.2 Hyperbolicity in the neighborhood of Lyapunov orbits.

The goal of this section is to prove that P is hyperbolic in the sense of Definition 4.3 in some neighborhood of points L_1^* and L_2^*.

Let us define

$$u_1 = (1, 2.5733011), \quad s_1 = (-1, 2.5733011),$$
$$u_2 = (1, 2.2817915), \quad s_2 = (-1, 2.2817915).$$

These vectors appear to be a good approximation for unstable (u_i) and stable eigenvectors (s_i) at L_i^* on the $(x,\dot{x})$-plane. Observe that $R(u_i) = -s_i$, this is in agreement with symmetry of $P_\pm$ stated in equation (2.7). We will also use (u_i, s_i) later, as the coordinate directions for good coordinate frame in the proof of hyperbolicity of P_+ and P_- in the neighborhood of L_i^*.

Let $H_i^1 = t(h_i, u_i^1, s_i^1)$ and $H_i^2 = t(h_i, u_i^2, s_i^2)$ for $i = 1,2$ denote h-sets on the $(x,\dot{x})$ plane, where

$$\begin{array}{ll} h_1 = (x_1, 0), & h_2 = (x_2, 0) \\ \alpha_1 = 3 \cdot 10^{-10}, & \alpha_2 = 4 \cdot 10^{-10} \\ u_1^1 = \alpha_1 u_1, & u_2^1 = \alpha_2 u_2 \\ s_1^1 = \alpha_1 s_1, & s_2^1 = \alpha_2 s_2 \\ u_1^2 = 2 \cdot 10^{-7} u_1, & u_2^2 = 1.2 \cdot 10^{-8} u_2, \\ s_1^2 = 2 \cdot 10^{-7} s_1, & s_2^2 = 2.8 \cdot 10^{-7} s_2 \end{array} \tag{5.57}$$

We assume that $H_1^1, H_1^2 \subset \Theta_+$ and $H_2^1, H_2^2 \subset \Theta_-$. Observe that $I_1 \subset H_1^1 \subset H_1^2$ and $I_2 \subset H_2^1 \subset H_2^2$, where sets I_i were defined in the proof of Theorem 5.1. Let

$$W_i = [-\eta_i, \eta_i] \times \{0\}, \quad i = 1,2 \tag{5.58}$$

where η_i where defined in Theorem 5.1. Let U_i, for $i = 1,2$ be given by

$$U_i = H_i^1 + W_i = \{(x+p, \dot{x}) \ : \ (x,\dot{x}) \in H_i^1, (p,0) \in W_i\} \tag{5.59}$$

The choices made in (5.57) are motivated by the following considerations: since we want exploit hyperbolicity of P around L_i^* it is desirable to choose stable and unstable directions as s_i and u_i. Sets H_i^1 (in fact U_i) will be used to establish hyperbolicity around L_i^*, hence it desirable to choose them very small, as we need to perform a costly rigorous computation of DP on U_i (a C^1-computation). Sets H_i^2 are used as a link in the chain of covering relations between small H_i^1 and relatively large sets N_i defined later in Section 6. Since the unstable eigenvalue is bigger than 10^3 (see proof of Lemma 5.5), we can choose H_i^2 to be of three order of magnitude larger than H_i^1 and still have a covering relation between them.

The following lemma was proved with a computer assistance

Lemma 5.3 *The maps $P_+ : U_1 \to \Theta_+$ and $P_- : U_2 \to \Theta_-$ are well defined. Moreover we have*

$$[DP_+(U_1)] \subset \begin{pmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_1 \end{pmatrix}, \qquad [DP_-(U_2)] \subset \begin{pmatrix} \mathbf{A}_2 & \mathbf{B}_2 \\ \mathbf{C}_2 & \mathbf{D}_2 \end{pmatrix} \tag{5.60}$$

where intervals $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$, $\mathbf{D}_i$ are given below

$$\begin{array}{lcllcl}
\mathbf{A}_1 & = & [695.659, 696.1085] & \mathbf{B}_1 & = & [270.3511, 270.4973] \\
\mathbf{C}_1 & = & [1789.9112, 1791.46231] & \mathbf{D}_1 & = & [695.61982, 696.12441] \\
\mathbf{A}_2 & = & [573.3983, 573.835] & \mathbf{B}_2 & = & [251.3098, 251.4675] \\
\mathbf{C}_2 & = & [1308.1679, 1309.5201] & \mathbf{D}_2 & = & [573.3613, 573.848]
\end{array}$$

Using the above lemma and symmetry R we can now prove the following

Lemma 5.4 *There exists exactly one fixed point $L_1^* = (x_1^*, 0) \in U_1$ for P_+. Moreover we have $|x_1^* - x_1| < \eta_1$.*

There exists exactly one fixed point $L_2^* = (x_2^*, 0) \in U_2$ for P_-. Moreover we have $|x_2^* - x_2| < \eta_2$.

Proof: We write down the proof for L_1^*, only. The proof for L_2^* is analogous.

An easy computation shows that

$$\det([DP_+(U_1)] - Id) < 0$$

hence from Theorem 4.2 it follows that there exists at most one fixed point for P_+ in U_1. Since $I_1 \subset U_1$ then we know from Theorem 5.1 that one such fixed point $L_1^* = (x_1^*, 0) \in I_1$ exists. The estimate for $|x_1^* - x_1|$ was also given in Theorem 5.1. ∎

Lemma 5.5 *There exist R-symmetric h-sets H_1 and H_2, such that $|H_1| \subset U_1$, $|H_2| \subset U_2$,t $L_1^* \in H_1$ and $L_2^* \in H_2$ and the following conditions hold*

1. *P_+ is hyperbolic on $|H_1|$*
2. *P_- is hyperbolic on $|H_2|$*

Proof: We will proceed as it was outlined in step 2 in section 4.2. First we need to find a coordinate frame (via an affine transformation) in which the inequality (4.41) is satisfied for ($f = P_+, U_f = U_1$) and ($f = P_-, U_f = U_2$). Form Lemma 5.3 it follows that P_+ is defined on U_1 and P_- is defined on U_2.

Observe that the transformation of $[DP_+(U_1)]$ ($[DP_-(U_2)]$) to new coordinates does not depend on the exact location L_1^* (L_2^*). In new coordinates $L_1^* = L_2^* = 0$, but we have to choose the coordinate directions in U_1 and U_2. It turns out that the vectors (u_i, s_i) which were used in the definition of H_i^1 are good for this purpose, as they are reasonably good approximations of unstable and stable directions of corresponding Poincaré map.

A short computation shows that in new coordinates we obtain

$$[DP_+(U_1)] \subset \begin{pmatrix} \boldsymbol{\lambda}_{1,1} & \boldsymbol{\varepsilon}_{1,1} \\ \boldsymbol{\varepsilon}_{1,2} & \boldsymbol{\lambda}_{1,2} \end{pmatrix}, \qquad [DP_-(U_2)] \subset \begin{pmatrix} \boldsymbol{\lambda}_{2,1} & \boldsymbol{\varepsilon}_{2,1} \\ \boldsymbol{\varepsilon}_{2,2} & \boldsymbol{\lambda}_{2,2} \end{pmatrix} \tag{5.61}$$

where

$$\begin{array}{lllllll} \boldsymbol{\lambda}_{1,1} & = & [1391.271, 1392.239] & \boldsymbol{\lambda}_{1,2} & = & [-0.482, 0.485] \\ \boldsymbol{\varepsilon}_{1,1} & = & [-0.494, 0.472] & \boldsymbol{\varepsilon}_{1,2} & = & [-0.483, 0.484] \\ \boldsymbol{\lambda}_{2,1} & = & [1146.751, 1147.69] & \boldsymbol{\lambda}_{2,2} & = & [-0.468, 0.47] \\ \boldsymbol{\varepsilon}_{2,1} & = & [-0.481, 0.457] & \boldsymbol{\varepsilon}_{2,2} & = & [-0.468, 0.47]. \end{array}$$

It is clear that $\lambda_{i,2} < 1 < \lambda_{i,1}$ and $\varepsilon_{i,1}\varepsilon_{i,2} < (1 - \lambda_{i,2})(\lambda_{i,1} - 1)$. Moreover

$$\frac{\varepsilon_{1,1}}{\lambda_{1,1} - 1} < 1 < \frac{1 - \lambda_{1,2}}{\varepsilon_{1,2}} \tag{5.62}$$

$$\frac{\varepsilon_{2,1}}{\lambda_{2,1} - 1} < 1 < \frac{1 - \lambda_{2,2}}{\varepsilon_{2,2}} \tag{5.63}$$

We define H_i for $i = 1, 2$ as follows

$$H_i = t(L_i^*, \alpha_i u_1, \alpha_i s_i), \tag{5.64}$$

where α_i were given in (5.57). Observe that by construction

$$|H_i| \subset |H_i^1| \subset U_i. \tag{5.65}$$

This shows that P_+ is hyperbolic on $|H_1|$ and P_- is hyperbolic on $|H_2|$. ∎

With a computer assistance we proved the following lemma

Lemma 5.6 *Let H_1 and H_2 be the h-sets obtained in Lemma 5.5, then*

- $H_1 \stackrel{P_+}{\Longrightarrow} H_1 \stackrel{P_+}{\Longrightarrow} H_1^2$
- $H_2^2 \stackrel{P_-}{\Longrightarrow} H_2 \stackrel{P_-}{\Longrightarrow} H_2$

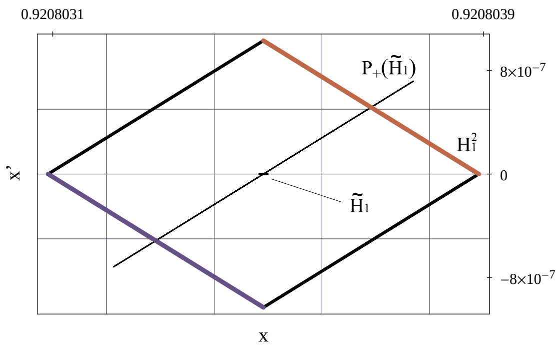

Figure 8: The Set H_1^2 (the large parallelogram), the fuzzy set $\tilde{H}_1$ (a small set in the center) and $P_+(\tilde{H}_1)$ (the nearly diagonal segment across H_1^2) illustrating covering relation: $\tilde{H}_1 \overset{P_+}{\Longrightarrow} \tilde{H}_1 \overset{P_+}{\Longrightarrow} H_1^2$. Vertical edges (when in color: red and blue) are marked by a bold line.

Proof: Consider the following fuzzy h-sets

$$\widetilde{H_i} = t(W_i, \alpha_i u_i, \alpha_i s_i), \tag{5.66}$$

where W_i are defined by equation (5.58). We assume that $|\widetilde{H_1}| \subset \Theta_+$ and $|\widetilde{H_2}| \subset \Theta_-$.

Observe that $H_i \in \widetilde{H_i}$. The fuzzy sets $\tilde{H}_i$ reflect out lack of knowledge of exact coordinates of L_i^*.

The following covering relations were established with a computer assistance (see Fig. 8)

$$\widetilde{H_1} \overset{P_+}{\Longrightarrow} \widetilde{H_1} \overset{P_+}{\Longrightarrow} H_1^2 \tag{5.67}$$

$$H_2^2 \overset{P_-}{\Longrightarrow} \widetilde{H_2} \overset{P_-}{\Longrightarrow} \widetilde{H_2}. \tag{5.68}$$

The assertion of the lemma follows now immediately from Def. 4.6. ∎

6 An existence of homo- and heteroclinic connections for Lyapunov orbits.

In this section we prove with a computer assistance Theorem 1.1. During the proof we define h-sets which will be used later in the construction of symbolic dynamics in the proof of Theorem 1.2.

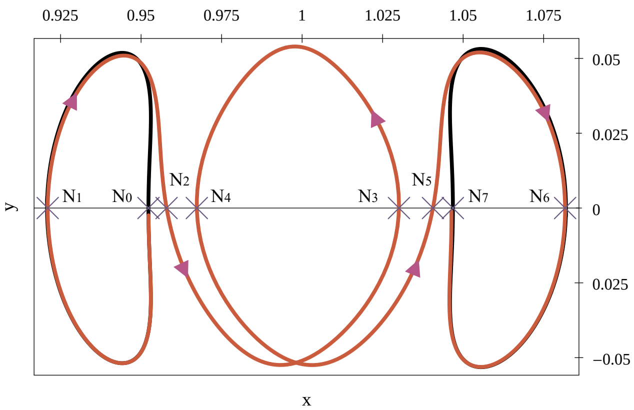

Figure 9: The location of sets N_i along a heteroclinic orbit in (x, y)-coordinates.

6.1 An existence of heteroclinic connection between Lyapunov orbits.

In order to prove an existence of heteroclinic connection between L_1^* and L_2^* we need to find a chain of covering relations which starts close to L_1^* (begins with H_1^2) and ends close to L_2^* (with H_2^2). For this sake we choose the sets N_i along a numerically constructed, (nonrigorous), heteroclinic orbit in the vicinity of the intersection of such orbit with the section Θ (see Fig. 9). Let $N_i = t(X_i, u_i, s_i)$ be h-sets, where

$$
\begin{aligned}
X_0 &= (0.9522928423486199945, 1.23 \cdot 10^{-5}) \\
X_1 &= (0.921005737890425169, 0.000520593281764688371 4) \\
X_2 &= (0.957916338594066441, 0.0219149736647649452 7) \\
X_3 &= (1.030069865952822683, 0.00330658676251664686) \\
X_4 &= (0.967306682018305608, 0.00370323016503655046 2) \\
X_5 &= (1.040628850444842879, 0.0231706345529880640 4) \\
X_6 &= (1.081670357450509545, 0.00059182264901723794 21) \\
X_7 &= (1.046819673646057103, 2.13365065043902489 \cdot 10^{-5})
\end{aligned}
$$

and

$$
\begin{aligned}
s_0 &= (-4 \cdot 10^{-6}, 1.45 \cdot 10^{-5}) & u_0 &= -R(s_0)/10 \\
s_1 &= (-4.5 \cdot 10^{-7}, \tfrac{7}{6} \cdot 10^{-6}) & u_1 &= -R(s_1)/10 \\
s_2 &= (-1.2 \cdot 10^{-7}, 2.92 \cdot 10^{-7}) & u_2 &= -R(s_2) \\
s_3 &= (-1.05 \cdot 10^{-7}, 2.92 \cdot 10^{-7}) & u_3 &= -R(s_3) \\
s_4 &= (-1 \cdot 10^{-7}, 2.9 \cdot 10^{-7}) & u_4 &= -R(s_4)/2 \\
s_5 &= (-1.44 \cdot 10^{-7}, 5.8 \cdot 10^{-7}) & u_5 &= -R(s_5)/6 \\
s_6 &= (-1.625 \cdot 10^{-7}, 3.75 \cdot 10^{-7}) & u_6 &= -R(s_6)/2 \\
s_7 &= (-8.3 \cdot 10^{-7}, 2.9 \cdot 10^{-6}) & u_7 &= -R(s_7)/5
\end{aligned}
$$

Vectors s_i were chosen to be a good approximation of the stable direction at X_i. Vectors u_i were chosen as symmetric image of s_i, but usually with different length.

Remark 6.1 *For our computations to succeed vectors u_i might have been chosen quite arbitrarily (but not too close to s_i). For example, numerical explorations show that if we choose $u_i = \alpha R(s_i)$ then we can reduce time of rigorous computation (Lemma 6.2) around 3 times in comparison to u_i parallel to x-axis.*

We assume that $N_0, N_2, N_4, N_6 \subset \Theta_-$ and $N_1, N_3, N_5, N_7 \subset \Theta_+$. With a computer assistance we proved

Lemma 6.2 *The maps*

$$P_{\frac{1}{2},+} : H_1^2 \cup N_1 \cup N_3 \cup N_5 \cup N_7 \longrightarrow \Theta_-$$

and

$$P_{\frac{1}{2},-} : N_0 \cup N_2 \cup N_4 \cup N_6 \longrightarrow \Theta_+$$

are well defined and continuous. Moreover, we have the following chain of covering relations

$$\begin{array}{l} H_1^2 \overset{P_{1/2,+}}{\Longrightarrow} N_0 \overset{P_{1/2,-}}{\Longrightarrow} N_1 \overset{P_{1/2,+}}{\Longrightarrow} N_2 \overset{P_{1/2,-}}{\Longrightarrow} N_3 \overset{P_{1/2,+}}{\Longrightarrow} N_4 \\ \qquad \overset{P_{1/2,-}}{\Longrightarrow} N_5 \overset{P_{1/2,+}}{\Longrightarrow} N_6 \overset{P_{1/2,-}}{\Longrightarrow} N_7 \overset{P_{1/2,+}}{\Longrightarrow} H_2^2 \end{array}$$

Figure 10 illustrates the chain of covering relations from Lemma 6.2 obtained by a nonrigorous procedure.

Now we are ready to prove the part of Theorem 1.1 concerning an existence of heteroclinic connections.

Theorem 6.3 *For PCR3BP with $C = 3.03$, $\mu = 0.0009537$ there exist two periodic solutions in the Jupiter region,L_1^* and L_2^* , called Lyapunov orbits, and there exists heteroclinic connections between them, in both directions.*

Proof: We prove only an existence of the connection from L_1^* to L_2^*. An existence of connection in the opposite direction is obtained by symmetry R.

¿From Lemmas 5.6 and 6.2 it follows that there exists the following chain of covering relations

$$\begin{array}{l} H_1 \overset{P_+}{\Longrightarrow} H_1 \overset{P_+}{\Longrightarrow} H_1^2 \overset{P_{1/2,+}}{\Longrightarrow} N_0 \overset{P_{1/2,-}}{\Longrightarrow} N_1 \overset{P_{1/2,+}}{\Longrightarrow} N_2 \overset{P_{1/2,-}}{\Longrightarrow} N_3 \overset{P_{1/2,+}}{\Longrightarrow} N_4 \\ \qquad \overset{P_{1/2,-}}{\Longrightarrow} N_5 \overset{P_{1/2,+}}{\Longrightarrow} N_6 \overset{P_{1/2,-}}{\Longrightarrow} N_7 \overset{P_{1/2,+}}{\Longrightarrow} H_2^2 \overset{P_-}{\Longrightarrow} H_2 \overset{P_-}{\Longrightarrow} H_2 \end{array}$$

The assertion follows now from Lemma 5.5 and Theorem 4.5. ∎

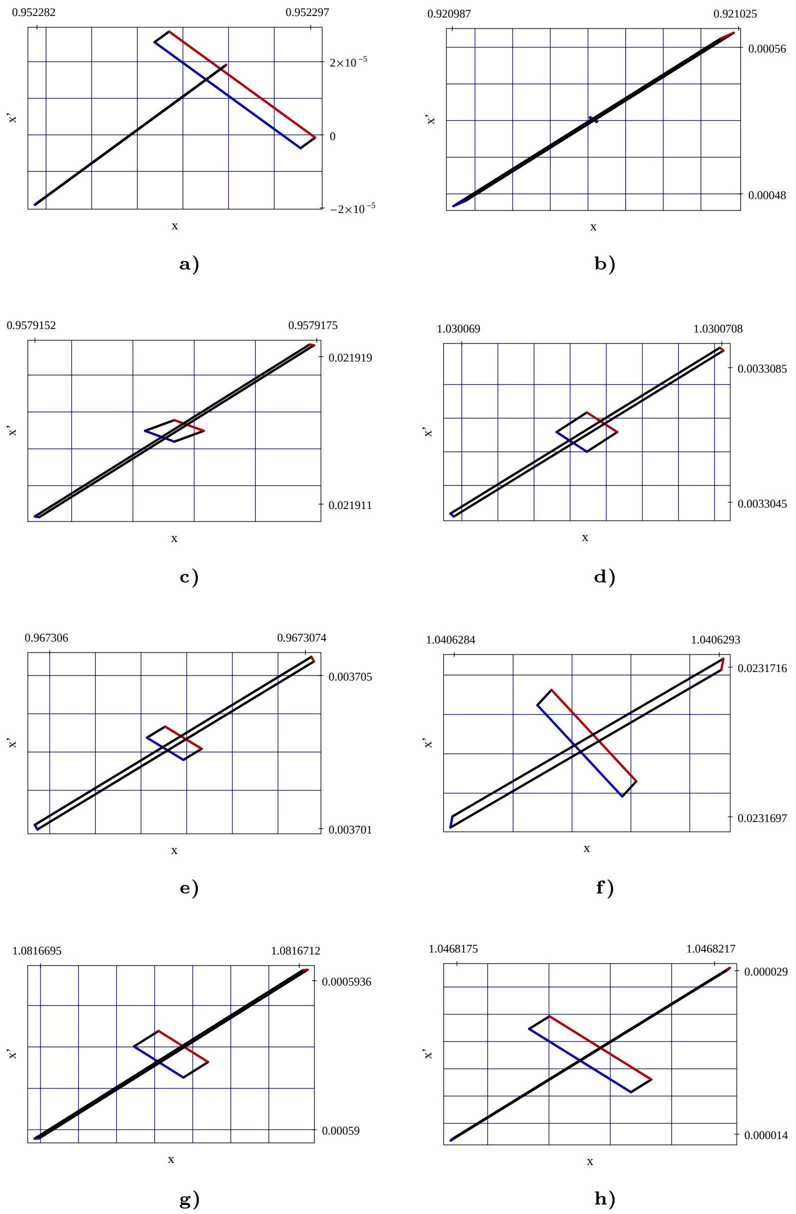

Figure 10: A chain of covering relations. a) $H_1^2 \overset{P_{1/2,+}}{\Longrightarrow} N_0$, b) $N_0 \overset{P_{1/2,-}}{\Longrightarrow} N_1$, c) $N_1 \overset{P_{1/2,+}}{\Longrightarrow} N_2$, d) $N_2 \overset{P_{1/2,-}}{\Longrightarrow} N_3$, e) $N_3 \overset{P_{1/2,+}}{\Longrightarrow} N_4$, f) $N_4 \overset{P_{1/2,-}}{\Longrightarrow} N_5$, g) $N_5 \overset{P_{1/2,+}}{\Longrightarrow} N_6$, h) $N_6 \overset{P_{1/2,-}}{\Longrightarrow} N_7$. These pictures are'nt produced by a rigorous procedure, as we checked the covering relations by less direct approach to reduce the computation time - see section 8 for details

6.2 Homoclinic connection in an exterior region.

In this section we establish an existence of an orbit homoclinic to L_2^* (see Figure 11). For this end we find a chain of covering relations, which starts close to L_2^* passes through the sets located in the exterior region and then ends close to L_2^*. For this sake we choose the sets E_i along a numerically constructed, (nonrigorous), homoclinic orbit in the vicinity of the intersection of such orbit with the section Θ (see Fig. 11).

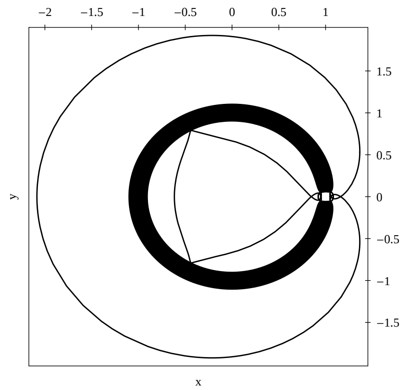

Figure 11: Homoclinic orbits to L_1^* and L_2^* Lyapunov orbits.

We define the following h-sets $E_i = t(Y_i, u_i, s_i)$, where

$$
\begin{aligned}
Y_0 &= (-2.08509704964865536, 0),\\
Y_1 &= (1.160261327316386816, -0.1812035059427922688),\\
Y_2 &= (1.059527808809695232, -0.03871458787165545984),\\
Y_3 &= (1.082284499686768768, -0.0008090412116073312256),\\
Y_4 &= (1.046834433386131072, -0.00002957990840481726976),\\
Y_5 &= (1.081929798158888576, -0.0000007068412578518833152)
\end{aligned}
$$

and

$$
\begin{aligned}
s_0 &= (-1\cdot 10^{-7}, 3\cdot 10^{-8}) & u_0 &= -R(s_0)\\
s_1 &= (1\cdot 10^{-7}, 8\cdot 10^{-8}) & u_1 &= -4R(s_1)\\
s_2 &= (-3\cdot 10^{-7}, 81\cdot 10^{-8}) & u_2 &= -R(s_2)/10\\
s_3 &= (-1\cdot 10^{-7}, 23\cdot 10^{-8}) & u_3 &= -R(s_3)/4\\
s_4 &= (-1\cdot 10^{-7}, 35\cdot 10^{-8}) & u_4 &= -R(s_4)/4\\
s_5 &= (-1\cdot 10^{-8}, 22817915\cdot 10^{-15}) & u_5 &= -R(s_5)/2
\end{aligned}
$$

We assume, that $E_0, E_2, E_4 \subset \Theta_+$ and $E_1, E_3, E_5 \subset \Theta_-$.

With a computer assistance we proved the following

Lemma 6.4 *The maps*

$$\begin{aligned} P_{\frac{1}{2},+} &: E_0 \cup E_2 \cup E_4 \to \Theta_- \\ P_{\frac{1}{2},-} &: E_1 \cup E_3 \to \Theta_+ \\ P_- &: E_5 \to \Theta_i \end{aligned}$$

are well defined and continuous. Moreover, we have the following chain of covering relations

$$E_0 \overset{P_{1/2,+}}{\Longrightarrow} E_1 \overset{P_{1/2,-}}{\Longrightarrow} E_2 \overset{P_{1/2,+}}{\Longrightarrow} E_3 \overset{P_{1/2,-}}{\Longrightarrow} E_4 \overset{P_{1/2,+}}{\Longrightarrow} E_5 \overset{P_-}{\Longrightarrow} H_2^2$$

∎

We are now ready to state the basic theorem in this section.

Theorem 6.5 *For PCR3BP with $C = 3.03$, $\mu = 0.0009537$ there exists a an orbit homoclinic to L_2^*.*

Proof: From Lemmas 6.4 and 5.6 it follows that

$$E_0 \overset{P_{1/2,+}}{\Longrightarrow} E_1 \overset{P_{1/2,-}}{\Longrightarrow} E_2 \overset{P_{1/2,+}}{\Longrightarrow} E_3 \overset{P_{1/2,-}}{\Longrightarrow} E_4 \overset{P_{1/2,+}}{\Longrightarrow} E_5 \overset{P_-}{\Longrightarrow} H_2^2 \overset{P_-}{\Longrightarrow} H_2. \tag{6.69}$$

Observe that from the definition of E_0 it follows that E_0 is R-symmetric. From Corollary 3.14, Lemma 5.6 and equation (6.69) we obtain

$$\begin{aligned} H_2 = R(H_2) \overset{P_-}{\Longleftarrow} R(H_2^2) \overset{P_-}{\Longleftarrow} R(E_5) \overset{P_{1/2,-}}{\Longleftarrow} R(E_4) \overset{P_{1/2,+}}{\Longleftarrow} \\ R(E_3) \overset{P_{1/2,-}}{\Longleftarrow} R(E_2) \overset{P_{1/2,+}}{\Longleftarrow} R(E_1) \overset{P_{1/2,-}}{\Longleftarrow} E_0 = R(E_0) \end{aligned} \tag{6.70}$$

¿From (6.69), (6.70), Lemmas 5.6 and Theorem 4.5 we obtain an orbit homoclinic to L_2^*. ∎

6.3 Homoclinic connection in interior region.

In this section we establish an existence of an orbit homoclinic to L_1^* (see Figure 11). For this sake we find a chain of covering relations, which starts close to L_1^* passes through the sets located in the interior region and then ends close to L_1^*. For this sake we choose the sets F_i along a numerically constructed, (nonrigorous), homoclinic orbit in the vicinity of the intersection of such orbit with the section Θ (see Fig. 11).

We define the following h-sets $F_i = t(Z_i, u_i, s_i)$ where

$$\begin{aligned} Z_0 &= (-0.6160415155975000064, 0), \\ Z_1 &= (0.8466850372287604736 0, 0.17563753764246766080), \\ Z_2 &= (0.94793695784874987520, 0.01522141990729746432), \\ Z_3 &= (0.92067611200358768640, 0.00032764933375860776), \\ Z_4 &= (0.95228425894935162880, 0.00001048139819208300) \end{aligned}$$

and

$$
\begin{array}{rclrcl}
s_0 & = & (-1 \cdot 10^{-7}, 25 \cdot 10^{-8}) & u_0 & = & -R(s_0) \\
s_1 & = & (1 \cdot 10^{-7}, 92 \cdot 10^{-9}) & u_1 & = & -2.2R(s_1) \\
s_2 & = & (-25 \cdot 10^{-9}, \frac{33}{4} \cdot 10^{-8}) & u_2 & = & -R(s_2)/5 \\
s_3 & = & (-1 \cdot 10^{-7}, 26 \cdot 10^{-8}) & u_3 & = & -R(s_3)/6 \\
s_4 & = & (-1 \cdot 10^{-7}, 37 \cdot 10^{-8}) & u_4 & = & -R(s_4)/6
\end{array}
$$

We assume, that $F_0, F_2, F_4 \subset \Theta_-$ and $F_1, F_3 \subset \Theta_+$.

With a computer assistance we proved the following

Lemma 6.6 *The maps*

$$
\begin{array}{rcl}
P_{\frac{1}{2},-} & : & F_0 \cup F_2 \cup F_4 \to \Theta_+ \\
P_{\frac{1}{2},+} & : & F_1 \cup F_3 \to \Theta_-
\end{array}
$$

are well defined and continuous. Moreover, we have the following covering relations

$$
F_0 \overset{P_{1/2,-}}{\Longrightarrow} F_1 \overset{P_{1/2,+}}{\Longrightarrow} F_2 \overset{P_{1/2,-}}{\Longrightarrow} F_3 \overset{P_{1/2,+}}{\Longrightarrow} F_4 \overset{P_{1/2,-}}{\Longrightarrow} H_1^2
$$

∎

We are now ready to state the basic theorem in this section.

Theorem 6.7 *For PCR3BP with $C = 3.03$, $\mu = 0.0009537$ there exists a an orbit homoclinic to L_1^*.*

Proof: From Lemmas 6.6 and 5.6 it follows that

$$
F_0 \overset{P_{1/2,-}}{\Longrightarrow} F_1 \overset{P_{1/2,+}}{\Longrightarrow} F_2 \overset{P_{1/2,-}}{\Longrightarrow} F_3 \overset{P_{1/2,+}}{\Longrightarrow} F_4 \overset{P_{1/2,-}}{\Longrightarrow} H_1^2 \overset{P_+}{\Longrightarrow} H_1 \qquad (6.71)
$$

Observe that from the definition of F_0 it follows that F_0 is R-symmetric. From Corollary 3.14, Lemma 5.6 and equation (6.71) we obtain

$$
H_1 = R(H_1) \overset{P_+}{\Longleftarrow} R(H_1^2) \overset{P_{1/2,+}}{\Longleftarrow} R(F_4) \overset{P_{1/2,-}}{\Longleftarrow} R(F_3) \overset{P_{1/2,+}}{\Longleftarrow} R(F_2) \overset{P_{1/2,-}}{\Longleftarrow} R(F_1) \overset{P_{1/2,+}}{\Longleftarrow} R(F_0) = F_0 \quad (6.72)
$$

¿From (6.71), (6.72), Lemmas 5.6 and Theorem 4.5 we obtain an orbit homoclinic to L_1^*. ∎

Proof of Theorem 1.1: We combine together Theorems 6.3, 6.5 and 6.7. ∎

7 Symbolic dynamics on four symbols

The goal of this section is to give a precise meaning and a proof to Theorem 1.2

As in previous sections in the symbol of covering relation we will drop the degree part, hence we will use $N \stackrel{f}{\Longrightarrow} M$ instead of $N \stackrel{f,w}{\Longrightarrow} M$ for some nonzero w.

From Lemmas 5.6 and 6.2 we know that there exists the following chain of covering relations

$$\begin{aligned} H_1 \stackrel{P_+}{\Longrightarrow} H_1 \stackrel{P_+}{\Longrightarrow} H_1^2 \stackrel{P_{1/2,+}}{\Longrightarrow} N_0 \stackrel{P_{1/2,-}}{\Longrightarrow} N_1 \stackrel{P_{1/2,+}}{\Longrightarrow} N_2 \stackrel{P_{1/2,-}}{\Longrightarrow} N_3 \stackrel{P_{1/2,+}}{\Longrightarrow} N_4 \\ \stackrel{P_{1/2,-}}{\Longrightarrow} N_5 \stackrel{P_{1/2,+}}{\Longrightarrow} N_6 \stackrel{P_{1/2,-}}{\Longrightarrow} N_7 \stackrel{P_{1/2,+}}{\Longrightarrow} H_2^2 \stackrel{P_-}{\Longrightarrow} H_2 \stackrel{P_-}{\Longrightarrow} H_2. \end{aligned} \tag{7.73}$$

From Lemmas 5.5 and 3.12 we have $R(H_i) = H_i$ for $i = 1, 2$.

From Lemmas 5.6, 6.2 and Corollary 3.14 it follows that

$$\begin{aligned} H_2 = R(H_2) \stackrel{P_-}{\Longleftarrow} R(H_2^2) \stackrel{P_{1/2,-}}{\Longleftarrow} R(N_7) \stackrel{P_{1/2,+}}{\Longleftarrow} R(N_6) \stackrel{P_{1/2,-}}{\Longleftarrow} R(N_5) \\ \stackrel{P_{1/2,+}}{\Longleftarrow} R(N_4) \stackrel{P_{1/2,-}}{\Longleftarrow} R(N_3) \stackrel{P_{1/2,+}}{\Longleftarrow} R(N_2) \stackrel{P_{1/2,-}}{\Longleftarrow} R(N_1) \\ \stackrel{P_{1/2,+}}{\Longleftarrow} R(N_0) \stackrel{P_{1/2,-}}{\Longleftarrow} R(H_1^2) \stackrel{P_+}{\Longleftarrow} R(H_1) = H_1 \end{aligned} \tag{7.74}$$

From Lemma 6.4 and the proof of Theorem 6.5 it follows that

$$E_0 \stackrel{P_{1/2,+}}{\Longrightarrow} E_1 \stackrel{P_{1/2,-}}{\Longrightarrow} E_2 \stackrel{P_{1/2,+}}{\Longrightarrow} E_3 \stackrel{P_{1/2,-}}{\Longrightarrow} E_4 \stackrel{P_{1/2,+}}{\Longrightarrow} E_5 \stackrel{P_-}{\Longrightarrow} H_2^2 \stackrel{P_-}{\Longrightarrow} H_2. \tag{7.75}$$

and

$$\begin{aligned} H_2 = R(H_2) \stackrel{P_-}{\Longleftarrow} R(H_2^2) \stackrel{P_-}{\Longleftarrow} R(E_5) \stackrel{P_{1/2,-}}{\Longleftarrow} R(E_4) \stackrel{P_{1/2,+}}{\Longleftarrow} \\ R(E_3) \stackrel{P_{1/2,-}}{\Longleftarrow} R(E_2) \stackrel{P_{1/2,+}}{\Longleftarrow} R(E_1) \stackrel{P_{1/2,-}}{\Longleftarrow} E_0 = R(E_0). \end{aligned} \tag{7.76}$$

From Lemma 6.6 and the proof of Theorem 6.7 it follows that

$$F_0 \stackrel{P_{1/2,-}}{\Longrightarrow} F_1 \stackrel{P_{1/2,+}}{\Longrightarrow} F_2 \stackrel{P_{1/2,-}}{\Longrightarrow} F_3 \stackrel{P_{1/2,+}}{\Longrightarrow} F_4 \stackrel{P_{1/2,-}}{\Longrightarrow} H_1^2 \stackrel{P_+}{\Longrightarrow} H_1 \tag{7.77}$$

and

$$\begin{aligned} H_1 = R(H_1) \stackrel{P_+}{\Longleftarrow} R(H_1^2) \stackrel{P_{1/2,+}}{\Longleftarrow} R(F_4) \stackrel{P_{1/2,-}}{\Longleftarrow} R(F_3) \stackrel{P_{1/2,+}}{\Longleftarrow} R(F_2) \stackrel{P_{1/2,-}}{\Longleftarrow} \\ R(F_1) \stackrel{P_{1/2,+}}{\Longleftarrow} R(F_0) = F_0. \end{aligned} \tag{7.78}$$

We will construct now the symbolic dynamics on four symbols. The construction is a little bit involved, because we have four different maps in all covering relations listed above.

We assign symbols as follows: 1 - the set H_1, 2 - H_2, 3 - E_0 and 4 - F_0. The covering relations allow for transitions $1 \to 1$, $1 \to 2$, $1 \to 4$, $2 \to 1$, $2 \to 2$, $2 \to 3$, $3 \to 2$ and $4 \to 1$. For each such transition $i \to j$ we associate a pair (j, i). This defines a set of admissible pairs Γ.

For any $(\alpha,\beta) \in \Gamma$ we define a map $f_{(\alpha,\beta)}$ as follows

$$f_{(\alpha,\beta)} = \begin{cases} P_+ & \text{if } (\alpha,\beta)=(1,1), \\ P_- \circ P_{1/2,+} \circ (P_{1/2,-} \circ P_{1/2,+})^4 \circ P_+ & \text{if } (\alpha,\beta)=(2,1), \\ P_+ \circ P_{1/2,-} \circ (P_{1/2,+} \circ P_{1/2,-})^4 \circ P_- & \text{if } (\alpha,\beta)=(1,2), \\ P_- & \text{if } (\alpha,\beta)=(2,2) \\ P_{\frac{1}{2},+} \circ (P_{1/2,-} \circ P_{1/2,+})^2 \circ P_{1/2,+} & \text{if } (\alpha,\beta)=(4,1) \\ P_{\frac{1}{2},-} \circ (P_{1/2,+} \circ P_{1/2,-})^2 \circ P_-^2 & \text{if } (\alpha,\beta)=(3,2) \\ P_+ \circ P_{1/2,-} \circ (P_{1/2,+} \circ P_{1/2,0})^2 & \text{if } (\alpha,\beta)=(1,4) \\ P_-^2 \circ P_{1/2,+} \circ (P_{1/2,-} \circ P_{1/2,+})^2 & \text{if } (\alpha,\beta)=(2,3) \end{cases}$$

Let $\Sigma_\Gamma \subset \{1,2,3,4\}^{\mathbb{Z}}$ be defined as follows $c \in \Sigma_\Gamma$ iff for every $i \in \mathbb{Z}$ $(c_i, c_{i+1}) \in \Gamma$. We set $S_1 = H_1$, $S_2 = H_2$, $S_3 = E_0$ and $S_4 = F_0$.

We can now formulate the theorem about an existence of symbolic dynamics on four symbols symbols

Theorem 7.1 *For any sequence $\alpha = \{\alpha_i\} \in \Sigma_\Gamma$ there exists a point $x_0 \in S_{\alpha_0}$, such that*

- *its trajectory exists for $t \in (-\infty, \infty)$*
- $x_n = f_{(\alpha_n,\alpha_{n-1})} \circ \cdots \circ f_{(\alpha_2,\alpha_1)} \circ f_{(\alpha_1,\alpha_0)}(x_0) \in S_{\alpha_n}$ *for $n > 0$*
- $x_n = f^{-1}_{(\alpha_{n+1},\alpha_n)} \circ \cdots \circ f^{-1}_{(\alpha_{-1},\alpha_{-2})} \circ f^{-1}_{(\alpha_0,\alpha_{-1})}(x_0) \in S_{\alpha_n}$ *for $n < 0$.*

Moreover, we have

periodic orbits: *If α is periodic with the principal period equal to k , then x_0 can be chosen so that $x_k = x_0$, hence its trajectory is periodic.*

homo- and heteroclinic orbits: *If α is such that $\alpha_k = i_-$ for $k \leq k_-$ and $\alpha_k = i_+$ for $k \geq k_+$, where $i_-, i_+ \in \{1,2\}$, then*

$$\lim_{n\to-\infty} x_n = L^*_{i_-}, \qquad \lim_{n\to\infty} x_n = L^*_{i_+}$$

Proof: From chains of covering relations (7.73),(7.74), (7.75), (7.76), (7.77), (7.78) and Theorem 3.6 we obtain the statement on periodic points for periodic α. To treat a nonperiodic α we approximate it with periodic sequences β_n with increasing periods to obtain sequence of points x^n and after eventually passing to a subsequence we obtain x_0 with desired properties.

The statement on homo- and heteroclinic orbits is an easy consequence of Theorem 4.5 and the hyperbolicity of $P_\pm$ on H_i established in Lemma 5.5 ∎

Our methods do not allow to make any claims about the uniqueness of x_0 for a given α. The only claims of this type we can make is if $\alpha_n = i$ for all $n \in \mathbb{Z}$ then $x_0 = L^*_i$.

8 Numerical aspects of the proof

In this section we give details of the computer assisted proofs of Lemmas 5.2, 5.3, 5.6 and 6.2. As in previous section in the symbol of covering relation we will drop the degree part, hence we will use $N \overset{f}{\Longrightarrow} M$ instead of $N \overset{f,w}{\Longrightarrow} M$ for some nonzero w.

8.1 The existence and continuity of Poincaré maps. Hyperbolicity on $\mathbf{U}_i$.

All proofs required to check first that suitable Poincaré maps ($P_\pm$, $P_{\frac{1}{2},\pm}$) are defined on some parallelograms (supports of our h-sets) on $\Theta_\pm$. For this end the parallelogram, Z, was represented as a finite union of small parallelograms, Z_i, and each of Z_i's was used as an initial condition for our routine computing the necessary Poincaré map, $P_{1/2,\pm}$ or $P\pm$. We divided horizontal edges on n equal parts (a horizontal grid) and vertical edges on m equal parts (a vertical grid) and hence we covered Z by $n \times m$ parallelograms. Our routine was constructed so that, if completed successfully, then we can claim that Z_i is contained in the domain of P and the computed image contains $P(Z_i)$. Our routine is based on the C^0 and C^1-Lohner algorithms [14, 23].

We had to prove the following assertions

1. (in Lemma 5.2) $P_{\frac{1}{2},+}$ is well defined and continuous on I_1 and $P_{\frac{1}{2},-}$ is well defined and continuous on I_2.
2. (in Lemma 5.3) P_+ is well defined and smooth on U_1, P_- is well defined and smooth on U_2.
3. (in Lemma 5.6 - equations (5.67) and (5.68)). P_- is well defined and continuous on H_2^2. Observe that since $\widetilde{H}_1 \subset U_1$, $\widetilde{H}_2 \subset U_2$, then the previous assertion guarantees an existence and continuity of P_+ on $\widetilde{H}_1$ and P_- on $\widetilde{H}_2$.
4. (in Lemmas 6.2, 6.4 and 6.6) $P_{\frac{1}{2},+}$ is well defined and continuous on H_1^2, N_1, N_3, N_5, N_7, E_0, E_2, E_4, F_1, F_3 . $P_{\frac{1}{2},-}$ is well defined and continuous on N_0, N_2, N_4, N_6, E_1, E_3, F_0, F_2 and F_4. P_- is well defined on E_5.

The first assertion follows easily from the second one. We reason as follows: since $I_i \subset U_i$, then an existence of P_- (P_+) on I_1 (I_2) implies that also $P_{\frac{1}{2},-}$ ($P_{\frac{1}{2},+}$) is defined.

To prove the second assertion we cover U_i by finite number (13×13) of parallelograms. Then we compute an image of each part and an enclosure of the derivative of the Poincaré map using a routine based on C^1-Lohner algorithm recently proposed in [23]. As a consequence we obtain an estimation of $DP_\pm$ (see Lemma 5.3). Parameter settings used in these computations are listed in Table 1. Let us stress also, that a successful termination of our routine proves

set	order	step	horizontal grid	vertical grid
U_1	5	0.007	13	13
U_2	5	0.007	13	13

Table 1: Parameter settings of the Taylor method used in C^1-computations - in the proof of Lemma 5.3

also that P_+ and P_- are defined on U_1 and U_2, respectively. From the standard theory it follows that $P_\pm$ are smooth on their domain.

To prove the third and fourth assertion we proceed in the similar way. We cover each set by finite number of parallelograms and compute an image of each parallelogram. Since an estimation of the derivative of the Poincaré map is not necessary we have used a C^0-Lohner algorithm [14, 23]. Parameter settings for these computations are listed in Table 2.

covering relations	order	step	h. grid	v. grid
$\widetilde{H}_1 \overset{P_+}{\Longrightarrow} H_1^2 \overset{P_{1/2,+}}{\Longrightarrow} N_0 \overset{P_{1/2,-}}{\Longrightarrow} N_1 \overset{P_{1/2,+}}{\Longrightarrow} N_2$ $N_2 \overset{P_{1/2,-}}{\Longrightarrow} N_3 \overset{P_{1/2,+}}{\Longrightarrow} N_4 \overset{P_{1/2,-}}{\Longrightarrow} N_5$ $N_5 \overset{P_{1/2,+}}{\Longrightarrow} N_6 \overset{P_{1/2,-}}{\Longrightarrow} N_7 \overset{P_{1/2,+}}{\Longrightarrow} H_2^2 \overset{P_-}{\Longrightarrow} \widetilde{H}_2,$	5	0.01	1	1
$E_0 \overset{P_{1/2,+}}{\Longrightarrow} E_1 \overset{P_{1/2,-}}{\Longrightarrow} E_2 \overset{P_{1/2,+}}{\Longrightarrow} E_3$ $E_3 \overset{P_{1/2,-}}{\Longrightarrow} E_4 \overset{P_{1/2,+}}{\Longrightarrow} E_5 \overset{P_-}{\Longrightarrow} H_2^2$	5	0.02	1	1
$F_0 \overset{P_{1/2,-}}{\Longrightarrow} F_1 \overset{P_{1/2,+}}{\Longrightarrow} F_2 \overset{P_{1/2,-}}{\Longrightarrow} F_3$ $F_3 \overset{P_{1/2,+}}{\Longrightarrow} F_4 \overset{P_{1/2,-}}{\Longrightarrow} H_1^2 \overset{P_+}{\Longrightarrow} H_1$	5	0.02	1	1

Table 2: Parameters of the Taylor method used in the proof of an existence of the Poincaré map in Lemma 6.2, Lemma 5.6, Lemma 6.4 and Lemma 6.6

8.2 Details of the proof of Lemma 5.2

To prove inequalities (5.51),(5.52) we had to compute rigorous enclosures for $P_{\frac{1}{2},+}(x_1 \pm \eta_1, 0)$ and $P_{\frac{1}{2},-}(x_2 \pm \eta_2, 0)$, respectively. The values of the time step and the order of the Taylor method used in our routine are listed in Table 3. Figures 6 and 7 display the actual enclosures obtained.

orbit	order of Taylor method	time step
L_1^*	20	0.05
L_2^*	19	0.055

Table 3: Settings used in the proof of (5.51) (L_1^*-row) and (5.52)(L_2^*-row)

8.3 How do we verify covering relations - details of proofs of Lemmas 6.2, 6.4 and 6.6

This is the most computationally demanding part of our programm.

In principle the same rigorous computations can be used to obtain both an existence of Poincaré maps and covering relations, but in practice this does'nt work, i.e. it will result in an enormous computation time (see the discussion in Sec. 6 of [5]).

It turns out that once an existence of Poincaré map is established, we can reduce the computations to the boundary of our h-sets and one interval inside, only (see Lemmas 8.3 and 8.4). Now, when we compute an image of an edge I, we still have to divide it into subintervals, but the number of subintervals of the order of square root of the number parallelograms need to achieve the same accuracy on the parallelogram build on two intervals of the linear size similar to that of I.

In order to establish an existence of covering relations we need to verify the assumptions of Theorem 3.9.

To facilitate a discussion about various conditions implying Theorem 3.9 we introduce the following

Definition 8.1 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map and let $N_1 = t(c_1, u_1, s_1)$ and $N_2 = t(c_2, u_2, s_2)$ be two h-sets.*

We say that f satisfies condition **ah, a0, a, b+, b-** *on N_1 and N_2 if*

ah: *there exists $q_0 \in [-1, 1]$, such that*

$$f(c_{N_1}([-1,1] \times \{q_0\})) \subset \text{int}\left(N_2^l \cup |N_2| \cup N_2^r\right)$$

a0: $f(|N_1|) \cap N_2^+ = \emptyset$

a: $f(|N_1|) \subset \text{int}(N_2^l \cup |N_2| \cup N_2^r)$

b+: $f(N_1^{le}) \subset \text{int}(N_2^l)$ *and* $f(N_1^{re}) \subset \text{int}(N_2^r)$

b-: $f(N_1^{le}) \subset \text{int}(N_2^r)$ *and* $f(N_1^{re}) \subset \text{int}(N_2^l)$

We say that f satisfies condition **b** *on N_1 and N_2 if either* **b+** *or* **b-** *is satisfied.*

Remark 8.2 *Observe that conditions (3.23), (3.24), (3.25) and (3.26) from Theorem 3.9 coincide with conditions* **ah**, **a0**, **b+** *and* **b-**, *respectively.*

Observe that condition **a** *implies conditions* **ah** *and* **a0**.

The following lemma gives sufficient conditions for an existence of covering relations for injective maps.

Lemma 8.3 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map and let N_1 and N_2 be two h-sets. Assume that f is injective on $|N_1|$ and f satisfies condition* **b** *on N_1 and N_2 and the following condition* **a'**

a': $f(\partial|N_1|) \subset \mathrm{int}(N_2^l \cup |N_2| \cup N_2^r)$.

Then

$$N_1 \overset{f}{\Longrightarrow} N_2.$$

Proof: From Remark 8.2 and Theorem 3.9 it follows that it is enough to verify condition **a**. This follows easily from **a'** and the Jordan theorem (see [5], page 180). ∎

Figures illustrating covering relations obtained in Lemmas 6.2 and 5.6 suggest that condition **a** is satisfied in all relations. Unfortunately the verification of condition **a** (or **a'**) pose the following difficulty: In the relation $N_1 \overset{f}{\Longrightarrow} N_2$ the set $|N_1|$ is mapped across of N_2, without touching its horizontal edges, but if $|N_2|$ is small then we need a very good estimation of image of horizontal edges of N_1. This forces us to make a very fine partition of the boundary of N_1, take small time steps and a high order in the numerical method resulting in very long computation times.

Above phenomenon is illustrated on Fig. 12, which shows enclosures obtained from our rigorous routines. On this picture we can see rigorous enclosure for an image of $P_{1/2,-}(\partial N_6)$. This image was obtained as follows: we divided the boundary of the set N_6 into some number of subintervals (see Table 4) and computed an image of each part via $P_{1/2,-}$. This picture shows, that much tighter enclosure of an image of horizontal edge was required compared to an enclosure for an image of vertical edges (for example edge $(N_6)^{le}$ was divided into 8 equal parts, but $(N_6)^{be}$ into 5 equal parts). In other covering relations this disproportion was often much bigger.

To deal with this problem we use the following lemma, in which we indirectly verify conditions **a0** and **ah** instead of **a'** . This approach allowed us to reduce the computation time by a factor of 5.

Lemma 8.4 *Let $N_1 = t(c_1, u_1, s_1)$, $N_2 = t(c_2, u_2, s_2)$ are h-sets, $f : |N_1| \longrightarrow \mathbb{R}^2$ an injection of class $\mathcal{C}^1$. Let γ be a horizontal line in $|N_1|$ connecting vertical edges given by*

$$\gamma : [-1,1] \ni t \longrightarrow c_1 + t \cdot u_1 \in |N_1| \tag{8.79}$$

Let $g = (g_1, g_2) = c_{N_2}^{-1} \circ f \circ \gamma$. Assume f satisfies condition **b** *on N_1 and N_2 and the following conditions hold:*

a1 $\frac{dg_1}{dt}(t) \neq 0$ *for $t \in (-1,1)$*

a2 *there exists $t_0 \in (-1,1)$ such that $f(\gamma(t_0)) \in \mathrm{int}(|N_2|)$*

a3 $f^{-1}(N_2^+) \cap |N_1| = \emptyset$

Then $N_1 \overset{f}{\Longrightarrow} N_2$.

Proof: We need to show that conditions **ah** and **a0** are satisfied.

Observe that condition **a0** follows immediately from condition **a3** and injectivity of the map f. Namely, by applying f to both sides of **a3** we obtain $N_2^+ \cap f(|N_1|) = \emptyset$.

We now show that condition **ah** is true.

For this end we consider $f \circ \gamma$ in the coordinates induced by the map c_{N_2}. In these coordinates

$$|N_2| = [-1,1] \times [-1,1] \tag{8.80}$$
$$N_2^r = [1,\infty) \times (-\infty,\infty) \tag{8.81}$$
$$N_2^l = (-\infty,-1] \times (-\infty,\infty) \tag{8.82}$$
$$f \circ \gamma = g \tag{8.83}$$

Without any loss of generality we can assume that

$$\frac{dg_1}{dt}(t) > 0, \quad \text{for} \quad t \in (-1,1), \tag{8.84}$$

Hence g_1 is a strictly increasing function and from condition **b** it follows that **b+** is satisfied.

We define two numbers

$$t^* = \min\{t > t_0 \mid f(\gamma(t)) \in \partial|N_2|\}, \tag{8.85}$$
$$t_* = \max\{t < t_0 \mid f(\gamma(t)) \in \partial|N_2|\}. \tag{8.86}$$

¿From conditions **a2**, **a3** and **b** it follows that these numbers are well defined $t_* < t^*$ and

$$f(\gamma([-1,t_*))) \subset \text{int}(N_2^l), \tag{8.87}$$
$$f(\gamma((t^*,1])) \subset \text{int}(N_2^r), \tag{8.88}$$
$$f(\gamma((t_*,t^*))) \subset \text{int}(|N_2|). \tag{8.89}$$

To finish the proof observe that from condition **b** it follows that

$$f(\gamma(\pm 1)) \in \text{int}\left(N_2^r \cup N_2^l\right)$$

∎

Remark 8.5 *Observe that* $c_{N_2}^{-1}(x) = A^{-1}(x - c_2)$, *where* $A = [u_2^T, s_2^T]$.

Hence

$$\frac{dg_1}{dt}(t) = \sum_{i,j=1}^{2} A_{1i}^{-1} df(\gamma(t))_{ij} u_{1,j}, \tag{8.90}$$

where $u_1 = (u_{1,1}, u_{1,2})$.

Remark 8.6 *Observe that, when* $N_1 = t(c_1, u, s)$ *and* $N_2 = t(c_2, \alpha u, s_2)$, *which means that the unstable coordinate direction for both h-sets coincide, then*

$$\frac{dg_1}{dt}(t) = \alpha(A^{-1} \cdot df(\gamma(t)) \cdot A)_{11}. \tag{8.91}$$

Hence it is enough to look at the $(1,1)$ *entry of* df *expressed in the coordinates of the h-set* N_2.

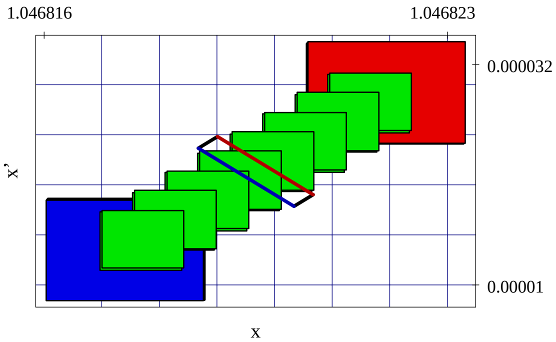

Figure 12: An example of the rigorous enclosure of an image of ∂N_6 in relation $N_6 \overset{P_{1/2,-}}{\Longrightarrow} N_7$.

In Table 4 we present settings used in the proof of Lemma 6.2. In particular the parameter *grid* gives the number of equal intervals into which we divide an edge. A positive time step means that Lemma 8.3 is used to verify a covering relation. A negative time step means that we use Lemma 8.4 and symbolizes the fact that we compute an inverse of the Poincaré map to verify condition **a3**. Parameter settings for the verification of **a1** and **a2** are given in Table 5. In this table *order(m)* and *step(m)* denote an order and a time step of Taylor method which we use to prove **a1** and *order(c)*, *step(c)* denote an order and a time step of Taylor method which we use to prove **a2**. Parameter *grid(m)* denotes a number of equal intervals used to cover the curve γ in condition **a1**.

To verify condition **a2** we usually compute image of the center of the set $(t_0 = 0)$. Only in the case $N_7 \overset{P_{1/2,+}}{\Longrightarrow} H_2^2$ we used $t_0 = 0.228$.

Parameters of the Taylor method used in the proof of conditions of Lemma 6.4 are listed in Tables 6 and 7. In the relation $E_5 \overset{P_-}{\Longrightarrow} H_2^2$ we do not compute an image of the center of set E_5 but an image of the point $C = Y_5 - 0.155u_5$.

Parameters settings used in the proof of Lemma 6.6 are listed in Tables 8 and 7.

8.4 Verification of covering relations for fuzzy set - details of the proof of Lemma 5.6

In this subsection we discuss how we verify covering relations for fuzzy h-sets. It is convenient to think about a fuzzy h-set $\widetilde{N}$ as an parallelogram with thickened edges. We define the support, left and right edges and left and right sides of a fuzzy set $\widetilde{N}$ as follows

$$\begin{array}{rclrcl} |\widetilde{N}| &=& \langle\bigcup_{M\in\widetilde{N}} |M|\rangle, & \partial\widetilde{N} &=& \bigcup_{M\in\widetilde{N}} \partial|M|, \\ \widetilde{N}^{le} &=& \langle\bigcup_{M\in\widetilde{N}} M^{le}\rangle, & \widetilde{N}^{re} &=& \langle\bigcup_{M\in\widetilde{N}} M^{re}\rangle \\ \widetilde{N}^{l} &=& \bigcap_{M\in\widetilde{N}} M^{l}, & \widetilde{N}^{r} &=& \bigcap_{M\in\widetilde{N}} M^{r}. \end{array}$$

covering relation	edges	grid	order	step
$H_1^2 \overset{P_{1/2},+}{\Longrightarrow} N_0$	$(H_1^2)^{be}$ and $(H_1^2)^{te}$	6	4	0.01
	$(H_1^2)^{re}$ and $(H_1^2)^{le}$	7	5	0.01
$N_0 \overset{P_{1/2},-}{\Longrightarrow} N_1$	N_1^{be} and N_1^{te}	1	5	-0.01
	N_0^{re} and N_0^{le}	40	8	0.04
$N_1 \overset{P_{1/2},+}{\Longrightarrow} N_2$	N_2^{be} and N_2^{te}	8	6	-0.01
	N_1^{re} and N_1^{le}	25	5	0.01
$N_2 \overset{P_{1/2},-}{\Longrightarrow} N_3$	N_3^{be} and N_3^{te}	6	6	-0.004
	N_2^{re} and N_2^{le}	5	6	0.004
$N_3 \overset{P_{1/2},+}{\Longrightarrow} N_4$	N_4^{be} and N_4^{te}	3	6	-0.005
	N_3^{re} and N_3^{le}	5	6	0.004
$N_4 \overset{P_{1/2},-}{\Longrightarrow} N_5$	N_5^{be} and N_5^{te}	2	6	-0.01
	N_4^{re} and N_4^{le}	15	6	0.006
$N_5 \overset{P_{1/2},+}{\Longrightarrow} N_6$	N_6^{be} and N_6^{te}	2	6	-0.01
	N_5^{re} and N_5^{le}	32	6	0.01
$N_6 \overset{P_{1/2},-}{\Longrightarrow} N_7$	N_6^{be} and N_6^{te}	8	6	0.01
	N_6^{re} and N_6^{le}	5	6	0.01
$N_7 \overset{P_{1/2},+}{\Longrightarrow} H_2^2$	$(H_2^2)^{be}$ and $(H_2^2)^{te}$	2	5	-0.01
	N_7^{re} and N_7^{le}	33	5	0.01

Table 4: Parameters of the Taylor method used in the proof of Lemma 6.2

covering relations	order(m)	step(m)	grid(m)	order(c)	step(c)
$N_0 \overset{P_{1/2},-}{\Longrightarrow} N_1 \overset{P_{1/2},+}{\Longrightarrow} N_2$	4	0.01	1	6	0.01
$N_2 \overset{P_{1/2},-}{\Longrightarrow} N_3$	4	0.004	4	6	0.004
$N_3 \overset{P_{1/2},+}{\Longrightarrow} N_4$	4	0.003	2	6	0.003
$N_4 \overset{P_{1/2},-}{\Longrightarrow} N_5$	4	0.004	1	6	0.004
$N_5 \overset{P_{1/2},+}{\Longrightarrow} N_6 \overset{P_{1/2},+}{\Longrightarrow} H_2^2$	4	0.01	1	6	0.01
$H_2^2 \overset{P_-}{\Longrightarrow} \widetilde{H}_2$	4	0.01	1	—	—

Table 5: Parameters of the Taylor method used in the proof of conditions **a1, a2** in Lemmas 6.2 and 5.6

covering relation	edges	grid	order	step
$E_0 \overset{P_{1/2},+}{\Longrightarrow} E_1$	E_1^{be} and E_1^{te}	33	7	-0.04
	E_0^{re} and E_0^{le}	12	7	0.04
$E_1 \overset{P_{1/2},-}{\Longrightarrow} E_2$	E_2^{be} and E_2^{te}	12	8	-0.02
	E_1^{re} and E_1^{le}	25	7	0.02
$E_2 \overset{P_{1/2},+}{\Longrightarrow} E_3$	E_3^{be} and E_3^{te}	1	7	-0.02
	E_2^{re} and E_2^{le}	50	7	0.03
$E_3 \overset{P_{1/2},-}{\Longrightarrow} E_4$	E_4^{be} and E_4^{te}	2	7	-0.02
	E_3^{re} and E_3^{le}	7	7	0.03
$E_4 \overset{P_{1/2},+}{\Longrightarrow} E_5$	E_5^{be} and E_5^{te}	2	7	-0.02
	E_4^{re} and E_4^{le}	20	7	0.03
$E_5 \overset{P_-}{\Longrightarrow} H_2^2$	$(H_2^2)^{be}$ and $(H_2^2)^{te}$	1	5	-0.02
	E_5^{re} and E_5^{le}	4	5	0.01

Table 6: Parameters of the Taylor method used in the proof of covering relations in Lemma 6.4.

covering relations	o(m)	s(m)	g(m)	o(c)	s(c)
$E_0 \overset{P_{1/2},+}{\Longrightarrow} E_1 \overset{P_{1/2},-}{\Longrightarrow} E_2 \overset{P_{1/2},+}{\Longrightarrow} E_3$ $E_3 \overset{P_{1/2},-}{\Longrightarrow} E_4 \overset{P_{1/2},+}{\Longrightarrow} E_5$	4	0.02	1	7	0.02
$E_5 \overset{P_-}{\Longrightarrow} H_2^2$	5	0.02	1	8	0.02
$F_0 \overset{P_{1/2},-}{\Longrightarrow} F_1$	5	0.01	2	8	0.02
$F_1 \overset{P_{1/2},+}{\Longrightarrow} F_2 \overset{P_{1/2},-}{\Longrightarrow} F_3 \overset{P_{1/2},+}{\Longrightarrow} F_4 \overset{P_{1/2},-}{\Longrightarrow} H_1^2$	4	0.01	1	8	0.02
$H_1^2 \overset{P_+}{\Longrightarrow} H_1$	4	0.01	1	–	–

Table 7: Parameters of the Taylor method used in the proof of conditions **a1, a2** in Lemma 6.4 and Lemma 6.6.

covering relation	edges	grid	order	step
$F_0 \overset{P_{1/2,-}}{\Longrightarrow} F_1$	F_1^{be} and F_1^{te}	50	6	-0.01
	F_0^{re} and F_0^{le}	20	8	0.02
$F_1 \overset{P_{1/2,+}}{\Longrightarrow} F_2$	F_2^{be} and F_2^{te}	9	7	-0.02
	F_1^{re} and F_1^{le}	330	7	0.02
$F_2 \overset{P_{1/2,-}}{\Longrightarrow} F_3$	F_3^{be} and F_3^{te}	1	7	-0.02
	F_2^{re} and F_2^{le}	35	7	0.03
$F_3 \overset{P_{1/2,+}}{\Longrightarrow} F_4$	F_4^{be} and F_4^{te}	1	7	-0.02
	F_3^{re} and F_3^{le}	10	7	0.03
$F_4 \overset{P_{1/2,-}}{\Longrightarrow} H_1^2$	$(H_1^2)^{be}$ and $(H_1^2)^{te}$	3	7	-0.02
	F_4^{re} and F_4^{le}	45	7	0.03
$H_1^2 \overset{P_+}{\Longrightarrow} H_1$	H_1^{be} and H_1^{te}	3	8	-0.02
	$(H_1^2)^{re}$ and $(H_1^2)^{le}$	7	7	0.03

Table 8: Parameters of the Taylor method used in the proof of covering relations for sets F_i.

where by $\langle Z \rangle$ we denoted a convex hull of the set Z. We introduce one more notation for allowed image of the h-set covering $\widetilde{N}$

$$\text{strip}(\widetilde{N}) = \bigcap_{M \in \widetilde{N}} \text{int}\,(M^l \cup |M| \cup M^r)$$

Lemmas 8.3 and 8.4 can be easily adopted to fuzzy h-sets. Namely we have

Lemma 8.7 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map and let $\widetilde{N}_1$ and $\widetilde{N}_2$ be two fuzzy h-sets. Assume that f is injective on $|N_1|$ and the following conditions* **af** *and* **bf** *are satisfied:*

af $f(|\widetilde{N}_1|) \subset \text{strip}(\widetilde{N}_2)$

bf *either* $f(\widetilde{N}_1^{le}) \subset \text{int}(\widetilde{N}_2^l)$ *and* $f(\widetilde{N}_1^{re}) \subset \text{int}(\widetilde{N}_2^r)$
or $f(\widetilde{N}_1^{le}) \subset \text{int}(\widetilde{N}_2^r)$ *and* $f(\widetilde{N}_1^{re}) \subset \text{int}(\widetilde{N}_2^l)$.

Then $\widetilde{N}_1 \overset{f}{\Longrightarrow} \widetilde{N}_2$. ∎

Lemma 8.8 *Let $\widetilde{N}_1 = t(W_1, u_1, s_1)$, $\widetilde{N}_2 = t(W_2, u_2, s_2)$ be fuzzy h-sets and $f : |\widetilde{N}_1| \longrightarrow \mathbb{R}^2$ an injection of class $\mathcal{C}^1$. Let γ be a fuzzy horizontal line in $|\widetilde{N}_1|$, given by*

$$\gamma : [-1, 1] \times W_1 \ni (t, w_1) \longrightarrow w_1 + t \cdot u_1 \in |\widetilde{N}_1| \qquad (8.92)$$

For $w_2 \in W_2$ let $N_{2,w_2} = t(w_2, u_2, s_2)$ and

$$g_{w_2}(t, w_1) = (g_{w_2,1}, g_{w_2,2})(t, w_1) = c_{N_{2,w_2}}^{-1} \circ f \circ \gamma(t, w_1).$$

Assume that W_1 is connected, f satisfies condition **bf** *on $\widetilde{N}_1$ and $\widetilde{N}_2$ and the following conditions hold:*

af1 $\frac{dg_{w_2,1}}{dt}(t, w_1) \neq 0$ *for* $t \in (-1,1)$, $w_1 \in W_1$, $w_2 \in W_2$

af2 *there exists* $t_0 \in (-1,1)$ *and* $w_1 \in W_1$ *such that* $f(\gamma(t_0, w_1)) \in \mathrm{int}(|\widetilde{N}_2|)$

af3 $f^{-1}(\widetilde{N}_2^+) \cap |\widetilde{N}_1| = \emptyset$

Then $\widetilde{N}_1 \stackrel{f}{\Longrightarrow} \widetilde{N}_2$. ∎

Remark 8.9 *Observe that (compare Remark 8.5) that* $\frac{dg_{w_2,1}}{dt}$ *does not depend on* w_2 *and is given by formula (8.90) with the matrix* A *depending only on* u_2 *and* s_2.

Let us describe how above conditions **af1, af2, af3** were verified for relations (5.67) and (5.68), which we rewrite below for the convenience of the reader

$$\widetilde{\widetilde{H_1}} \stackrel{P_+}{\Longrightarrow} \widetilde{\widetilde{H_1}} \stackrel{P_+}{\Longrightarrow} H_1^2 \tag{8.93}$$

$$H_2^2 \stackrel{P_-}{\Longrightarrow} \widetilde{\widetilde{H_2}} \stackrel{P_-}{\Longrightarrow} \widetilde{\widetilde{H_2}}. \tag{8.94}$$

Let us recall (see Section 5.2) that the h-sets entering above covering relations are given by

$$\begin{array}{rcl} \widetilde{H}_i &=& t(W_i, \alpha_i u_i, \alpha_i u_i) \\ H_1^2 &=& t(h_1, 2\cdot 10^{-7} u_1, 2 \cdot 10^{-7} s_1) \\ H_2^2 &=& t(h_2, 1.2 \cdot 10^{-8} u_2, 2.8 \cdot 10^{-8} s_2) \end{array}$$

For all covering relations, $N \stackrel{P_\pm}{\Longrightarrow} M$, listed above the unstable vectors for both h-sets entering the relations are proportional, hence we can apply Remark 8.6 and look on $(1,1)$-entry of $dP_\pm$ expressed in c_M-coordinates.

Observe that, since $\widetilde{H}_i \subset U_i$ then from Lemma 5.3 and equation (5.61) we obtain an enclosure for $[DP_\pm(|\widetilde{H}_i)]$ expressed in $\widetilde{H}_i$-coordinates. From inspection of $\lambda_{i,1}$ it follows that condition **af1** holds for covering relations $\widetilde{H}_i \Longrightarrow \widetilde{H}_i$ and $\widetilde{H}_1 \Longrightarrow H_1^2$. Since $|H_2^2|$ is not contained in U_2 we had to verify condition **af1** for the relation $H_2^2 \stackrel{P_-}{\Longrightarrow} \widetilde{H}_2$. Parameter settings for this computation is added as the last row to Table 5.

Condition **af2** is clearly satisfied with $w_2 = L_1^*$ for relations (5.67) and $w_2 = L_2^*$ for relations (5.68).

Conditions **af3** and **bf** must be verified in direct computations. Parameter settings for these computations are given in Table 9.

It turns out that some inclusions involved in condition **b** can be verified at the same time. For example, to prove that $\widetilde{H}_1 \stackrel{P_+}{\Longrightarrow} \widetilde{H}_1 \stackrel{P_+}{\Longrightarrow} H_1^2$ we need to verify condition **b+** for both relations.

$$P_+(\widetilde{H}_1^{re}) \subset \widetilde{H}_1^r, \quad P_+(\widetilde{H}_1^{le}) \subset \widetilde{H}_1^l, \tag{8.95}$$

$$P_+(\widetilde{H}_1^{re}) \subset (H_1^2)^r, \quad P_+(\widetilde{H}_1^{le}) \subset (H_1^2)^l. \tag{8.96}$$

It is sufficient to show that

$$P_+(\widetilde{H}_1^{re}) \subset \widetilde{H}_1^r \cap (H_1^2)^r \tag{8.97}$$
$$P_+(\widetilde{H}_1^{le}) \subset \widetilde{H}_1^l \cap (H_1^2)^l \tag{8.98}$$

Similarly, to prove that $H_2^2 \overset{P_-}{\Longrightarrow} \widetilde{H}_2 \overset{P_-}{\Longrightarrow} \widetilde{H}_2$ we must verify condition **af3** for both relations. Namely, we have to check that

$$P_-^{-1}(\widetilde{H}_2^+) \cap |\widetilde{H}_2| = \emptyset, \tag{8.99}$$
$$P_-^{-1}(\widetilde{H}_2^+) \cap |H_2^2| = \emptyset. \tag{8.100}$$

Since $|\widetilde{H}_2| \subset |H_2^2|$ (compare (5.57 and (5.66)) it is sufficient to show (8.100) only.

covering relations	edges	grid	order	time step
$\widetilde{H}_1 \overset{P_+}{\Longrightarrow} \widetilde{H}_1$, $\widetilde{H}_1 \overset{P_+}{\Longrightarrow} H_1^2$	$(\widetilde{H}_1)^{be}$ and $(\widetilde{H}_1)^{te}$	2	6	-0.01
	$(\widetilde{H}_1)^{re}$ and $(\widetilde{H}_1)^{le}$	3	6	0.01
	$(H_1^2)^{be}$ and $(H_1^2)^{te}$	2	5	-0.01
$H_2^2 \overset{P_-}{\Longrightarrow} \widetilde{H}_2$, $\widetilde{H}_2 \overset{P_-}{\Longrightarrow} \widetilde{H}_2$	$(\widetilde{H}_2)^{be}$ and $(\widetilde{H}_2)^{te}$	4	8	-0.02
	$(H_2^2)^{re}$ and $(H_2^2)^{le}$	32	5	0.01
	$(\widetilde{H}_2)^{re}$ and $(\widetilde{H}_2)^{le}$	2	8	0.02

Table 9: Parameters of the Taylor method used in the proof of the covering relations for fuzzy sets.

9 Concluding remarks, future work

There are several directions in which this research can be extended.

First, all the methods presented in this paper are not restricted to the particular parameters of the *Oterma* comet, other parameters may require slight changes in the definition of the sets on which covering relations are build, but the method itself will be the same. Basically this method can be applied to prove a symbolic dynamics in any system for which numerical simulations indicate an existence of some kind of hyperbolic behavior, for example here we have homo- and heteroclinic chains.

Another interesting problem is the question of an existence of a hyperbolic invariant set claimed in [11], where the authors assumed an existence of transversal homo- and heteroclinic connections between Lyapunov orbits and then followed the standard dynamical system theory argument from the Birkhoff-Smale homoclinic theorem. Since we did'nt computed here unstable and stable manifolds, we cannot use these arguments. Observe also that an rigorous computation of stable and unstable manifolds for our problem appears to be very difficult (requires very extensive C^1-computations). Hence developing tools which avoid a direct computation of invariant manifolds is of interest. In this context we formulate the following conjecture.

Conjecture 9.1 *Let f be a diffeomorphism. Let N_0, N_1 be h-sets. Assume that f is hyperbolic on N_0 and N_1 (in the sense of Def.4.3). Assume that we have the following sequences of covering relations*

$$N_0 \overset{f}{\Longrightarrow} N_0 \overset{f}{\Longrightarrow} A_1 \overset{f}{\Longrightarrow} A_2 \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} A_s \overset{f}{\Longrightarrow} N_1$$
$$N_1 \overset{f}{\Longrightarrow} N_1 \overset{f}{\Longrightarrow} B_1 \overset{f}{\Longrightarrow} B_2 \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} B_r \overset{f}{\Longrightarrow} N_0,$$

then there exists $k \geq 1$ and $S \subset |N_0| \cup |N_1|$, such that

- *$f^k(S) = S$, i.e. S is an invariant set for f^k*
- *S is hyperbolic (in the standard sense - see for example [9])*
- *the map $\pi : S \to \Sigma_2 = \{0, 1\}$ given by $\pi(x)_i = j$ iff $f^{ki}(x) \in |N_j|$ is one-to-one.*

∎

Another interesting problem is a question of stability of obtained results with respect to various extension of PCR3BP. By this we mean the following:

- Does the symbolic dynamics persists if the Jupiter orbit become an ellipse with a small eccentricity (which is the case in nature)? This can be seen as a small periodic perturbation to the ODE describing PCR3BP. We believe that an answer is positive. Obviously in this context one can consider a more general question:
Assume that we obtained a symbolic dynamics for an ODE $x' = f(x)$ using covering relations. Does this symbolic dynamics persists for an nonautonomous ODE $x' = f(x) + \epsilon(t, x)$ if $\epsilon(t, x)$ is small ?
- What about an restricted three body problem in three dimensions? One obvious observation is that plane (x, y) is invariant for full 3D problem, hence we have symbolic dynamics also in a spatial problem. We would like to pose a more general question: *Does there exists a symbolic dynamics for 3D problem such the corresponding orbits are not all contained in the Sun-Jupiter plane?* Some preliminary numerical explorations in this direction can be found in paper [8],
- What about full 3-body problem? Will the symbolic dynamics established here continue to very small but nonzero mass of a comet? Some results in this direction for non-degenerate periodic orbits and generic bifurcations can be found in [15].

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