COMPUTER ASSISTED PROOF OF CHAOTIC DYNAMICS IN THE RÖSSLER MAP

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Abstract. In this paper we present the proof of the existence of symbolic dynamics for third iterate of the Rössler map. We combine an abstract topological results based on the fixed point index and covering relations with computer assisted rigorous computations.

1. Introduction

In this paper we are interested in the Rössler map $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$\mathcal{R}_1(x, y) = 3.8x(1 - x) - 0.1y \qquad \mathcal{R}_2(x, y) = 0.2(y - 1.2)(1 - 1.9x). \tag{1.1}$$

This two-dimensional "walking-stick diffeomorphism" was introduced in [6] as a simplicification of Poincaré cross section of the Rössler differential equation

$$\dot{x} = -y - z, \quad \dot{y} = x + 0.25y + w$$
$$\dot{z} = 3 + xz, \quad \dot{w} = -0.5z + 0.05w$$

In the numerical simulations of discrete dynamical system induced by (1.1) one observes the attracting set (see Fig.1) with very rich and chaotic dynamics. By Devaney's definition [3] the map is chaotic iff it is topologically transitive and if the set of periodic points for this map is dense. In this paper we show the existence of the periodic points of an arbitrary period. Moreover, we prove the existence of symbolic dynamics on three symbols. We use computer only to rigorously verify some inclusions.

The proof of the main Theorems is given in Section 4. In Section 5 we discuss the topological entropy of the Rössler map.

2. Main result

We will use the following standard notations and definitions. Let (X, ρ) be a metric space. For any $Z \subset X$ by $\mathrm{int}(Z)$, $\mathrm{cl}(Z)$, $\mathrm{bd}(Z)$, we denote the interior, closure and the boundary of the set Z respectively.

For a map $f : X \longrightarrow X$ and $N \subset X$, we denote by $\mathrm{Inv}(f, N)$ the maximal invariant part of N, i.e.

$$\mathrm{Inv}(f, N) := \bigcap_{i \in \mathbb{Z}} \left(f|_N\right)^{-i}(N) \tag{2.1}$$

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For fixed $K \in \mathbb{N}$, $\Sigma_K := \{0, 1, \ldots, K-1\}^{\mathbb{Z}}$ is a topological space with Tichonov topology [4]. We consider the shift map $\sigma : \Sigma_K \longrightarrow \Sigma_K$ given by

$$\sigma((c_i)_{i\in\mathbb{Z}}) := (c_{i+1})_{i\in\mathbb{Z}} \tag{2.2}$$

Let $A = [\alpha_{ij}]$ be any $K \times K$ matrix, where $\alpha_{ij} \in \mathbb{R}_+ \cup \{0\}$ for all $i, j = 0, 1, \ldots, K-1$. We define the subset of admissible sequences $\Sigma_A \subset \Sigma_K$ as

$$\Sigma_A := \left\{ c = (c_i)_{i\in\mathbb{Z}} \;\middle|\; \alpha_{c_i c_{i+1}} > 0 \right\} \tag{2.3}$$

It is easy to see that $\sigma(\Sigma_A) = \Sigma_A$ and $\sigma : \Sigma_A \longrightarrow \Sigma_A$ is a well defined homeomorphism. We denote by N_0, N_1, N_2 parallelograms on the plane $\mathbb{R}^2$ given by

$$N_0 := conv\{a_0, b_0, c_0, d_0\}$$
$$N_1 := conv\{a_1, b_1, c_1, d_1\}$$
$$N_2 := conv\{a_2, b_2, c_2, d_2\}$$

where

$$\begin{aligned}
a_0 &:= \{\ 0.6230\ ,\ 0.1000\ \}, & b_0 &:= \{\ 0.6590\ ,\ 0.0920\ \} \\
c_0 &:= \{\ 0.6600\ ,\ 0.1320\ \}, & d_0 &:= \{\ 0.6240\ ,\ 0.1400\ \} \\
a_1 &:= \{\ 0.7094\ ,\ 0.0808\ \}, & b_1 &:= \{\ 0.7670\ ,\ 0.0680\ \} \\
c_1 &:= \{\ 0.7680\ ,\ 0.1080\ \}, & d_1 &:= \{\ 0.7104\ ,\ 0.1208\ \} \\
a_2 &:= \{\ 0.9250\ ,-0.0070\ \}, & b_2 &:= \{\ 0.8950\ ,-0.0370\ \} \\
c_2 &:= \{\ 0.9100\ ,-0.0520\ \}, & d_2 &:= \{\ 0.9400\ ,-0.0220\ \}
\end{aligned}$$

Let $N = N_0 \cup N_1 \cup N_2$ and A be the transition matrix (see Fig.1 and Lemma 4.4)

$$A := \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Let $\mathcal{R}^n$ denote the n-th iterate of the Rössler map. It is the aim of this paper to prove the following theorems:

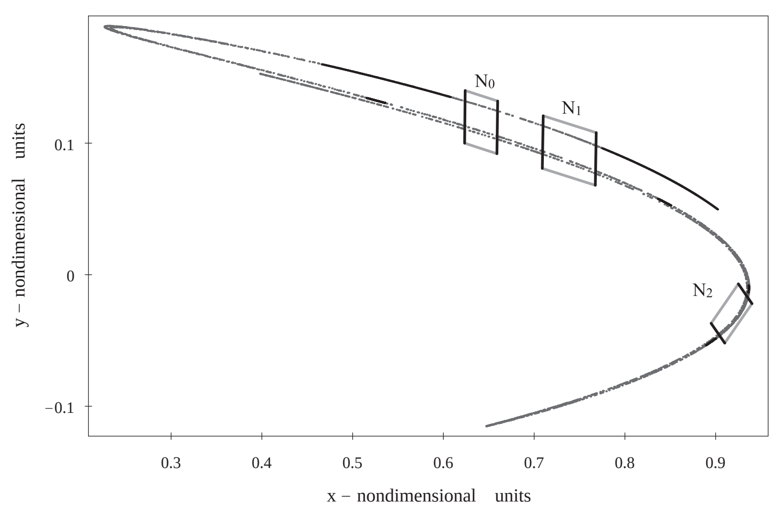

FIGURE 1. Attractor, sets N_0, N_1, N_2 and images of vertical edges.

Theorem 2.1.

(1) *There exists a continuous projection* $\pi : Inv(\mathcal{R}^3, N) \longrightarrow \Sigma_3$ *such that*

$$\pi \circ \mathcal{R}^3 = \sigma \circ \pi \tag{2.4}$$

$$\Sigma_A \subset \pi(Inv(\mathcal{R}^3, N)) \tag{2.5}$$

Moreover, the preimage of any periodic sequence contains a periodic point for $\mathcal{R}^3$ *with the same period.*

(2) *For any* $n \geq 1$ *there exists a periodic point for* $\mathcal{R}^3$ *of the basic period* n.

Theorem 2.2. *There exists a continuous projection* $\pi : Inv(\mathcal{R}^6, N_0 \cup N_1) \longrightarrow \Sigma_2$ *such that*

$$\pi \circ \mathcal{R}^6 = \sigma \circ \pi \tag{2.6}$$

$$\Sigma_2 \subset \pi(Inv(\mathcal{R}^6, N_0 \cup N_1)) \tag{2.7}$$

Moreover, the preimage of any periodic sequence contains a periodic point for $\mathcal{R}^6$ *with the same period.*

3. Topological tools

The proof of the main theorems is based on fixed point index and covering relations, which were introduced in [8] and [9].

Definition 3.1. [1, Def. 1] *A triple set (or t-set) is a triple* $N = (|N|, N^l, N^r)$ *of closed subsets of* $\mathbb{R}^2$ *satisfying the following properties:*

(1) $|N|$ *is a parallelogram,* N^l *and* N^r *are half-planes*
(2) $N^l \cap N^r = \emptyset$
(3) *the sets* $N^{le} := N^l \cap |N|$ *and* $N^{re} := N^r \cap |N|$ *are two nonadjacent edges of* $|N|$

We call $|N|$, N^l, N^r, N^{le}, *and* N^{re} *the support, the left side, the right side, the left edge, and the right edge of the t-set* N *respectively.*

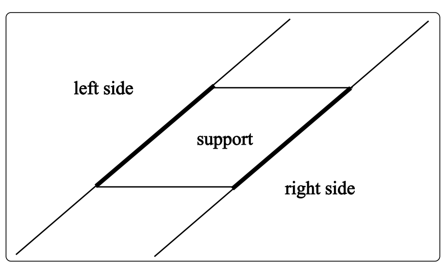

Figure 2. Support, right side, left side and vertical edges of t-set.

A typical t-set is presented in Fig.2. The definition of the triple set is in fact too restrictive. We can take any t-set homeomorphic to this definition.

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a map, and N, M be triple sets.

Definition 3.2. [1, Def. 2] *We say that* N f*-covers* M *if the following conditions hold*

a: $f(|N|) \subset int(M^l \cup |M| \cup M^r)$

b: *either* $f(N^{le}) \subset int(M^l)$ *and* $f(N^{re}) \subset int(M^r)$
or $f(N^{le}) \subset int(M^r)$ *and* $f(N^{re}) \subset int(M^l)$

We write $N \overset{f}{\Longrightarrow} M$.

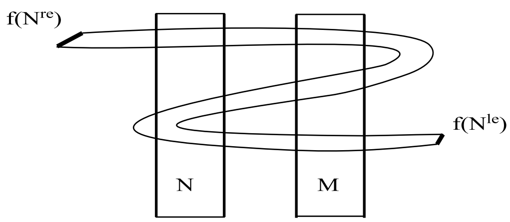

FIGURE 3. An example of covering relations $N \overset{f}{\Longrightarrow} N$, $N \overset{f}{\Longrightarrow} M$.

The next theorem plays a significant role in the proof of Theorems 2.1 and Theorem 2.2 (for more details see [1, Theorem 1] or [9, Theorem 1]).

Theorem 3.3. *Let* $M_0, M_1, \ldots, M_{n-1}$ *be triple sets and let* $f : \bigcup_{i=0}^{n-1} M_i \longrightarrow \mathbb{R}^2$ *be a continuous map. Suppose that*

$$M_0 \overset{f}{\Longrightarrow} M_1 \overset{f}{\Longrightarrow} M_2 \cdots \overset{f}{\Longrightarrow} M_{n-1} \overset{f}{\Longrightarrow} M_0 = M_n. \tag{3.1}$$

Then, there exists the point $x \in int|M_0|$ *such that* $f^k(x) \in int|M_k|$, *for* $k = 1, \ldots, n$ *and* $x = f^n(x)$.

4. PROOFS

Given triple sets $M_1, M_2, \ldots, M_n$, let $M = \bigcup_{i=1}^n |M_i|$. Suppose $f : M \longrightarrow \mathbb{R}^2$ is a continuous function.

Definition 4.1. *The transition matrix* T_{ij}, $i, i = 1, \ldots, n$, *is defined, as follows:*

$$T_{ij} = \begin{cases} 1 & \text{if } M_j \overset{f}{\Longrightarrow} M_i \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Definition 4.2. *Assume* $|M_i| \cap |M_j| = \emptyset$ *for* $i \neq j$ *and* $f : M \longrightarrow f(M)$ *is a homeomorphism. The projection* $\pi : Inv(f, M) \longrightarrow \Sigma_n$ *is defined by the condition*

$$\pi(u)_j = i \iff f^j(u) \in M_i \quad \text{for} \quad u \in Inv(f, M) \tag{4.2}$$

Lemma 4.3. *Assume that* M_i *are t-sets, for* $i = 1, \ldots, n$ *and* $M = \bigcup_{i=1}^n |M_i|$. *Let* $f : M \longrightarrow f(M)$ *be a homeomorphism,* T *be the transition matrix of covering relations. Then, the projection* π *is continuous and*

$$\Sigma_T \subset \pi(Inv(f, M)). \tag{4.3}$$

Proof. The set Σ_T inherits the topology from Σ_n. First of all, we prove that π is continuous. Let

$$C_m^s := \{(c_i)_{i\in\mathbb{Z}} \mid c_m = s\}, \quad \text{where} \quad s \in \{1, 2, \ldots, n\} \tag{4.4}$$

Since f is a homeomorphism, we have

$$\pi^{-1}(C_m^s) = \{u \in \mathrm{Inv}(f, M) \mid f^m(u) \in M_s\} = f^{-m}(\mathrm{Inv}(f, M) \cap M_s) \tag{4.5}$$

and $\pi^{-1}(C^s_m)$ is an open set because $\text{Inv}(f,M)\cap M_s$ is open in $\text{Inv}(f,M)$. It follows that π is continuous.

To prove (4.3) let us observe (see [7, Theorem 5.12]) that the set of periodic sequences $Per(\Sigma_T)$ is dense in Σ_T . By Theorem 3.3 $Per(\Sigma_T) \subset \pi(\text{Inv}(f,M))$. By continuity of π we have the compactness of $\pi(\text{Inv}(f,M))$, hence

$$\Sigma_T = \text{cl}(Per(\Sigma_T)) \subset \pi(\text{Inv}(f,M))$$

□

With computer assistance we proved

Lemma 4.4. *We have the following covering relations*

$$\begin{aligned} &N_1 \overset{\mathcal{R}^3}{\Longrightarrow} N_1 \overset{\mathcal{R}^3}{\Longrightarrow} N_0 \\ &N_0 \overset{\mathcal{R}^3}{\Longrightarrow} N_2 \overset{\mathcal{R}^3}{\Longrightarrow} N_0 \\ &N_2 \overset{\mathcal{R}^3}{\Longrightarrow} N_1 \end{aligned} \tag{4.6}$$

Proof. We implemented our algorithm in Borland C++. Rigorous numerical computations and round-off errors (see [5]) will not be presented in this paper.

All required inclusions were checked on the Intel Celeron 333A processor within less than 1 second. □

Proof of Theorem 2.1. The first assertion is a direct consequence of Lemma 4.4 and Lemma 4.3. To prove the second assertion, using Lemma 4.4, one can observe that the following covering relations hold:

$$N_1 \overset{\mathcal{R}^3}{\Longrightarrow} N_1 \tag{4.7}$$

$$N_0 \overset{\mathcal{R}^3}{\Longrightarrow} N_2 \overset{\mathcal{R}^3}{\Longrightarrow} N_0 \tag{4.8}$$

$$N_1 \overset{\mathcal{R}^3}{\Longrightarrow} N_0 \overset{\mathcal{R}^3}{\Longrightarrow} N_2 \overset{\mathcal{R}^3}{\Longrightarrow} N_1 \tag{4.9}$$

From (4.7), (4.8), (4.9) and Theorem 3.3 we obtain the periodic points for $\mathcal{R}^3$ of basic period $1,2,3$. If we combine (4.7) with (4.9) we have periodic points for $\mathcal{R}^3$ of every period $n \geq 4$.

□

Proof of Theorem 2.2. Let us observe that Lemma 4.4 implies

$$\begin{aligned} &N_0 \overset{\mathcal{R}^3}{\Longrightarrow} N_2 \overset{\mathcal{R}^3}{\Longrightarrow} N_0 \\ &N_0 \overset{\mathcal{R}^3}{\Longrightarrow} N_2 \overset{\mathcal{R}^3}{\Longrightarrow} N_1 \\ &N_1 \overset{\mathcal{R}^3}{\Longrightarrow} N_1 \overset{\mathcal{R}^3}{\Longrightarrow} N_0 \\ &N_1 \overset{\mathcal{R}^3}{\Longrightarrow} N_1 \overset{\mathcal{R}^3}{\Longrightarrow} N_1 \end{aligned}$$

Now, Theorem 2.2 is a simply consequence of Lemma 4.3.

□

5. TOPOLOGICAL ENTROPY

Let $f : X \longrightarrow X$ be a continuous function on a compact metric space. By $h(f)$ we denote the topological entropy of f. For any invariant set $S \subset X$ we have $h(f|_S) \leq h(f)$ [7, p. 167]. The next lemma follows directly from [7, Theorem 7.2] and [7, Theorem 7.13].

Lemma 5.1. *Assume $f : X \longrightarrow X$ admits the semi-conjugacy with the shift of n symbols, i.e. there exists invariant subset $S \subset X$ and the continuous surjection $\pi : S \longrightarrow \Sigma_T$, such that*

$$\pi \circ f = \sigma \circ \pi,$$

where T is the transition matrix of this conjugacy. Then $h(f) \geq \ln(\lambda)$, where λ is the greatest positive eigenvalue of T.

Let W be the rectangle on the plane containing N given by

$$W = [\,0.01\,,\ 0.99\,] \times [-0.33\,,\ 0.27\,] \subset \mathbb{R}^2.$$

We want to estimate the topological entropy of $\mathcal{R}^3|_W$. Below we show that the restriction $\mathcal{R}^3|_W : W \longrightarrow W$ is well defined.

Lemma 5.2. *Set W is a positively invariant, i.e. $\mathcal{R}(W) \subset W$.*

Proof. Let $(x\,,\ y) \in W$. We have

$$\begin{aligned}
\mathcal{R}_1(x,y) &= 3.8x(1-x) - 0.1y \leq 3.8 \cdot \left(\frac{1}{2}\right)^2 + 0.1 \cdot 0.33 \\
&= 0.983 < 0.99 \\
\mathcal{R}_1(x,y) &= 3.8x(1-x) - 0.1y \geq 3.8 \cdot 0.99 \cdot 0.01 - 0.1 \cdot 0.27 \\
&= 0.01062 > 0.01
\end{aligned}$$

and similarly

$$\begin{aligned}
\mathcal{R}_2(x,y) &= 0.2(y-1.2)(1-1.9x) \leq 0.2(-1.53)(1-1.9 \cdot 0.99) \\
&= 0.269586 < 0.27 \\
\mathcal{R}_2(x,y) &= 0.2(y-1.2)(1-1.9x) \geq 0.2(-0.33-1.2)(1-1.9 \cdot 0.01) \\
&= -0.300186 > -0.33
\end{aligned}$$

□

Let us observe that Lemma 5.2 guarantees the existence of a compact, connected invariant set.

We apply Lemma 5.1 to estimate the entropy of the Rössler map. In our case the characteristic polynomial of the transition matrix A has the following form $p(\lambda) = -\lambda(\lambda^2 - \lambda - 1)$. Maximal positive eigenvalue of A, $\lambda_0 = \frac{1}{2}(1+\sqrt{5})$. It follows that topological entropy

$$h(\mathcal{R}^3|_W) \geq \ln(\lambda_0) > 0.48$$

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