Rigorous numerics for homoclinic dynamics

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1 Introduction

Homoclinic and heteroclinic connections of hyperbolic objects play an important role in the study of dynamical systems from a global point of view. They were used in the design of space missions using libration point dynamics [1], among which the Genesis [2] has been the first one to make use of a heteroclinic connection.

For the design of such missions, the circular Restricted Three Body Problem (RTBP) is the natural problem to start with. Of special interest are the L_1 and L_2 libration points because of their suitability to place stationary satellites.

It turns out that the existence of homoclinic and heteroclinic orbits as well as chaotic dynamics for a given parameter values of the system is rather difficult to prove by means of an analytic approach. In this report we propose a method that allows us to verify the existence of homoclinic and heteroclinic orbits for maps and flows by means of a computer assisted proof. The method is geometric in the spirit and it assumes that the map is of class C^1 and that we can compute rigorous bounds for values and derivatives of the map. The method has been applied to the Planar Circular Restricted Three Body Problem (PCR3BP).

Let S and J be two bodies called Sun and Jupiter, of masses $m_s = 1-\mu$ and $m_j = \mu$, $\mu \in (0,1)$, respectively. They rotate in the plane on circles counter clockwise about their common center of mass and with the angular velocity normalized to one. Choose a rotating coordinate system, so that the origin is at the center of mass and the Sun and the Jupiter are fixed on the x-axis at $(-\mu, 0)$ and $(1-\mu, 0)$, respectively. In this coordinate frame the equations of motion of a massless particle called the comet or the spacecraft under the gravitational action of the Sun and the Jupiter are (see [3] and references given there)

$$\ddot{x} - 2\dot{y} = \Omega_x(x,y), \qquad \ddot{y} + 2\dot{x} = \Omega_y(x,y), \tag{1}$$

where

$$\Omega(x,y) = \frac{x^2+y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1-\mu)}{2}$$
$$r_1 = \sqrt{(x+\mu)^2 + y^2}, \qquad r_2 = \sqrt{(x-1+\mu)^2 + y^2}.$$

Equations (1) are called the equations of the Planar Circular Restricted Three-Body Problem (PCR3BP). They have a first integral called the *Jacobi integral*, which is given by

$$C(x,y,\dot{x},\dot{y}) = -(\dot{x}^2 + \dot{y}^2) + 2\Omega(x,y).$$

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We consider the PCR3BP on the hypersurface

$$\mathcal{M}(\mu, C) = \{(x, y, \dot{x}, \dot{y}) \mid C(x, y, \dot{x}, \dot{y}) = C\}$$

and we restrict our attention to the following parameter values $C = 3.03$, $\mu = 0.0009537$ - the parameter values for the *Oterma* comet in the Sun-Jupiter system (see [3]).

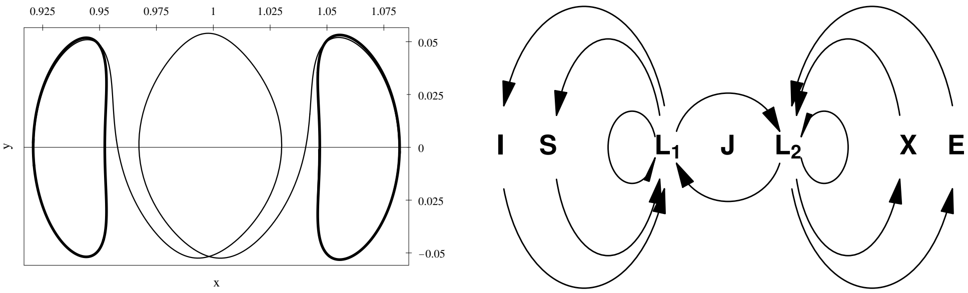

Figure 1: Left: the primary heteroclinic connection between L_1 and L_2 orbits. Right: the graph of symbolic dynamics for the PCR3BP.

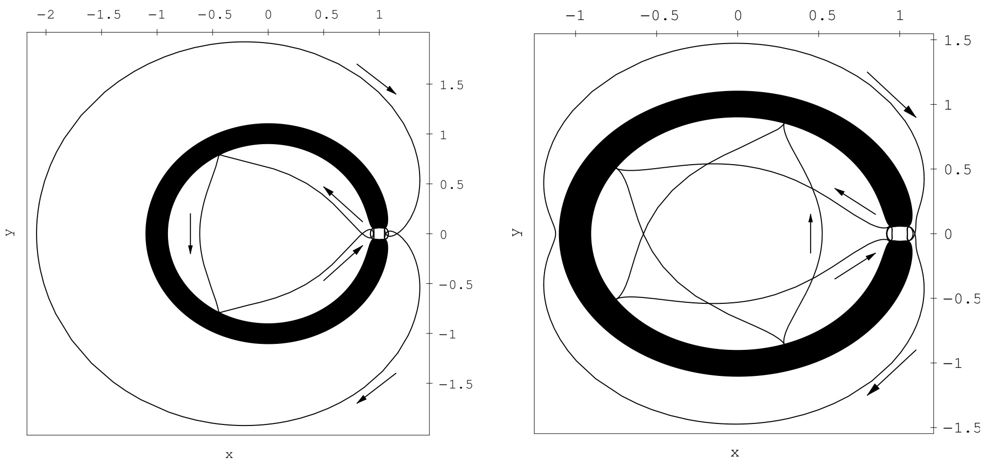

Figure 2: Homoclinic orbits to L_1 and L_2 orbits. The projection of the energy surface $\mathcal{M}(\mu, C)$ for $\mu = 0.0009537$ and $C = 3.03$ onto (x, y) coordinates does not contain the C-shaped black region. In this region a comet cannot move due to its energy level.

Theorem 1 *[5, 6] For the parameter values* $C = 3.03$, $\mu = 0.0009537$ *there are two periodic solutions of the equation (1) around the libration points* L_1^* *and* L_2^* *called the Lyapunov orbits and denoted by* L_1 *and* L_2*, respectively. Moreover, there exist*

- *heteroclinic orbits in both directions connecting* L_1 *and* L_2 *– see Fig. 1 left panel,*
- *two geometrically different homoclinic solutions in the Sun region to the* L_1 *orbit, denoted by* S *and* I *– see Fig. 2,*

- *two geometrically different homoclinic solutions in the exterior region to the L_2 orbit, denoted by X and E – see Fig. 2.*

Moreover, for any biinfinite path on the graph presented in Fig. 1 (right panel)

$$\mathbf{a} = (a_i)_{i\in\mathbb{Z}} \in \{L_1, L_2, X, E, I, S\}^{\mathbb{Z}}$$

- *there exists a true orbit of the PCR3BP which stays close (explicitly given estimation) to the orbits $\{L_1, L_2, X, E, I, S\}$ with the order given by the sequence $\mathbf{a}$,*
- *if the sequence $\mathbf{a}$ is periodic then the corresponding solution to PCR3BP can be chosen to be periodic,*
- *if the sequence $\mathbf{a}$ has the form*

$$(\ldots, L_k, L_k, a_0, a_1, \ldots, a_N, L_s, L_s, \ldots)$$

  then there exists a solution $u(t)$ of the equation (1) such that

  – *the omega limit set $\omega(u) = L_s$,*
  – *the alpha limit set $\alpha(u) = L_k$,*
  – *there is a part of trajectory of u which stays close to the orbits $(a_0, a_1, \ldots, a_N)$ with the preserved order.*

2 Topological tools.

The main idea of the proof of Theorem 1 is to use the method of covering relations in order to prove the existence of orbits which intersect given subsets of the phase space with a desired order. The existence of homoclinic and heteroclinic orbits requires an argument for the convergence of an orbit. This will be verified by means of the cone conditions and presented in the next section.

Definition 1 *[7] Let N be a parallelogram with distinguished left and right halfplanes N^r and N^l and corresponding right and left edges N^{re}, and N^{le} as presented in Fig. 3 (left panel). We will call such an object an h-set.*

Let N, M be h-sets and let $f\colon N \to \mathbb{R}^2$ be continuous.

Definition 2 *[7] We say that N f-covers M and denote this by $N \overset{f}{\Longrightarrow} M$ if*

- $f(N) \subset \mathrm{int}(M \cup M^r \cup M^l)$
- *$f(N^{re})$ and $f(N^{le})$ are mapped into different halfplanes $\mathrm{int}(M^r)$ and $\mathrm{int}(M^l)$.*

The geometry of this concept is presented in Fig. 3 - right panel.

The following theorem is the main tool used in this paper for proving the existence of chaotic dynamics.

Theorem 2 *[7] Assume $\{N_i\}_{i=1,\ldots,K}$ are pairwise disjoint h-sets. Let $(i_j)_{\mathbb{Z}}$ be a biinfinite sequence such that*

$$N_{i_j} \overset{f}{\Longrightarrow} N_{i_{j+1}}.$$

Then there exists a sequence $(x_j)_{j\in\mathbb{Z}}$ such that $f(x_j) = x_{j+1}$ and $x_j \in \mathrm{int}(N_{i_j})$ for $j \in \mathbb{Z}$. If the sequence $(i_j)_{\mathbb{Z}}$ is periodic then x_0 can be chosen to be a periodic point for f with the same principal period.

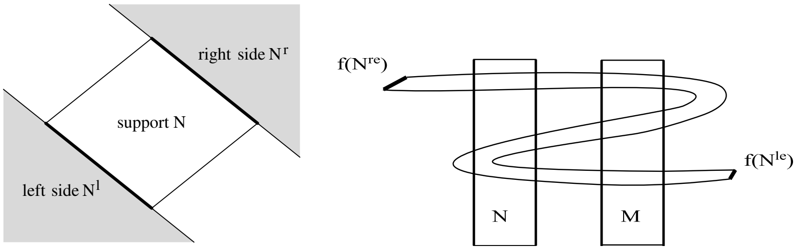

Figure 3: Left: an h-set. Right: geometry of the covering relations. In this case $N \stackrel{f}{\Longrightarrow} M$ and $N \stackrel{f}{\Longrightarrow} N$.

3 Cone conditions.

Let $Q\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a quadratic form. An h-set N with associated quadratic form Q_N will be called an h-set with cones and denoted by (N, Q_N).

Definition 3 *[4] Let $f\colon N \to \mathbb{R}^2$ be a smooth map. Let (N, Q_N) and (M, Q_M) be h-sets with cones. We will say that f satisfies the cone conditions with respect to the pair (N, M) if for $x, y \in N$ holds*

$$Q_M(f(x) - f(y)) > Q_N(x - y).$$

The following theorem can be used to prove the convergence of a trajectory to a fixed point.

Theorem 3 *[4] Let (N, Q_N) be an h-set with cones. Assume that $N \stackrel{f}{\Longrightarrow} N$ and f satisfies the cone conditions with respect to the pair (N, N). Then*

- *the map f has unique fixed point x_* in N,*
- *if $x \in N$ is such that $f^k(x) \in N$ for $k \in \mathbb{N}$ then $\lim_{k\to\infty} f^k(x) = x_*$,*
- *if $(x_k)_{k\leq 0}$ is such that $f(x_{k-1}) = x_k$ and $x_k \in N$ for $k \leq 0$ then $\lim_{k\to-\infty} x_k = x_*$.*

4 Proof of Theorem 1.

The proof of Theorem 1 consists of the following steps which were verified by means of verified numerics.

- After fixing the Jacobi constants C and the mass ratio μ corresponding to the Oterma comet the motion is on three dimensional isoenergetic manifold $\mathcal{M}(\mu, C)$. On this manifold we choose a Poincaré section $\Pi = \mathcal{M}(\mu, C) \cap \{y = 0\}$.
- We construct two h-sets with cones (H_1, Q_1), (H_2, Q_2) on Π and verified that Theorem 3 apply to a suitable Poincaré map P and the sets (H_i, Q_i), $i = 1, 2$. This proves the existence of two periodic orbits (called Lyapunov orbits) L_1 and L_2 which intersect the sets H_1 and H_2, respectively, at exactly one point.
- We construct six chains of h-sets along homoclinic and heteroclinic orbits presented in Fig 1 and Fig. 2 such that

$$H_{k_i} \stackrel{P}{\Longrightarrow} M_{i,1} \stackrel{P}{\Longrightarrow} \cdots \stackrel{P}{\Longrightarrow} M_{i,N_i} \stackrel{P}{\Longrightarrow} H_{s_i},$$

for $i = 1, \ldots, 6$, where P is a suitable Poincaré map. The cases $k_i \neq s_i$ correspond to heteroclinic connections between L_1 and L_2 in both directions. The cases $k_i = s_i = 1$ and $k_i = s_i = 2$ correspond to two pairs of homoclinic orbits in the interior and the exterior regions, respectively.

Now the assertion follows from Theorems 2 and 3.

We would like to point out here that the verification of the cone conditions required verified integration of the variational equations associated to the equations for the PCR3BP.

References and Notes

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